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Existence Theorems for Certain Nonlinear Equations

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Introduction

The aim of this paper is to give theorems on the existence of solutions of equations of the forms

$$\text{(a)} \qquad Lu + Au = f$$

and

$$\text{(b)} \qquad u + LAu = f,$$

where L is a linear mapping and A is a nonlinear mapping.

It is known that if L is a linear maximal monotone mapping and A is a bounded pseudomonotone mapping, then (a) and (b) have solutions (see [4; THEOREM 1] and [7; THEOREM 1]). H. Brezis [3] introduced a class of nonlinear mappings, called of type M, of a Banach space into its dual space, which contains the class of pseudomonotone mappings, and then showed in [2] that if L is a linear monotone mapping which is V-regular and A is a bounded mapping of type M, then (a) has a solution. We shall show that if L is a linear maximal monotone mapping and A is a bounded mapping of type M, then (a) and (b) admit solutions; thus the above two results are corollaries to our theorem.

§1. Definitions and notation

Let V be a real reflexive Banach space and V^* its dual space with the dual norm. We denote the norm of $x \in V$ by $\|x\|_V$, the norm of $x^* \in V^*$ by $\|x^*\|_{V^*}$ and the natural pairing between V^* and V by $\langle , \rangle$. We use the symbols "$\xrightarrow{s}$", "$\xrightarrow{W}$" and "$\xrightarrow{W^*}$" to denote the convergence in the strong, weak and weak* topology respectively.

Let T be a multivalued mapping of V into V^* (i.e., to each $x \in V$, a subset Tx of V^* is assigned). The sets $D(T)=\{x \in V;\ Tx \neq \phi\}$, $R(T)=\bigcup_{x\in V} Tx$ and $G(T)=\{(x, x^*) \in V\times V^*;\ x^* \in Tx\}$ are called the domain, the range and the graph of T respectively. The inverse T^{-1} of T is the multivalued mapping defined by $T^{-1}x^*=\{x \in V;\ x^* \in Tx\}$ with the domain $D(T^{-1})=R(T)$.

A multivalued mapping T of V into V^* is called monotone if for any

$(x_i, x_i^*) \in G(T)$ $(i=1, 2)$,

$$<x_1^*-x_2^*, x_1-x_2>\geq 0,$$

and it is called maximal monotone if there is no proper monotone extension of T. A multivalued mapping T is called coercive if

$$\inf_{x^*\in Tx}\frac{<x^*, x>}{\|x\|_V}\to\infty \text{ as } \|x\|_V\to\infty,\ x\in D(T).$$

A singlevalued mapping T of $D(T)=V$ into V^* is called of type M([3]) if it satisfies the following conditions (M_1) and (M_2).

(M_1) If $\{x_i\}$ is a net such that $\|x_i\|_V\leq K$, $x_i\xrightarrow{W}x$ in V, $Tx_i\xrightarrow{W^*}x^*$ in V^* and $\limsup_i<Tx_i, x_i>\leq<x^*, x>$, then $Tx=x^*$.

(M_2) The restriction of T to any finite dimensional subspace of V is continuous with respect to the weak* topology.

We generalize the notion of mappings of type M to the multivalued case. For a multivalued mapping T of $D(T)=V$ into V^*, we consider the following conditions.

(m_1) If $\{x_i\}$ and $\{x_i^*\}$ are nets such that $x_i^*\in Tx_i$, $\|x_i\|_V\leq K$, $x_i\xrightarrow{W}x$ in V, $x_i^*\xrightarrow{W^*}x^*$ in V^* and $\limsup_i<x_i^*, x_i>\leq<x^*, x>$, then $x^*\in Tx$.

(m_2) The restriction of T to any finite dimensional subspace F of V is upper semicontinuous with respect to the weak* topology, that is, for any $x\in F$ and any weak*-neighborhood U^* of Tx, there exists a neighborhood U of x in F such that $U^*\supset T(U)=\bigcup_{x\in U}Tx$.

(m_3) For each $x\in V$, Tx is a bounded closed convex subset of V^*.

REMARK 1. It is easy to see that if T is bounded, that is, T maps bounded subsets of V to bounded subsets of V^*, then (m_1) and (m_3) imply (m_2), and if V is finite dimensional, then (m_2) and (m_3) imply (m_1).

Let J be the duality mapping of V, that is, J be defined by $Jx=\{x^*\in V^*;$ $<x^*, x>=\|x\|_V^2, \|x^*\|_{V^*}=\|x\|_V\}$ for each $x\in V$. In general, J is multivalued. The inverse J^{-1} is the duality mapping of V^*. It is known that if V^* is strictly convex, then Jx consists of a single element for each $x\in V$.

§ 2. Multivalued mappings satisfying (m_1), (m_2) and (m_3)

In the rest of this paper we assume that V is a real reflexive Banach space.

THEOREM 1. *Let A be a multivalued mapping of $D(A)=V$ into V^* satisfying (m_1), (m_2) and (m_3). Let C be a bounded closed convex subset of V containing the origin 0 in its interior. Suppose that*

$$<x^*, x> \geq 0 \qquad \textit{for any } x \in \partial C \textit{ and any } x^* \in Ax,$$

where ∂C is the boundary of C. Then the set $S=\{x \in C; 0 \in Ax\}$ is non-empty and weakly compact.

To prove this theorem we use the following lemma.

LEMMA 1. *Let V be finite dimensional and A be a multivalued mapping of $D(A)=V$ into V^* such that*

(1) *Ax is a bounded closed convex subset of V^* for each $x \in V$,*

(2) *A is upper semicontinuous.*

Let C be a bounded closed convex subset of V. Then there exist $x_0 \in C$ and $x_0^* \in Ax_0$ such that

$$<x_0^*, x_0-x> \leq 0 \qquad \textit{for all } x \in C.$$

We omit the proof of LEMMA 1, since this lemma is a special case of THEOREM 6 in [6].

PROOF OF THEOREM 1: Let $\{V_\alpha; \alpha \in \Lambda\}$ be the family of all finite dimensional subspaces of V, j_α the canonical injection of V_α into V and j_α^* the adjoint of j_α. We define an order "$\leq$" in the index set Λ by inclusion of corresponding subspaces, that is, for $\alpha, \beta \in \Lambda$, $\alpha \leq \beta$ if and only if $V_\alpha \subset V_\beta$. Then Λ is a directed set. For each $\alpha \in \Lambda$, we set $A_\alpha = j_\alpha^* A j_\alpha$. It is easy to see that each A_α satisfies conditions (1) and (2) in LEMMA 1. Therefore there exist $x_\alpha \in C \cap V_\alpha$ and $x_\alpha^* \in Ax_\alpha$ such that

$$(2.1) \qquad <x_\alpha^*, x_\alpha - x> \leq 0 \qquad \textit{for all } x \in C \cap V_\alpha.$$

In the case where $x_\alpha \in C-\partial C$, we have

$$(2.2) \qquad <x_\alpha^*, x> = 0 \qquad \textit{for all } x \in V_\alpha.$$

In the case where $x_\alpha \in \partial C$, noting that $<x_\alpha^*, x_\alpha> \geq 0$ by hypothesis and $<x_\alpha^*, x_\alpha> \leq 0$ by (2.1), we obtain $<x_\alpha^*, x_\alpha> = 0$. Thus also in this case we have (2.2). Hence,

$$(2.3) \qquad j_\alpha^* x_\alpha^* = 0.$$

Since C is weakly compact, there exists a subnet $\{x_{\alpha_i}\}$ of $\{x_\alpha\}_{\alpha \in \Lambda}$ such that $x_{\alpha_i} \xrightarrow{W} x_0 \in C$. By (2.3), $x_{\alpha_i}^* \xrightarrow{W^*} 0$ and $\limsup_i <x_{\alpha_i}^*, x_{\alpha_i}> = 0$. Condition (m_1) implies $0 \in Ax_0$. Thus S is non-empty. The weak compactness of S follows from (m_1). q.e.d.

COROLLARY. *Let A be a multivalued mapping of $D(A)=V$ into V^**

satisfying (m_1), (m_2) *and* (m_3). *Suppose that* A *is coercive. Then* $R(A)=V^*$.

PROOF. Let y^* be an arbitrary element of V^*. We define a mapping A' by $A'x=Ax-y^*$. It is easy to see that A' satisfies conditions (m_1), (m_2) and (m_3). By the coerciveness of A, there exists a positive numberr such that $<x^*, x>\geq 0$ for any $x \in \partial B_r$ and any $x^* \in A'x$, where $B_r=\{x \in V; \|x\|_V \leq r\}$. Therefore by THEOREM 1 we obtain $R(A') \ni 0$, that is, $R(A) \ni y^*$. Thus $R(A)=V^*$. q.e.d.

§ 3. Nonlinear functional equations

In this section we shall show the existence of solutions of nonlinear functional equations of the forms (a) and (b).

The existence of solutions of (a) is given by the following theorem.

THEOREM 2. *Let* C *be a bounded closed convex subset of* V *containing the origin* 0 *in its interior. Let* L *be a multivalued maximal monotone mapping of* $D(L) \subset V$ *into* V^* *such that the graph* $G(L)$ *is a linear subspace in* $V \times V^*$ *and* A *be a bounded multivalued mapping of* $D(A)=V$ *into* V^* *satisfying* (m_1) *and* (m_3). *Suppose that*

$$(3.1) \qquad <x^*, x>\geq 0 \qquad \text{for any } x \in \partial C \text{ and any } x^* \in Ax.$$

Then the set $S=\{x \in C; Lx+Ax \ni 0\}$ *is non-empty and weakly compact.*

The method of proof is based on that of THEOREM 19 in [3]. To prove the above theorem, we prepare three lemmas.

LEMMA 2. *Let* T *be a multivalued maximal monotone mapping of* $D(T) \subset V$ *into* V^*. *Then* $x_n^* \in Tx_n$, $n=1, 2, \cdots$, $x_n \xrightarrow{s} x_0$ *and* $x_n^* \xrightarrow{W^*} x_0^*$ *imply that* $x_0 \in D(T)$ *and* $x_0^* \in Tx_0$.

PROOF. From the monotonicity of T it follows that

$$<x_n^*-x^*, x_n-x>\geq 0 \qquad \text{for any } (x, x^*) \in G(T).$$

Letting $n \to \infty$, we have

$$<x_0^*-x^*, x_0-x>\geq 0 \qquad \text{for any } (x, x^*) \in G(T).$$

The maximal monotonicity of T implies that $(x_0, x_0^*) \in G(T)$. q.e.d.

REMARK 2. From LEMMA 2 it follows that the graph $G(L)$ of the mapping L in THEOREM 2 is strongly closed in $V \times V^*$. Since $G(L)$ is a linear subspace, it is weakly closed in $V \times V^*$.

LEMMA 3. *Let V and V^* be strictly convex and L be as in* THEOREM 2. *For $\varepsilon>0$, we set $L_\varepsilon=(L^{-1}+\varepsilon J^{-1})^{-1}$. Then L_ε is a singlevalued, bounded, demicontinuous (i.e., $x_n \xrightarrow{s} x$ implies that $L_\varepsilon x_n \xrightarrow{W^*} L_\varepsilon x$) and maximal monotone mapping with the domain $D(L_\varepsilon)=V$ and $L_\varepsilon 0=0$.*

PROOF. It is evident that L^{-1} is a maximal monotone mapping with the domain $D(L^{-1})=R(L)\subset V^*$. From a result in [5] it follows that $L^{-1}+\varepsilon J^{-1}$: $D(L^{-1})=R(L)\to V$ is a maximal monotone coercive mapping and $R(L^{-1}+\varepsilon J^{-1})=V$. Thus $D(L_\varepsilon)=V$ and L_ε is a bounded maximal monotone mapping.

Let x^* and y^* be contained in $L_\varepsilon x$. Then $L^{-1}x^*+\varepsilon J^{-1}x^*\ni x$ and $L^{-1}y^*+\varepsilon J^{-1}y^*\ni x$. Therefore there exist $x'\in L^{-1}x^*$ and $y'\in L^{-1}y^*$ such that $x'+\varepsilon J^{-1}x^*=y'+\varepsilon J^{-1}y^*=x$. We have

$$
\begin{aligned}
0&=<x^*-y^*,\ x'+\varepsilon J^{-1}x^*-y'-\varepsilon J^{-1}y^*>\\
&=<x^*-y^*,\ x'-y'>+\varepsilon<x^*-y^*, J^{-1}x^*-J^{-1}y^*>\\
&\geq\varepsilon[\|x^*\|_{V^*}^2-<x^*,\ J^{-1}y^*>-<y^*,\ J^{-1}x^*>+\|y^*\|_{V^*}^2]\\
&\geq\varepsilon(\|x^*\|_{V^*}-\|y^*\|_{V^*})^2.
\end{aligned}
$$

Hence $\|x^*\|_{V^*}=\|y^*\|_{V^*}$ and $<y^*, J^{-1}x^*>=\|y^*\|_{V^*}^2$. This implies $x^*=y^*$. Thus L_ε is singlevalued.

Let $x_n^*=L_\varepsilon x_n$ and $x_n\xrightarrow{s} x_0$. By the boundedness of L_ε, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k}^*\xrightarrow{W^*} x_0^*$. From LEMMA 2 it follows that $x_0^*=L_\varepsilon x_0$. This implies that $x_n^*\xrightarrow{W^*} x_0^*=L_\varepsilon x_0$. Thus L_ε is demicontinuous.

Since $G(L)$ is linear, $0\in L^{-1}0$. Therefore we have $0\in L^{-1}0=L^{-1}0+\varepsilon J^{-1}0=(L^{-1}+\varepsilon J^{-1})0$. Thus $0=L_\varepsilon 0$. q.e.d.

LEMMA 4. *Let V and V^* be strictly convex, and let C, A and L be as in* THEOREM 2; (3.1) *is assumed as well. For each $\varepsilon>0$ and each $\alpha\in\Lambda$, we set*

$$A_{\varepsilon,\alpha}=j_\alpha^*(L_\varepsilon+A)j_\alpha,$$

where $L_\varepsilon=(L^{-1}+\varepsilon J^{-1})^{-1}$ and Λ, j_α and j_α^* are as in the proof of THEOREM 1. *Then each $A_{\varepsilon,\alpha}$ is bounded and satisfies conditions (m_1), (m_3) and the boundary condition:*

(3.2) $<x^*, x>\geq 0$ *for any $x\in\partial(C\cap V_\alpha)$ and any $x^*\in A_{\varepsilon,\alpha}x$.*

PROOF. It is clear that each $A_{\varepsilon,\alpha}$ is bounded and satisfies (m_3). Since $<L_\varepsilon x, x>\geq 0$ for any $x\in V$ by LEMMA 3, we have (3.2), by making use of (3.1).

To verify condition (m_1) for $A_{\varepsilon,\alpha}$, it is sufficient to show that $G(A_{\varepsilon,\alpha})$ is closed in $V_\alpha\times V_\alpha^*$. Let $\{x_n\}\subset V_\alpha$ and $\{y_n^*\}$ be sequences such that $y_n^*\in A_{\varepsilon,\alpha}x_n$,

$x_n \to x$ in V_α and $y_n^* \to y^*$ in V_α^*. For each n we have $y_n^* = j_\alpha^*(L_\varepsilon x_n + x_n^*)$ for some $x_n^* \in Ax_n$. Since A is bounded, there is a subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$ such that $x_{n_k}^* \xrightarrow{W^*} x^*$ in V^*. Since $x_n \xrightarrow{s} x$ in V and $L_\varepsilon x_n \xrightarrow{W^*} L_\varepsilon x$ by LEMMA 3, we obtain

$$0 = \lim_k < y_{n_k}^* - j_\alpha^* L_\varepsilon x_{n_k}, x_{n_k} - x >$$

$$= \lim_k < x_{n_k}^*, x_{n_k} - x >$$

$$= \lim_k < x_{n_k}^*, x_{n_k} > - < x^*, x >$$

and $y^* = j_\alpha^*(L_\varepsilon x + x^*)$. By condition (m_1) for A, we have $x^* \in Ax$, and hence $y^* \in j_\alpha^*(L_\varepsilon + A) j_\alpha x = A_{\varepsilon,\alpha} x$. Thus $G(A_{\varepsilon,\alpha})$ is closed in $V_\alpha \times V_\alpha^*$. q.e.d.

PROOF OF THEOREM 2: Since V is reflexive, there exists a norm on V equivalent to the initial norm with respect to which V and V^* are strictly convex (see [1]). Thus, we may assume that V and V^* are strictly convex from the beginning.

For each $\varepsilon > 0$ and each $\alpha \in \Lambda$ we consider the mapping $A_{\varepsilon,\alpha}$ which is given in LEMMA 4. By REMARK 1 and THEOREM 1, there exist $x_{\varepsilon,\alpha} \in C \cap V_\alpha$ and $x_{\varepsilon,\alpha}^* \in Ax_{\varepsilon,\alpha}$ such that

$$(3.3) \qquad < L_\varepsilon x_{\varepsilon,\alpha} + x_{\varepsilon,\alpha}^*, x > = 0 \qquad \textit{for all } x \in V_\alpha.$$

By the weak compactness of C and the boundedness of A and L_ε, there exists a cofinal subdirected set $\{\alpha_i\}$ of Λ such that $x_{\varepsilon,\alpha_i} \xrightarrow{W} x_\varepsilon \in C$, $x_{\varepsilon,\alpha_i}^* \xrightarrow{W^*} x_\varepsilon^*$ and $L_\varepsilon x_{\varepsilon,\alpha_i} \xrightarrow{W^*} X_\varepsilon^*$. From (3.3) it follows that $x_\varepsilon^* + X_\varepsilon^* = 0$ in V^*.

There exists α such that $x_\varepsilon \in V_\alpha$. Therefore by (3.3) we have

$$0 = \limsup_i [< L_\varepsilon x_{\varepsilon,\alpha_i}, x_{\varepsilon,\alpha_i} - x_\varepsilon > + < x_{\varepsilon,\alpha_i}^*, x_{\varepsilon,\alpha_i} - x_\varepsilon >]$$

and by the monotonicity of L_ε

$$\liminf_i < L_\varepsilon x_{\varepsilon,\alpha_i}, x_{\varepsilon,\alpha_i} - x_\varepsilon > \geq \lim_i < L_\varepsilon x_\varepsilon, x_{\varepsilon,\alpha_i} - x_\varepsilon > = 0.$$

Thus it follows that

$$(3.4) \qquad \limsup_i < x_{\varepsilon,\alpha_i}^*, x_{\varepsilon,\alpha_i} > \leq < x_\varepsilon^*, x_\varepsilon >.$$

Now condition (m_1) for A and (3.4) imply that $x_\varepsilon^* \in Ax_\varepsilon$.

Set $X_{\varepsilon,\alpha}^* = L_\varepsilon x_{\varepsilon,\alpha}$. Then $x_{\varepsilon,\alpha} \in L^{-1} X_{\varepsilon,\alpha}^* + \varepsilon J^{-1} X_{\varepsilon,\alpha}^*$ and hence $x_{\varepsilon,\alpha} - \varepsilon J^{-1} X_{\varepsilon,\alpha}^* \in L^{-1} X_{\varepsilon,\alpha}^*$. By the monotonicity of L^{-1} and the fact $0 \in L^{-1}0$, we have $0 \leq < X^*_{\varepsilon,\alpha}, x_{\varepsilon,\alpha} - \varepsilon J^{-1} X^*_{\varepsilon,\alpha} >$. Using this relation and (3.3) we infer that

$$-< x^*_{\varepsilon,\alpha}, x_{\varepsilon,\alpha} > \geq \varepsilon < X^*_{\varepsilon,\alpha}, J^{-1} X^*_{\varepsilon,\alpha} > = \varepsilon \|X^*_{\varepsilon,\alpha}\|^2_{V^*}. \tag{3.5}$$

Since $\{x_{\varepsilon,\alpha}\} \subset C$ and A is bounded, $-< x^*_{\varepsilon,\alpha}, x_{\varepsilon,\alpha} > \leq K$ where K is a constant which is independent of ε and α. Therefore from (3.5) it follows that

$$\sqrt{\varepsilon}\|J^{-1} X^*_{\varepsilon,\alpha}\|_V = \sqrt{\varepsilon}\|X^*_{\varepsilon,\alpha}\|_{V^*} \leq \sqrt{K}. \tag{3.6}$$

Therefore we may assume that $\sqrt{\varepsilon} J^{-1} X^*_{\varepsilon,\alpha_i} \xrightarrow{W} \rho_\varepsilon$. Then $\|\rho_\varepsilon\|_V \leq \sqrt{K}$. Since $X^*_{\varepsilon,\alpha_i} \in L(x_{\varepsilon,\alpha_i} - \varepsilon J^{-1} X^*_{\varepsilon,\alpha_i})$, $X^*_{\varepsilon,\alpha_i} \xrightarrow{W^*} -x^*_\varepsilon$ and $x_{\varepsilon,\alpha_i} - \varepsilon J^{-1} X^*_{\varepsilon,\alpha_i} \xrightarrow{W} x_\varepsilon - \sqrt{\varepsilon}\rho_\varepsilon$, we have $-x^*_\varepsilon \in L(x_\varepsilon - \sqrt{\varepsilon}\rho_\varepsilon)$ by Lemma 2 and Remark 2.

Since $\{x_\varepsilon; \varepsilon > 0\} \subset C$ and $\{x^*_\varepsilon; \varepsilon > 0\}$ is bounded by the boundedness of A, there exists a sequence $\{\varepsilon_k\}$ tending to 0 such that $x_{\varepsilon_k} \xrightarrow{W} x_0 \in C$ and $x^*_{\varepsilon_k} \xrightarrow{W^*} x^*_0$. Then $\sqrt{\varepsilon_k}\rho_{\varepsilon_k} \xrightarrow{s} 0$ and $x_{\varepsilon_k} - \sqrt{\varepsilon_k}\rho_{\varepsilon_k} \xrightarrow{W} x_0$. By Lemma 2 and Remark 2 again, we have $-x^*_0 \in Lx_0$, and hence, using the monotonicity of L,

$$< -x^*_\varepsilon + x^*_0, x_\varepsilon - \sqrt{\varepsilon}\rho_\varepsilon - x_0 > \geq 0.$$

It follows that

$$\limsup_k < x^*_{\varepsilon_k}, x_{\varepsilon_k} > \leq \lim_k < x^*_0, x_{\varepsilon_k} - \sqrt{\varepsilon_k}\rho_{\varepsilon_k} - x_0 > + \lim_k < x^*_0, x_{\varepsilon_k} > = < x^*_0, x_0 >. \tag{3.7}$$

Condition (m_1) and (3.7) imply that $x^*_0 \in Ax_0$. Thus $0 = -x^*_0 + x^*_0 \in Lx_0 + Ax_0$, that is, $S \neq \phi$. The weak compactness of S follows from condition (m_1) for A and the maximal monotonicity of L. q.e.d.

As an immediate consequence of Theorem 2 we have

Corollary 1. *Let A be a bounded coercive mapping of type M of $D(A) = V$ into V^* and L a linear maximal monotone mapping of $D(L) \subset V$ into V^*. Then for any given $f \in V^*$ the equation $Lx + Ax = f$ has a solution and the set of all solutions is weakly compact.*

Proof. For $f \in V^*$ we define a mapping A_f by $A_f x = Ax - f$. By the coerciveness of A there exists a positive number r such that $< A_f x, x > \geq 0$

for all $x \in \partial B_r$. Therefore by THEOREM 2 the equation $Lx+A_f x=0$ has a solution in B_r. The weak compactness of the set of all solutions follows from the coerciveness of A, (M_1) and the maximal monotonicity of L. q.e.d.

The next three corollaries give the existence of solutions of equations of type (b).

COROLLARY 2. *Let A be a bounded mapping of type M of $D(A)=V$ into V^*, and let C be a bounded closed convex subset of V containing the origin 0 in its interior. Suppose that $<Ax, x>\geq 0$ for all $x \in \partial C$. Let L be a linear maximal monotone mapping of $D(L)\subset V^*$ into V. Then the set $S=\{x \in C; x+LAx=0\}$ is non-empty and weakly compact.*

PROOF. Since $S=\{x \in C; L^{-1}x+Ax \ni 0\}$, L^{-1} is maximal monotone and the graph $G(L^{-1})$ is linear, THEOREM 2 implies that S is non-empty and weakly compact. q.e.d.

REMARK 3. COROLLARY 2 is a generalization of THEOREM 19 in [3].

COROLLARY 3. *Let A be a bounded mapping of type M of $D(A)=V$ into V^* such that for each $x_0 \in V$*

$$\frac{<A(x+x_0), x>}{||x||_V} \to \infty \text{ as } ||x||_V \to \infty$$

and L a linear maximal monotone mapping of $D(L)\subset V^*$ into V. Then $R(I+LA)=V$.

PROOF. For any $x_0 \in V$ we define a mapping A_{x_0} by $A_{x_0}x=A(x_0+x)$. Clearly A_{x_0} is a bounded coercive mapping of type M. By the coerciveness of A_{x_0} there exists a positive number r such that $<A_{x_0}x, x>\geq 0$ for all $x \in \partial B_r$. Therefore, by COROLLARY 2, $y+LA_{x_0}y=0$ has a solution in B_r, that is, $x+LAx=x_0$ has a solution. q.e.d.

COROLLARY 4. *Let L be a linear maximal monotone mapping of $D(L)\subset V^*$ into V and A a coercive mapping of type M of $D(A)=V$ into V^*. Suppose that A^{-1} is coercive and for each $x^* \in R(A)$, $A^{-1}x^*$ is closed and convex in V. Then $R(I+LA)=V$.*

PROOF. By the COROLLARY of THEOREM 1 we have $R(A)=V^*$. By hypotheses A^{-1} is a coercive bounded mapping of $D(A^{-1})=V^*$ into V. It is easy to see that A^{-1} satisfies conditions (m_1) and (m_3). Therefore by THEOREM 2 there exists a point $x^* \in V^*$ such that $A^{-1}x^*+Lx^* \ni 0$, that is, there exists $x \in A^{-1}x^*$ such that $x+LAx=0$. Just as in COROLLARY 1 we can show the existence of a solution of $x+LAx=x_0$ for each $x_0 \in V$. q.e.d.

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