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A Note on the Admissible Tests and Classifications in Multivariate Analysis

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0. Summary

Kiefer and Schwartz [1] provided a general method of proving admissibility of tests in normal multivariate analysis. Using their method, in this paper, we prove admissibility of certain test procedures for the equality of a covariance matrix to a given one in a normal population. The test procedures include the likelihood ratio tests and the modified likelihood ratio tests. Admissibility of certain classification procedures are also proved.

1. Notation and preliminary results

The average of the columns of a matrix X will be denoted by $\bar{X}$. The parameter space will be denoted by $\Omega=\{\theta\}=H_0+H_1$ (or H_1+H_2). The entire random matrix will be denoted by V and its columns are defined to be independently distributed, each p-variate normal. *A priori* probability measure or positive constant multiples thereof will be denoted by Π. We only require $\Pi(\Omega)<\infty$. If $\Pi=\Pi_0+\Pi_1$ with Π_i a finite measure on H_i, we have the following lemma mentioned in Kiefer and Schwartz [1], where $f_v(v;\theta)$ denotes the density function of V:

LEMMA 1.1. *Every Bayes critical region for* 0-1 *loss function is of the form*

$$\left\{v:\int f_V(v;\theta)\Pi_1(d\theta)\Big/\int f_V(v;\theta)\Pi_0(d\theta)>c\right\}\cup L_c\cup L \tag{1.1}$$

for some $c\,(0\leqq c\leqq\infty)$, *where* L_c *is a measurable subset of the set obtained from the set in brackets in* (1.1) *by replacing* $>$ *by* $=$, *and* L *is a measurable subset of the set* $M=\left\{v:\int f_V(v;\theta)\Pi(d\theta)=0\right\}$.

In our applications every L_c and L have probability zero for any θ in Ω, so that our Bayes procedures will be shown to be admissible by the following well-known lemma:

LEMMA 1.2. *If a Bayes procedure is essentially unique for a problem with* *respect to a priori distribution Π, then it is admissible.*

In calculating the critical region of the form (1.1), the following lemmas of Kiefer and Schwartz [1] are useful for the integrability of *a priori* densities. We always use integration with respect to the ordinary Lebesgue measure in the n-dimensional Euclidean space, E^n.

LEMMA 1.3. *Let η be a $p\times q$ matrix. Then*

$$\int_{E^{pq}} |I_p+\eta\eta'|^{-h/2}d\eta<\infty, \tag{1.2}$$

if and only if $h>p+q-1$.

LEMMA 1.4. *Let η be a $p\times q$ matrix with $q\geqq p$. If $p-1<q+t<h-p+1$, then*

$$\int_{E^{pq}} |\eta\eta'|^{t/2}|I_p+\eta\eta'|^{-h/2}d\eta<\infty. \tag{1.3}$$

In Lemma 1.4, the condition $p-1<q+t$ assures the integrability of (1.3) when $|\eta\eta'|$ tends to zero.

2. A class of admissible tests for a covariance matrix

Let $p\times N$ matrix $V^*=(V_1^*, V_2^*, \cdots, V_N^*)$ be a random sample from a multivariate normal distribution with unknown mean vector μ and unknown covariance matrix Σ (nonsingular). From this sample we want to test the hypothesis $H_0:\Sigma=\Sigma_0$ against the alternatives H_1: $\Sigma\neq\Sigma_0$, where the mean μ is unspecified and Σ_0 is a given $p\times p$ positive definite matrix. This problem is reduced to the following canonical form; $V=(V_1, V_2, \cdots, V_n, W)$ $(N=n+1)$ with $EV_i=0(p\times 1)$ $(i=1, \cdots, n)$, $EW\neq 0(p\times 1)$ and all the columns of V are independently distributed with common unknown covariance matrix Σ. We treat the problem in this reduced form. If we write $S=\sum_{i=1}^{n} V_iV_i'$, then the following theorem holds:

THEOREM 2.1. *For given $p\times p$ positive definite matrix B_0 and nonnegative definite matrices $B_1, \cdots, B_{m_1+m_2}$, a test with the following critical region*

$$\frac{\mathrm{etr}(\Sigma_0^{-1}-B_0)S}{\left[\prod_{i=1}^{m_1}|B_i+S|^{q_i}\right]\left[\prod_{i=m_1+1}^{m_1+m_2}|B_i+S|^{q_i+t_i}\right]}\geqq c \tag{2.1}$$

is an admissible Bayes test, provided that (i) *$q_i\geqq p$ for $i=m_1+1, \cdots, m_1+m_2$ where $q_1, \cdots, q_{m_1+m_2}$ are positive integers,* (ii) *$p-1<q_i+t_i$ for $i=m_1+1, \cdots, m_1$*

$+m_2$ *and* (iii) $\sum_{i=1}^{m_1+m_2} q_i + \sum_{i=m_1+1}^{m_1+m_2} \max(0, t_i) < n-p+1$. *When* $m_1=0$ *and* $m_2=1$, *the condition* (iii) *is improved to* $q_1+t_1<n-p+1$.

PROOF. By Lemma 3.1 in Kiefer and Schwartz [1], it is sufficient to prove the theorem deleting W. Let $\Sigma^{-1}=B_0+\sum_{i=1}^{m_1+m_2}\eta_i\eta_i'$ under H_1, for $\eta_i(p\times q_i)$. We set

$$(2.2)\qquad d\Pi_1(\eta)/d\eta=\Big[\prod_{i=m_1+1}^{m_1+m_2}|\eta_i\eta_i'|^{t_i/2}\Big]\,|B_0+\sum_{i=1}^{m_1+m_2}\eta_i\eta_i'|^{-n/2}\operatorname{etr}\Big\{-\frac{1}{2}\sum_{i=1}^{m_1+m_2}B_i\eta_i\eta_i'\Big\}$$

for the Lebesgue density function of $\Pi_1(\eta)$. The integrability of (2.2) with each of $|\eta_i\eta_i'|$ $(i=m_1+1, \cdots, m_1+m_2)$ tending to zero is shown by Lemma 1.4. We show the integrability when each $|\eta_i\eta_i'|$ $(i=1, \cdots, m_1+m_2)$ tends to infinity. We may assume that each $|\eta_i\eta_i'|$ is larger than one. Then

$$|\eta_i\eta_i'|^{t_i/2}<|\sum_{i=1}^{m_1+m_2}\eta_i\eta_i'|^{\frac{1}{2}\max(0,t_i)}.$$

When we replace $|\eta_i\eta_i'|^{t_i/2}$ in (2.2) by $|\sum_{i=1}^{m_1+m_2}\eta_i\eta_i'|^{\frac{1}{2}\max(0,t_i)}$, the new function is integrable by the assumption (iii) and Lemma 1.4, when each $|\eta_i\eta_i'|$ tends to infinity. Therefore (2.2) is also integrable.

For this *a priori* distribution, Bayes critical region (1.1) is computed, namely,

$$\frac{\displaystyle\int\Big[\prod_{i=m_1+1}^{m_1+m_2}|\eta_i\eta_i'|^{t_i/2}\Big]\operatorname{etr}\Big\{-\frac{1}{2}(B_0+\sum_{i=1}^{m_1+m_2}\eta_i\eta_i')S\Big\}\cdot\operatorname{etr}\Big\{-\frac{1}{2}(\sum_{i=1}^{m_1+m_2}B_i\eta_i\eta_i')\Big\}d\eta}{\operatorname{etr}\Big\{-\frac{1}{2}\Sigma_0^{-1}S\Big\}}$$

$$=\Big[\operatorname{etr}\Big\{\frac{1}{2}(\Sigma_0^{-1}-B_0)S\Big\}\Big]\int\Big[\prod_{i=m_1+1}^{m_1+m_2}|\eta_i\eta_i'|^{t_i/2}\Big]\operatorname{etr}\Big\{-\frac{1}{2}\sum_{i=1}^{m_1+m_2}(B_i+S)\eta_i\eta_i'\Big\}d\eta$$

$$=\frac{\operatorname{etr}\Big\{\frac{1}{2}(\Sigma_0^{-1}-B_0)S\Big\}}{\Big[\prod_{i=1}^{m_1}|B_i+S|^{q_i/2}\Big]\Big[\prod_{i=m_1+1}^{m_1+m_2}|B_i+S|^{(q_i+t_i)/2}\Big]}$$

$$\cdot\int\Big[\prod_{i=m_1+1}^{m_1+m_2}|\eta_i^*\eta_i^{*\prime}|^{t_i/2}\Big]\operatorname{etr}\Big\{-\frac{1}{2}\sum_{i=1}^{m_1+m_2}\eta_i^*\eta_i^{*\prime}\Big\}d\eta^*,$$

where $\eta_i^*=(B_i+S)^{\frac{1}{2}}\eta_i$. Since the integral in the last line is constant, we obtain the theorem.

If every B_i $(i=1, \cdots, m_1+m_2)$ is positive definite, then (2.2) is integrable without the condition (iii). But in our applications, we set $B_i=0$. Therefore assumption (iii) can not be omitted in the theorem. Theorem 2.1 gives a class of admissible critical regions, among which the following two cases are important:

COROLLARY 2.1. *The likelihood ratio test*

$$(\mathrm{etr}\,\Sigma_0^{-1}S)/|S|^N \geqq c \tag{2.3}$$

is admissible Bayes, when $n>p$.

COROLLARY 2.2. *The modified likelihood ratio test*

$$(\mathrm{etr}\,\Sigma_0^{-1}S)/|S|^n \geqq c \tag{2.4}$$

is admissible Bayes, when $n>p$.

PROOF. Let $m_1=1$, $m_2=0$, $q_1=1$, $B_1=0(p\times p)$ and $B_0=[n/(n+1)]\Sigma_0^{-1}$ for Corollary 2.1, $B_0=[(n-1)/n]\Sigma_0^{-1}$ for Corollary 2.2. Then we obtain the corollaries.

We can generalize Theorem 2.1 to the k sample case. Let $V^{(i)}=(V_1^{(i)}, \cdots, V_{n_i}^{(i)}, W^{(i)})$ $(i=1, \cdots, k)$ where $\mathrm{E}V_t^{(i)}=0(p\times 1)$, $\mathrm{E}W^{(i)}=\nu_i(p\times 1)\neq 0$, $\mathrm{E}[V_t^{(i)}V_t^{(i)\prime}]=\mathrm{E}[(W^{(i)}-\nu_i)(W^{(i)}-\nu_i)']=\Sigma_i$ for $t=1, \cdots, n_i$, and let the columns of $V=(V^{(1)}, \cdots, V^{(k)})$ be independent. We consider the problem of testing H_0: $\Sigma_j=\Sigma_{0j}$ $(j=1, \cdots, k)$ against H_1: $\Sigma_i\neq\Sigma_{0i}$ for some i.

THEOREM 2.2. *For given $p\times p$ positive definite matrix B_{0j} and non-negative definite matrices $B_{1j}, \cdots, B_{m_{1j}+m_{2j}j}$, a test with the following critical region*

$$\prod_{j=1}^{k}\left\{\frac{\mathrm{etr}\,(\Sigma_{0j}^{-1}-B_{0j})\,S_j}{[\prod_{i=1}^{m_{1j}}|B_{ij}+S_j|^{q_{ij}}][\prod_{i=m_{1j}+1}^{m_{1j}+m_{2j}}|B_{ij}+S_j|^{q_{ij}+t_{ij}}]}\right\}\geqq c \tag{2.5}$$

is an admissible Bayes test, provided that (i) *$q_{ij}\geqq p$ for $i=m_{1j}+1, \cdots, m_{1j}+m_{2j}$ where $q_{1j}, \cdots, q_{m_{1j}+m_{2j}j}$ are positive integers,* (ii) *$p-1<q_{ij}+t_{ij}$ for $i=m_{1j}+1, \cdots, m_{1j}+m_{2j}$ and* (iii) *$\sum_{i=1}^{m_{1j}+m_{2j}}q_{ij}+\sum_{i=m_{1j}+1}^{m_{1j}+m_{2j}}\max(0, t_{ij})<n_j-p+1$ hold for all $j=1, \cdots, k$. When $m_{1j}=0$ and $m_{2j}=1$, the condition* (iii) *is improved to $q_{1j}+t_{1j}<n_j-p+1$.*

The admissibility of the likelihood ratio test (resp., the modified likelihood ratio test) is proved by putting, in Theorem 2.2, $m_{1j}=0$, $m_{2j}=1$, $B_{1j}=0(p\times p)$ $(j=1, \cdots, k)$ and further $q_{1j}+t_{1j}=c_1(n_j+1)$ (resp., $=c_1n_j$) for $j=1, \cdots, k$, where c_1 is slightly larger than $(p-1)/\min_j(n_j+1)$ (resp., $(p-1)/\min_j n_j$) and $B_{0j}=(1-c_1)\Sigma_{0j}^{-1}$. To satisfy the integrability condition (ii) and (iii), it is required that $\min_j n_j > 2(p-1)$. This technique is also due to Kiefer and Schwartz [1].

COROLLARY 2.3. *The likelihood ratio test*

$$\prod_{j=1}^{k} \frac{\operatorname{etr} \Sigma_{0j}^{-1} S_j}{|S_j|^{N_j}} \geqq c \tag{2.6}$$

is admissible Bayes, when $\min_j n_j > 2(p-1)$.

COROLLARY 2.4. *The modified likelihood ratio test*

$$\prod_{j=1}^{k} \frac{\operatorname{etr} \Sigma_{0j}^{-1} S_j}{|S_j|^{n_j}} \geqq c \tag{2.7}$$

is admissible Bayes, when $\min_j n_j > 2(p-1)$.

Unbiasedness of the modified likelihood ratio test was shown by Sugiura and Nagao [4], and monotonicity of the power function by Nagao [2]. But our approach is not successful to show the admissibility of the likelihood ratio test for $\mu = \mu_0$ and $\Sigma = \Sigma_0$ discussed in [4].

3. Classification

3.1. Equality of mean vectors and covariance matrices. Suppose $V = (V^{(1)}, V^{(2)}, V^{(3)})$, where $V^{(j)} = (V_1^{(j)}, \cdots, V_{n_j}^{(j)})$, each $V_t^{(j)}$ being $p \times r$, the columns of V being independent, the columns of $V^{(j)}$ having common unknown covariance matrix $\Sigma^{(j)}$ and $\mathrm{E} V_t^{(j)} = \xi^{(j)}$. Then we consider the classification problem that H_1: $\xi^{(3)} = \xi^{(1)}$, $\Sigma^{(3)} = \Sigma^{(1)}$, against H_2: $\xi^{(3)} = \xi^{(2)}$, $\Sigma^{(3)} = \Sigma^{(2)}$. Putting

$$Y^{(j)} = (n_j + n_3)^{-\frac{1}{2}} (n_j \bar{V}^{(j)} + n_3 \bar{V}^{(3)}),\ Z^{(j)} = n_{3-j}^{\frac{1}{2}} \bar{V}^{(3-j)},$$

$$U^{(j)} = (Y^{(j)}, Z^{(j)}),\ S^{(j)} = VV' - U^{(j)} U^{(j)\prime},$$

$$T^{(j)} = V^{(j)} V^{(j)\prime} - n_j \bar{V}^{(j)} \bar{V}^{(j)\prime},$$

Kiefer and Schwartz [1] proved the admissibility of the procedure which accepts H_1 or H_2 according as

$$|S^{(2)} - T^{(1)}|\,|T^{(1)}| / |S^{(1)} - T^{(2)}|\,|T^{(2)}| > \text{ or } < c, \tag{3.1}$$

when $(n_i - 1)r > p$ for $i = 1, 2$. We generalize this result to the following:

THEOREM 3.1. *Suppose* $p - 1 < s_1 < (n_1 + n_3 - 1)r - p + 1, p - 1 < s_2 < (n_2 - 1)r$

$-p+1$, $p-1<s_3<(n_1-1)r-p+1$ *and* $p-1<s_4<(n_2+n_3-1)r-p+1$, *then the procedure which accepts* H_1 *or* H_2 *according as*

$$(3.2) \qquad |S^{(2)}-T^{(1)}|^{s_4}|T^{(1)}|^{s_3}/|S^{(1)}-T^{(2)}|^{s_1}|T^{(2)}|^{s_2} > \text{ or } < c$$

is admissible Bayes.

PROOF. Let $s_j = q_j + t_j$ (q_j: positive integer and $q_j \geqq p$) and

$$\Sigma^{(j)}=(I_p+\eta_j\eta_j')^{-1} \text{ under } H_1 \text{ for } \eta_j(p\times q_j) \qquad (j=1, 2),$$

$$\Sigma^{(j)}=(I_p+\eta_j\eta_j')^{-1} \text{ under } H_2 \text{ for } \eta_j(p\times q_{2+j}) \qquad (j=1, 2).$$

And let

$$(3.3) \qquad \begin{aligned} &d\Pi_1(\eta)/d\eta \\ &\quad =|\eta_1\eta_1'|^{t_1/2}|\eta_2\eta_2'|^{t_2/2}|I_p+\eta_1\eta_1'|^{-(n_1+n_3-1)r/2}|I_p+\eta_2\eta_2'|^{-(n_2-1)r/2}, \\ &d\Pi_2(\eta)/d\eta \\ &\quad =|\eta_1\eta_1'|^{t_3/2}|\eta_2\eta_2'|^{t_4/2}|I_p+\eta_1\eta_1'|^{-(n_1-1)r/2}|I_p+\eta_2\eta_2'|^{-(n_2+n_3-1)r/2}, \end{aligned}$$

for the Lebesgue densities of $\Pi_i(\eta)$. Then they are intergrable by the conditions of Theorem 3.1. Kiefer and Schwartz [1] considered the case of $t_j=0$ and $q_j=1$. By the same argument as theirs (from equation (6.5)) we obtain the procedure in the theorem.

COROLLARY 3.1. *The procedure which accepts* H_1 *or* H_2 *according as*

$$(3.4) \qquad |S^{(2)}-T^{(1)}|^{n_2+n_3}|T^{(1)}|^{n_1}/|S^{(1)}-T^{(2)}|^{n_1+n_3}|T^{(2)}|^{n_2} > \text{ or } < c$$

is admissible Bayes, when $\min((n_1-1)r, (n_2-1)r)>2(p-1)$.

PROOF. Letting $s_1=c_1(n_1+n_3)$, $s_2=c_1n_2$, $s_3=c_1n_1$ and $s_4=c_1(n_2+n_3)$, where c_1 is slightly larger than $(p-1)/\min(n_1, n_2)$, the conditions in Theorem 3.1 are satisfied, which implies (3.4). This is the likelihood ratio criterion.

3.2. Equality of covariance matrices with known mean vector. Suppose $V=(V^{(1)}, V^{(2)}, V^{(3)})$ where each $V^{(j)}=(V_1^{(j)}, \cdots, V_{n_j}^{(j)})$ $(j=1, 2)$ is a random sample from a normal population with known mean vector μ and unknown covariance matrix $\Sigma^{(j)}$, and $V^{(3)}(p\times 1)$ is taken from a normal population with mean vector μ and unknown covariance matrix $\Sigma^{(3)}$. We consider the classification problem of testing H_1: $\Sigma^{(3)}=\Sigma^{(1)}$ against H_2: $\Sigma^{(3)}=\Sigma^{(2)}$. Letting $S_j=\sum_{t=1}^{n_j}(V_t^{(j)}-\mu)(V_t^{(j)}-\mu)'$, we have the following:

THEOREM. 3.2. *Suppose* $p-1<s_1<n_1-p+2$, $p-1<s_2<n_2-p+1$, $p-1<s_3<n_2-p+2$ *and* $p-1<s_4<n_1-p+1$, *then the procedure which accepts* H_1 *or* H_2 *according as*

$$(3.5)\quad |S_2+(V^{(3)}-\mu)(V^{(3)}-\mu)'|^{s_3}|S_1|^{s_4}/|S_1+(V^{(3)}-\mu)(V^{(3)}-\mu)'|^{s_1}|S_2|^{s_2} > \text{ or } < c$$

is admissible Bayes.

PROOF. In the proof of Theorem 3.1, replace (3.3) by the following

$$(3.6)\quad \begin{aligned} d\Pi_1(\eta)/d\eta &= |\eta_1\eta_1'|^{t_1/2}|\eta_2\eta_2'|^{t_2/2}|I_p+\eta_1\eta_1'|^{-(n_1+1)/2}|I_p+\eta_2\eta_2'|^{-n_2/2},\\ d\Pi_2(\eta)/d\eta &= |\eta_1\eta_1'|^{t_3/2}|\eta_2\eta_2'|^{t_4/2}|I_p+\eta_1\eta_1'|^{-n_1/2}|I_p+\eta_2\eta_2'|^{-(n_2+1)/2}, \end{aligned}$$

which are integrable. For this *a priori* distribution, Lemma 1.1 and Lemma 1.2 give the admissible Bayes rule (3.5).

We obtain the following corollary by the same argument as for Corollary 3.1:

COROLLARY 3.2. *The procedure which accepts* H_1 *or* H_2 *according as*

$$(3.7)\quad |S_2+(V^{(3)}-\mu)(V^{(3)}-\mu)'|^{n_2+1}|S_1|^{n_1} / |S_1+(V^{(3)}-\mu)(V^{(3)}-\mu)'|^{n_1+1}|S_2|^{n_2} > \text{ or } < c$$

is admissible Bayes, when $\min(n_1, n_2)>2(p-1)$. *It is the likelihood ratio criterion.*

In (3.6), by letting $t_i=0$ and $q_i=1$ $(i=1, \cdots, 4)$, we obtain the following corollary from Lemma 1.3:

COROLLARY 3.3. *The procedure which accepts* H_1 *or* H_2 *according as*

$$(3.8)\quad \{1+(V^{(3)}-\mu)'S_2^{-1}(V^{(3)}-\mu)\}/\{1+(V^{(3)}-\mu)'S_1^{-1}(V^{(3)}-\mu)\} > \text{ or } < c$$

is admissible Bayes, when $n_j>p$ $(j=1, 2)$.

This problem was disscussed in Okamoto [3]. He proposed the classification procedure using the difference between the numerator and the denominator in the left hand side of (3.8). In his paper the case where μ is unknown was also discussed. But we have not been able to show whether his procedures are admissible or not.

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