HIROSHIMA MATH. J.
1 (1971), 171-176

Some Oscillation Criteria for nth Order Nonlinear Delay-Differential Equations

Hiroshi ONOSE

(Received September 2, 1971)

1. Introduction.

Let us consider the nth order nonlinear delay-differential equation

$$(1) \qquad x^{(n)}(t)+\sum_{i=1}^{m} f_i(t)F_i(x_{d_{i,0}}(t),\ x'_{d_{i,1}}(t),\cdots,\ x^{(n-1)}_{d_{i,n-1}}(t))=0,$$

where

$$x^{(k)}_{d_{i,k}}(t)=x^{(k)}(t-d_{i,k}(t))$$

and the delays $d_{i,k}(t)$ are assumed to be continuous functions, nonnegative and bounded by some constant M on the half-line $[t_0,+\infty)$. In the special case where $d_{i,k}(t)=0$ for $i=1, 2,\cdots, m$, $k=0, 1,\cdots, n-1$, equation (1) clearly reduces to the ordinary differential equation

$$(2) \qquad x^{(n)}+\sum_{i=1}^{m} f_i(t)F_i(x,\ x',\cdots,\ x^{(n-1)})=0.$$

Let F be the family of solutions of (1) which are indefinitely continuable to the right. A solution $x(t)$ in F is said to be oscillatory if it has no last zero, i. e., if $x(t_1)=0$ for some t_1, then there exists some t_2, $t_2>t_1$, for which $x(t_2)=0$; otherwise a solution in F is nonoscillatory.

The purpose of this paper is to investigate the oscillatory properties of (1), giving sufficient conditions that all solutions of (1) in F are oscillatory in the case where n is even and are oscillatory or monotone in the case where n is odd. Our results generalize to arbitrary $n\geqq 2$ recent results of Staikos and Petsoulas [6] for the case $n=2$. It is to be noted that, still in the reduced case of the ordinary differential equation (2), our results improve previous results due to Kartsatos [1] and the present author [4], [5].

The author wishes to thank Professor T. Kusano for his interest in this work and for several helpful suggestions.

2. Oscillation Theorems.

We shall prove the following theorems.

THEOREM 1. *Assume for equation* (1) *that*

(i) $f_i(t)\geqq 0$ *for every* $t\in[t_0, \infty)$, $i=1, 2, \cdots, m$;

(ii) $\operatorname{sgn} F_i(x_1, x_2, \cdots, x_n)=\operatorname{sgn} x_1$ *and* $F_i(-x_1, -x_2, \cdots, -x_n)=-F(x_1, x_2, \cdots, x_n)$ *for every* $(x_1, x_2, \cdots, x_n)\in R^n$, $i=1, 2, \cdots, m$;

(iii) *there is an index* j *such that*

(a) $F_j(\lambda x_1, \lambda x_2, \cdots, \lambda x_n)=\lambda^{2p+1}F_j(x_1, x_2, \cdots, x_n)$ *for every* $(x_1, x_2, \cdots, x_n)\in R^n$, $\lambda\in R$ *and some integer* $p\geqq 0$;

(b) $\int_{t_0}^{\infty} f_j(t)dt=\infty$.

Then if n *is even, each solution of* (1) *in* F *is oscillatory, while if* n *is odd, each solution in* F *is either oscillatory or tends monotonically to zero together with its first* $n-1$ *derivatives.*

THEOREM 2. *In addition to the hypotheses* (i) *and* (ii) *of Theorem* 1, *assume that*

(iii′) *there exists an index* j *such that*

(a′) *for any* k, $2\leqq k\leqq n$, *and any* $c\geqq 0$
$\liminf F_j(x_1, x_2, \cdots, x_n)>0$ *or* ∞, *as* $x_1\to\infty, \cdots, x_{k-1}\to\infty$, $x_k\to c$, $x_{k+1}\to 0, \cdots, x_n\to 0$;

(b′) $\int_{t_0}^{\infty} f_j(t)dt=\infty$.

Then each solution of (1) *in* F *is oscillatory when* n *is even, and each solution in* F *is either oscillatory or tends to zero together with its first* $n-1$ *derivatives when* n *is odd.*

REMARK. Theorem 1 is a generalization of a recent result due to Staikos and Petsoulas for the case $n=2$ [6, Theorem 1]. When equation (1) is reduced to equation (2) it still generalizes the corresponding results that the author has established in [4] and [5]. Theorem 2 is an extension of a theorem of Kartsatos [1, Theorem 3] concerning oscillations of the equations of the form

$$x^{(2n)}+f(t)F(x, x')=0.$$

3. Proofs.

We begin by stating two lemmas which inform us of the possible behavior of a nonoscillatory function defined on the half-line $[t_0, \infty)$.

LEMMA 1. *Suppose* $\phi(t)\in C^n[t_0, \infty)$, $\phi(t)\geqq 0$ *and* $\phi^{(n)}(t)\leqq 0$ *on* $[t_0, \infty)$. *Then exactly one of the following is true:*

(1) $\phi'(t), \cdots, \phi^{(n-1)}(t)$ *tend monotonically to zero as* $t\to\infty$;

(11) *there is an odd integer* k, $1\leqq k\leqq n-1$, *such that* $\lim_{t\to\infty} \phi^{(n-j)}(t)=0$ *for* $1\leqq j\leqq k-1$, $\lim_{t\to\infty} \phi^{(n-k)}(t)\geqq 0$ (*finite*), $\lim_{t\to\infty} \phi^{(n-k-1)}(t)>0$ *and* $\phi(t), \phi'(t), \cdots, \phi^{(n-k-2)}(t)$ *tend to* ∞ *as* $t\to\infty$.

For the proof we refer to the papers by Kiguradze [2, Lemma 1], Kneser [3, pp. 410, 418-419] and the author [4, p. 111], [5, p. 877].

LEMMA 2. *Let $\phi(t)$ be a function such that $\phi \in C^n[t_0, \infty)$, $\phi(t)>0$ and $\phi^{(n)}(t)\leqq 0$ on $[t_0, \infty)$, and let $d_i(t)$, $i=0, 1,\cdots, n-1$, be continuous functions, nonnegative and bounded by some common constant M on $[t_0, \infty)$. Then*

$$\lim_{t\to\infty}\frac{\phi^{(i)}(t-d_i(t))}{\phi(t-d_0(t))}=0 \text{ for } 1\leqq i\leqq n-1, \tag{3}$$

unless $\phi(t)$ and its first $n-1$ derivatives tend to zero as $t\to\infty$. The exceptional case may arise only when n is odd.

PROOF. Suppose that the case 1 of Lemma 1 holds. Then, as the proof of Lemma 1 shows, $\phi(t)$ is monotone non-decreasing or non-increasing on $[t_0, \infty)$ according as n is even or odd. Hence, noteing that $\phi^{(i)}(t-d_i(t))\to 0$ as $t\to\infty$ for $1\leqq i\leqq n-1$, the assertion follows unless $\lim\limits_{t\to\infty} \phi(t)=0$, which is possible only when n is odd.

Suppose now that the case 11 of Lemma 1 holds. It is clear that (3) is true for $n-k\leqq i\leqq n-1$. If $i\leqq n-k-1$, $\phi^{(i)}(t)$ is (ultimately) non-decreasing, so that we have

$$0\leqq \lim_{t\to\infty}\frac{\phi^{(i)}(t-d_i(t))}{\phi(t-d_0(t))}\leqq \lim_{t\to\infty}\frac{\phi^{(i)}(t)}{\phi(t-M)}. \tag{4}$$

By using L′ Hospital′ s rule we easily obtain

$$\lim_{t\to\infty}\frac{\phi^{(i)}(t)}{\phi(t-M)}=0 \text{ for } 1\leqq i\leqq n-k-1.$$

Thus it follows from (4) that (3) holds also for $1\leqq i\leqq n-k-1$.

This completes the proof of the lemma.

PROOF OF THEOREM 1. Suppose (1) has a nonoscillatory solution $x(t)$ in F. Since, by condition (ii), $-x(t)$ is again a solution of (1), we can assume that $x(t)>0$ for $t\geqq t_1$, t_1 being sufficiently large. From (1),

$$x^{(n)}(t)=-\sum_{i=1}^{m} f_i(t)F_i(x_{d_{i,0}}(t), x'_{d_{i,1}}(t),\cdots,x^{(n-1)}_{d_{i,n-1}}(t))$$

and so our hypotheses imply that $x^{(n)}(t)\leqq 0$ for $t\geqq t_2\geqq t_1+M$.

If n is even, it follows from Lemma 2 that

$$\lim_{t\to\infty}\frac{x^{(i)}(t-d_{j,i}(t))}{x(t-d_{j,0}(t))}=0 \text{ for } i=1, 2,\cdots, n-1; \tag{5}$$

it is not difficult to see that

$$\lim_{t\to\infty}\frac{x(t-d_{j,0}(t))}{x(t)}=1.$$

Let $y=x^{(n-1)}/x$. Then, in view of the fact that $x'(t)$ and $x^{(n-1)}(t)$ are ultimately nonnegative, we have

$$y'(t)=\frac{x^{(n)}(t)}{x(t)}-\frac{x'(t)x^{(n-1)}(t)}{x^2(t)}\leqq\frac{x^{(n)}(t)}{x(t)}$$

for $t\geqq t_3\geqq \max(t_1+M, t_2)$. Integrating the above inequality over $[t_3, t]$ and using (1), we have

$$(6)\qquad y(t)-y(t_3)\leqq-\int_{t_3}^{t}\frac{f_j(s)}{x(s)}F_j(x_{d_{j,0}}(s), x'_{d_{j,1}}(s),\cdots, x^{(n-1)}_{d_{j,n-1}}(s))\, ds$$

$$\leqq-[x_{d_{j,0}}(t_3)]^{2p}\int_{t_3}^{t}f_j(s)\frac{x_{d_{j,0}}(s)}{x(s)}F_j\left(1,\frac{x'_{d_{j,1}}(s)}{x_{d_{j,0}}(s)},\cdots,\frac{x^{(n-1)}_{d_{j,n-1}}(s)}{x_{d_{j,0}}(s)}\right)ds,$$

where we have used condition (iii) (a) and the monotonicity of $x(t)$. Using (iii) (b) and (5), we derive the contradiction $-y(t_3)\leqq-\infty$ by letting $t\to\infty$ in (6).

This completes the theorem for the case of even n.

We now turn to the case where n is odd. Let $x(t)$ be a nonoscillatory solution of (1) in F. The case I I of Lemma 1 is impossible for $x(t)$, because the same argument as above leads to a contradiction. Suppose $x(t)$ satisfies

$$(7)\qquad \lim_{t\to\infty}x(t)=c>0,\ \lim_{t\to\infty}x^{(i)}(t)=0 \text{ for } 1\leqq i\leqq n-1.$$

Integrating the inequality

$$x^{(n)}(t)\leqq-f_j(t)F_j(x_{d_{j,0}}(t), x'_{d_{j,1}}(t),\cdots, x^{(n-1)}_{d_{j,n-1}}(t))$$

which follows from (1) over $[t^*, t]$ yields

$$(8)\quad x^{(n-1)}(t^*)-x^{(n-1)}(t)\geqq\int_{t^*}^{t}f_j(s)F_j(x_{d_{j,0}}(s), x'_{d_{j,1}}(s),\cdots, x^{(n-1)}_{d_{j,n-1}}(s))\, ds.$$

We see that

$$\lim_{t\to\infty}F_j(x_{d_{j,0}}(t), x'_{d_{j,1}}(t),\cdots, x^{(n-1)}_{d_{j,n-1}}(t))=F_j(c, 0,\cdots, 0)>0$$

and hence in view of (iii) (b)

$$\int_{t^*}^{\infty} f_j(s)F_j(x_{d_{j,0}}(s),\ x'_{d_{j,1}}(s), \cdots,\ x^{(n-1)}_{d_{j,n-1}}(s))\,ds=\infty.$$

If we let t tend to infinity in (8), we have the contradiction $x^{(n-1)}(t^*)\geqq\infty$. Thus we can conclude that a nonoscillatory solution of (1) in F, if it exists, tends to zero together with its first $n-1$ derivatives as $t\to\infty$.

This finishes the proof of Theorem 1.

PROOF OF THEOREM 2. Suppose $x(t)$ is a nonoscillatory solution of (1) in F. By condition (ii) we can assume that $x(t)>0$ for t sufficiently large, say $t\geqq t_1$. From (1) we have

$$x^{(n)}(t)\leqq -f_j(t)F_j(x_{d_{j,0}}(t),\ x'_{d_{j,1}}(t), \cdots,\ x^{(n-1)}_{d_{j,n-1}}(t)), \tag{9}$$

which implies $x^{(n)}(t)\leqq 0$ for $t\geqq t_2=t_1+M$.

An integration of (9) from t_2 to t and by Lemma 1, we have

$$x^{(n-1)}(t_2)\geqq \int_{t_2}^{t} f_j(s)F_j(x_{d_{j,0}}(s),\ x'_{d_{j,1}}(s), \cdots,\ x^{(n-1)}_{d_{j,n-1}}(s))\ ds \tag{10}$$

for $t\geqq t_2$.

We distinguish two cases:

Case 1. There exists k, $0<k\leqq n-1$, such that $\lim_{t\to\infty} x^{(i)}(t)=\infty$ for $0\leqq i\leqq k-1$, $\lim_{t\to\infty} x^{(k)}=c>0$ and $\lim_{t\to\infty} x^{(i)}(t)=0$ for $k+1\leqq i\leqq n-1$. Then, because of (iii′) (a′),

$$\liminf_{t\to\infty} F_j(x_{d_{j,0}}(t),\ x'_{d_{j,1}}(t), \cdots,\ x^{(n-1)}_{d_{j,n-1}}(t))>\varepsilon$$

for some positive constant ε, so that there exists a $t_3\geqq t_2$ such that

$$F_j(x_{d_{j,0}}(t),\ x'_{d_{j,1}}(t), \cdots,\ x^{(n-1)}_{d_{j,n-1}}(t))\geqq\varepsilon \text{ for all } t\geqq t_3.$$

It is obvious that inequality (10) remains valid if we replace t_2 by t_3. Thus we obtain

$$x^{(n-1)}(t_2)\geqq\varepsilon\int_{t_3}^{t} f_j(s)\,ds,$$

and consequently $x^{(n-1)}(t_2)=\infty$, a contradiction.

Case 2. $\lim_{t\to\infty} x(t)=c>0$ and $\lim_{t\to\infty} x^{(i)}(t)=0$ for $1\leqq i\leqq n-1$.

If $c<\infty$, then by the continuity of F_j, for any given positive ε with $\varepsilon<F_j(c, 0, \cdots, 0)$ there is a $t_3\geqq t_2$ such that

$$F_j(x_{d_{j,0}}(t),\ x'_{d_{j,1}}(t), \cdots,\ x^{(n-1)}_{d_{j,n-1}}(t))\geqq F_j(c,\ 0, \cdots,\ 0)-\varepsilon \text{ for } t\geqq t_3.$$

Then, from (10) with t_2 replace by t_3, we find

$$x^{(n-1)}(t_2) \geqq [F_j(c, 0, \cdots, 0) - \varepsilon] \int_{t_3}^{t} f_j(s) ds \text{ for } t \geqq t_3,$$

which again leads to a contraciction. If $c = \infty$, then by (iii′) (a′) we have also a contradiction.

This completes the proof of Theorem 2.

References

[1] A. G. Kartsatos, Some theorems on oscillation of certain nonlinear second-order ordinary differential equations, Arch. Math., **18** (1967), 425-429.

[2] I. I. Kiguradze, Oscillation properties of solutions of certain ordinary differential equations, Soviet Math. Dokl., **3** (1962), 649-652.

[3] A. Kneser, Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen, Math. Ann., **42** (1893), 409-435.

[4] H. Onose, Oscillatory property of certain nonlinear ordinary differential equations, Proc. Japan Acad., **44** (1968), 110-113.

[5] H. Onose, Oscillatory property of certain nonlinear ordinary differential equations II, Proc. Japan Acad., **44** (1968), 876-878.

[6] V. A. Stakos and A. G. Petsoulas, Some oscillation criteria for second order nonlinear delay-differential equations, J. Math. Anal. Appl., **30** (1970), 695-701.

Faculty of Engineering
Ibaraki University
Hitachi