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The Unbounded Growth of Solutions of Parabolic Equations with Unbounded Coefficients

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Dedicated to President Y.K. Tai on his 70th birthday

1. In 1966, Besala and Fife [1] studied the asymptotic behavior of solutions of the Cauchy problem for a parabolic differential operator

$$L=\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial}{\partial x_i}+c-\frac{\partial}{\partial t} \tag{1}$$

with non-negative Cauchy data not identically equal to zero. More recently Kuroda [4] also discussed an analogous problem under somewhat different conditions on the coefficients of such a parabolic differential operator and proved the following theorem:

Assume that the coefficients of (1) are defined for all $(x, t) \in R^n \times (0, \infty)$ and satisfy for some $\lambda \in (0, 1]$ the following hypotheses:

$$k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \tag{2}$$

for any real vector $\xi=(\xi_1,\cdots, \xi_n) \in R^n$,

$$|b_i| \leqq K_2(|x|^2+1)^{1/2}, \quad 1\leqq i \leqq n, \tag{3}$$

$$-k_3(|x|^2+1)^{\lambda}+k_4 \leqq c \leqq K_3(|x|^2+1)^{\lambda}, \tag{4}$$

where $k_1(>0)$, K_1, $K_2(\geqq 0)$, $k_3(\geqq 0)$, $k_4(\geqq 0)$ and $K_3(>0)$ are constants.

Let the following inequality hold:

$$-2\Big(\frac{r_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)\lambda K_1 n-4\Big(\frac{r_0}{2\lambda\sqrt{k_1}}+\frac{nK_2}{4\lambda k_1}\Big)^2\lambda^2 k_1+k_4>0 \tag{5}$$

where we have set $r_0=\Big(k_3+\dfrac{n^2K_2^2}{4k_1}\Big)^{1/2}$. If a non-negative function $u(x, t)$ continuous in $R^n \times [0, \infty)$ satisfies (i) $Lu \leqq 0$ in $R^n \times (0, \infty)$ in the usual sense, and (ii) $u(x, 0) \geqq 0$ and $u(x, 0) \not\equiv 0$ for $x \in R^n$ and $u(x, t) \geqq -\mu \exp(\nu(|x|^2+1)^{\lambda})$ for some positive constants μ and ν, then $u(x, t)$ grows to infinity exponentially as t tends to infinity and this exponential growth of $u(x, t)$ is uniform in any compact subset of R^n.

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In this result, if we take $\lambda=1$, then we get the result due to Kusano [5]. Now let us consider an interesting example of the Cauchy problem:

$$
(6)\quad \begin{cases} L_0 u = \dfrac{\partial^2 u}{\partial x^2} + ax\dfrac{\partial u}{\partial x} + [-2a^2(1+x^2)+2a+3a^2]u - \dfrac{\partial u}{\partial t} = 0, \ (a>0) \\ u(x,0) = \exp(-ax^2) \end{cases}
$$

in $R^1 \times (0, \infty)$. An easy computation shows that the unique solution of this problem is given explicitly by $u(x, t) = \exp(-ax^2 + a^2 t)$, which tends to infinity as $t \to \infty$. It is easy to see that the above fact can not be concluded from the theorem of Kuroda, for the left-hand side of (5) is negative by taking $n=1$, $k_1=K_1=1$, $K_2=a$, $k_3=2a^2$, $k_4=2a+3a^2$, and $\lambda=1$.

The main purpose of this paper is to prove a theorem (Theorem 1) which includes the behavior of the solution of the Cauchy problem (6). We shall also state another theorem (Theorem 2) of similar nature which corresponds to the case where $\lambda=0$ in Theorem 1.

2. Suppose the coefficients of L in (1) satisfy the conditions (2), (3), (4) and let a function $u(x, t)$ non-negative and continuous in $R^n \times [0, \infty)$ have the following properties:

(i) $Lu \leqq 0$ in $R^n \times (0, \infty)$ in the usual sense,
(ii) $u(x, 0)$ is non-negative and not identically equal to zero,
(iii) For each $T>0$ there are positive constants M_T and a_T such that $u(x, t) \geqq -M_T \exp[a_T(|x|^2+1)^\lambda]$ in $R^n \times [0, T]$.

We establish the following Theorem 1.

Theorem 1. *Let L be a parabolic differential operator of the form* (1) *with coefficients satisfying* (2), (3) *and* (4) *in $R^n \times (0, \infty)$ for $\lambda \in (0, 1]$ and let the inequality*

$$
(8)\qquad k_4 - 2\lambda K_1 \beta n - 4\lambda^2 k_1 \beta^2 > 0
$$

hold, where $-\beta$ is the negative root of the quadratic equation

$$
(9)\qquad 4\lambda^2 k_1 X^2 - 2\lambda n K_2 X - k_3 = 0.
$$

Assume that $u(x, t)$ satisfies the condition (7). *Then $u(x, t)$ grows to infinity exponentially as $t \to \infty$ and this exponential growth of $u(x, t)$ is uniform on every compact subset in R^n.*

3. Now we shall give the proof of Theorem 1.

Proof. From the assumption for $u(x, t)$, we see by Bodanko's maximum principle [2] and the strong maximum principle due to Nirenberg [3] that $u(x, t) > 0$ in $R^n \times (0, \infty)$. Furthermore, for each $T_0 > 0$ there exist positive constants μ and ν such that

$$u(x, T_0) \geqq \mu \exp[-\nu(|x|^2+1)^\lambda] \tag{10}$$

for $x \in R^n$ and some $\lambda \in (0, 1]$, ν being greater than the positive root of (9). Let $T_0 > 0$ and the corresponding μ, ν be fixed. First we construct a function of the form

$$H(x, t) = \mu \exp[-\varphi(t)(|x|^2+1)^\lambda + \psi(t)] \tag{11}$$

satisfying $LH \geqq 0$ in $R^n \times (T_0, \infty)$, where $\varphi(t) > 0$ and $\psi(t)$ are C^1-functions for $t > T_0$. Obviously the conditions (2), (3) and (4) imply

$$\begin{aligned}\frac{LH}{H} &= 4\lambda^2(|x|^2+1)^{2\lambda-2}\varphi^2(t)\sum_{i,j=1}^n a_{ij}x_ix_j - 4\lambda(\lambda-1)(|x|^2+1)^{\lambda-2} \\ &\quad \times \varphi(t)\sum_{i,j=1}^n a_{ij}x_ix_j - 2\lambda(|x|^2+1)^{\lambda-1}\varphi(t)\sum_{i=1}^n (a_{ii}+b_ix_i) + c \\ &\quad + (|x|^2+1)^\lambda \varphi'(t) - \psi'(t) \\ &\geqq (|x|^2+1)^\lambda[\varphi'(t) + 4\lambda^2k_1\varphi^2(t) - 2\lambda nK_2\varphi(t) - k_3] \\ &\quad + [k_4 - 4\lambda^2k_1\varphi^2(t) - 2\lambda nK_1\varphi(t) - \psi'(t)].\end{aligned}$$

We can easily verify that the function

$$\varphi(t) = \frac{\alpha(\beta+\nu)e^{4\lambda^2k_1(\alpha+\beta)(t-T_0)} + \beta(\nu-\alpha)}{(\beta+\nu)e^{4\lambda^2k_1(\alpha+\beta)(t-T_0)} - (\nu-\alpha)}$$

is a solution of the differential equation

$$\begin{cases} 4\lambda^2k_1\varphi^2(t) - 2\lambda nK_2\varphi(t) - k_3 + \varphi'(t) = 0 \\ \varphi(T_0) = \nu \end{cases}$$

in (T_0, ∞), where $-\alpha$ denotes the positive root of (9) and that for this $\varphi(t)$ the function

$$\begin{aligned}\psi(t) &= (k_4 - 4\lambda^2k_1\alpha^2)(t-T_0) - \frac{nK_1}{2\lambda k_1}\log\frac{(\beta+\nu)e^{4\lambda^2k_1(\alpha+\beta)(t-T_0)} - (\nu-\alpha)}{(\alpha+\beta)e^{4\lambda^2k_1\beta(t-T_0)}} \\ &\quad - (\alpha-\beta)\log\frac{(\beta+\nu)e^{4\lambda^2k_1(\alpha+\beta)(t-T_0)} - (\nu-\alpha)}{e^{4\lambda^2k_1(\alpha+\beta)(t-T_0)}} \\ &\quad + \frac{(\alpha+\beta)(\nu-\alpha)}{(\beta+\nu)e^{4\lambda^2k_1(\alpha+\beta)(t-T_0)} - (\nu-\alpha)} - (\beta-\alpha)\log(\alpha+\beta) - (\nu-\alpha).\end{aligned}$$

satisfies

$$\begin{cases} k_4 - 4\lambda^2 k_1 \varphi^2(t) - 2\lambda n K_1 \varphi(t) - \psi'(t) = 0 \\ \psi(T_0) = 0 \end{cases}$$

in (T_0, ∞). Hence the function $H(x, t)$ satisfies the differential inequality $LH \geqq 0$ in $R^n \times [T_0, \infty)$ and the initial condition $H(x, T_0) = \mu \exp[-\nu(|x|^2 + 1)^\lambda]$ for $x \in R^n$. To see how $u(x, t)$ behaves as $t \to \infty$ we apply Bodanko's maximum principle again to $u(x, t) - H(x, t)$. Then it follows readily that $u(x, t) \geqq H(x, t)$ in $R^n \times (T_0, \infty)$. Since $\varphi(t)$ is bounded for $t > T_0$, the limiting behavior of $H(x, t)$ as $t \to \infty$ is determined by the factor $\exp[\psi(t)]$, which grows exponentially to infinity as $t \to \infty$ provided (8) holds. Thus the desired unbounded growth of $u(x, t)$ is obtained. It is obvious that the divergence is uniform on every compact x -set.

4. Recently the author [6] treated an analogous problem for an operator of the form (1) whose coefficients satisfy the following conditions;

$$k_1(|x|^2+1)|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq K_1 (|x|^2+1)|\xi|^2 \tag{12}$$

for any real vector $\xi = (\xi_1, \cdots, \xi_n)$,

$$|b_i| \leqq K_2(|x|^2+1)^{1/2}, \quad 1 \leqq i \leqq n, \tag{13}$$

$$-k_3(\log(|x|^2+1)+1)^2 + k_4 \leqq c \leqq K_3 (\log(|x|^2+1)+1)^2, \tag{14}$$

for some constants $k_1(>0)$, K_1, $K_2(\geqq 0)$, $k_3(>0)$, $k_4(\geqq 0)$ and $K_3(>0)$.

Now we shall supplement Theorem 1 in § 2 by giving a result corresponding to the case $\lambda = 0$ in Theorem 1. We assume that the coefficients of L in (1) satisfy the conditions (12), (13) and (14) in $R^n \times (0, \infty)$. Let a continuous function $u(x, t)$ in $R^n \times (0, \infty)$ have the following properties:

(15)
- (i) $Lu \leqq 0$ in $R^n \times (0, \infty)$ in the usual sense,
- (ii) $u(x, 0)$ is non-negative and not identically equal to zero,
- (iii) there exist positive constants M_T and a_T such that $u(x, t) \geqq -M_T (\exp[a_T \log(|x|^2+1)+1]^2)$ in $R^n \times [0, T]$ for each $T > 0$.

By the quite similar method, we can prove the following. We may omit the proof of it.

Theorem 2. *Let the differential operator L in* (1) *satisfy the conditions* (12), (13) *and* (14) *in $R^n \times (0, \infty)$ and let $u(x, t)$ be a function satisfying* (15). *If it holds that*

$$k_4 - 8K_1\beta - 16k_1\beta^2 > 0,$$

where $-\beta$ is the negative root of the quadratic equation

$$16k_1X^2-4(K_1+K_2)nX-k_3=0,$$

then $u(x, t)$ grows exponentially as $t\to\infty$ and this exponential growth of $u(x, t)$ is uniform with respect to $x \in R^n$.

REMARK 1. In Theorem 1, consider the case $\lambda=1$. If we take $n=1$, $k_1=K_1=1$, $K_2=a$, $k_3=2a^2$, $k_4=2a+3a^2$, our theorem can be applied to the equation $L_0u=0$ stated in § 1. In this case, a is the positive constant. Hence the condition (8) is equivalent to $a+2a^2>0$. Thus Theorem 1 includes the example (6) as a special case.

REMARK 2. In Theorem 1, consider the case $\lambda=1$. If we take $k_1=K_1=1$, $K_2=0$, $k_3=k^2$ and $k_4=k^2+l$, our theorem can be applied to the equation

$$L_1u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+(-k^2|x|^2+l)u-\frac{\partial u}{\partial t}=0$$

in $R^n\times(0, \infty)$. Let $u(x, t)$ be continuous in $R^n\times[0, \infty)$ and satisfy $L_1u\leqq 0$ in $R^n\times(0, \infty)$, and $u(x, 0)\geqq 0$ and $u(x, 0)\not\equiv 0$ for $x \in R^n$. Theorem 1 implies that, if the condition $l>kn$ to which (8) reduces is fulfilled, then $u(x, t)$ grows exponentially to infinity as the time variable tends to infinity.

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