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Corrections to "The Reduced Symmetric Product of a Complex Projective Space and the Embedding Problem"

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There is a mistake in §5 of my previous note [6], and the results (2) and (3) of theorem 5.5 on pages 28 and 39 are incorrect. This theorem should be replaced by

THEOREM 5.5. *Let* $n \geq 4$.

(1) *There exists a unique isotopy class of embeddings of* CP^n *in* R^{4n}.

(2) *There exist countable isotopy classes of embeddings of* CP^n *in* R^{4n-1}.

(3) *There exists a unique isotopy class of embeddings of* CP^n *in* R^{4n-2} *for* $n \neq 2^r$.

This note contains some corrections of [6, §5] and the proof of (2) and (3) of the above.

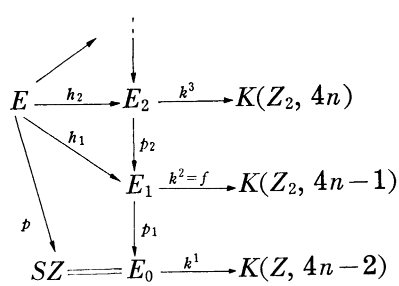
1. SOME CORRECTIONS. In this note, denote simply by SZ the quotient manifold $SZ_{n+1,2}$ of [6. (1.3)] and let λ be the real line bundle associated with the double covering $Z_{n+1,2} \longrightarrow SZ$. (In [6, §5], we consider also λ as the real line bundle associated with the double covering $CP^n \times CP^n - \Delta \longrightarrow (CP^n)^*$.) Let $\mathscr{B}$ be the S^{m-1}-bundle associated with $m\lambda$ and let $\mathscr{B}(\pi_i(S^{m-1}))$ be the bundle of coefficients with fiber $\pi_i(S^{m-1})$ associated with $\mathscr{B}$. Then the obstructions for the existence of a non-zero cross section of $m\lambda$ are the elements of $H^{i+1}(SZ; \mathscr{B}(\pi_i(S^{m-1})))$ and the obstructions for two given non-zero cross sections being homotopic are the elements of $H^i(SZ; \mathscr{B}(\pi_i(S^{m-1})))$. If m is even, then the bundle of coefficients $\mathscr{B}(\pi_i(S^{m-1}))$ is trivial since $m\lambda$ is orientable, and so the above cohomology groups with local coefficients coincide with the ordinary cohomology groups.

Therefore *the cohomology groups* $H^*(SZ; \pi_i(S^{m-1}))$ *for odd* m *in* [6, §5, pp. 38–39] *should be replaced by* $H^*(SZ; \mathscr{B}(\pi_i(S^{m-1})))$.

2. PROOF OF THEOREM 5.5. (2). By [4, §37.5] and [6, Prop. 5.2 (2)], it is sufficient to show that $H^{4n-2}(SZ; \mathscr{B}(\pi_{4n-2}(S^{4n-2}))) = Z$. Since $(4n-1)\lambda$ is unorientable, the bundle of coefficients $\mathscr{B}(\pi_{4n-2}(S^{4n-2}))$ with fiber $\pi_{4n-2}(S^{4n-2}) = Z$ is not trivial by [4, §38. 12]. Let $\mathscr{B}'$ be the tangent sphere bundle of SZ. Because SZ is a $(4n-2)$-dimensional unorientable manifold by [6, Th. 4.15], the bundle of coefficients $\mathscr{B}'(\pi_{4n-3}(S^{4n-3}))$ with fiber $\pi_{4n-3}(S^{4n-3}) = Z$ is not

trivial by [4, §38. 12]. Since $\pi_1(SZ)=Z_2$, which is easily seen, two bundles of coefficients $\mathscr{B}(\pi_{4n-2}(S^{4n-2}))$ and $\mathscr{B}'(\pi_{4n-3}(S^{4n-3}))$ with fiber Z are equivalent. Therefore we obtain $H^{4n-2}(SZ; \mathscr{B}(\pi_{4n-2}(S^{4n\ 2})))=H^{4n-2}(SZ; \mathscr{B}'(\pi_{4n-3}(S^{4n-3})))$. Referring to [4, §39. 5], we have $H^{4n-2}(SZ; \mathscr{B}'(\pi_{4n-3}(S^{4n-3})))=Z$ and so $H^{4n-2}(SZ; \mathscr{B}(\pi_{4n-2}(S^{4n-2})))=Z$.

3. PROOF OF THEOREM 5.5. (3). Consider the S^{4n-3}-bundle $p: E\longrightarrow SZ$ associated with $(4n-2)\lambda$. It is sufficient to show that there exists a unique homotopy class of cross sections of this sphere bundle. Since $(4n-2)\lambda$ is orientable, there exists a Postnikov system $\{E_i, p_i, h_i\}_{i\geq 1}$ where $p_i: E_i\longrightarrow E_{i-1}$ is the principal fibration with fiber $K(\pi_{4n-4+i}(S^{4n-3}), 4n-4+i)$ induced by $k^i: E_{i-1}\longrightarrow K(\pi_{4n-4+i}(S^{4n-3}), 4n-3+i)$ and $h_i: E\longrightarrow E_i$ is a $(4n-3+i)$-equivalence(*) and a lifting of h_{i-1} $(E_0=SZ, h_0=p)$.



Since h_2 is a $(4n-1)$-equivalence and SZ is a $(4n-2)$-dimensional manifold [6, Th. 4.15], $[SZ, E; id]$ is equivalent, as a set, to $[SZ, E_2; id]$ by [2, Th. 3.2] where $[X, Y; id]$ denotes the set of homotopy classes of cross sections of a fibration $Y\longrightarrow X$. Using the methods of [3], we shall showe that $[SZ, E_2; id]$ consists of one element.

Let $F=\Omega K(Z, 4n-2)=K(Z, 4n-3)$ and let $C=K(Z_2, 4n-1)$ which is considered as a topological group. Since the first invariant k^1 represents the Euler class of $(4n-2)\lambda$, which is zero for $n\neq 2^r$, we have $E_1=F\times SZ$. Let

$$m: F\times E_1=F\times(F\times SZ)\longrightarrow E_1=F\times SZ$$

be the action defined by

$$m(\nu, (\mu, x))=(\nu^{\vee}\mu, x) \qquad \text{for} \quad x\in SZ, \quad \nu, \mu\in F$$

where $\nu^{\vee}\mu$ is the composite of loops ν and μ in F [3, §§2-3]. Let $f=k^2: E_1=F\times SZ\longrightarrow C$ and let

$$f_1: (F\times E_1, *\times E_1)\longrightarrow(C, *)$$

$$\tilde{f}_2: PF\times E_1\longrightarrow PC, \quad f_2: \Omega F\times E_1\longrightarrow\Omega C,$$

(*) A map $g: X\longrightarrow Y$ is called an n-equivalence for $n\geq 1$ if $g_*: \pi_q(X)\longrightarrow\pi_q(Y)$ is isomorphic for $q<n$ and epimorphic for $q=n$.

denote the maps defined by

$$f_1(\nu, y)=f(m(\nu, y))\cdot[f(m(*, y))]^{-1}$$
$$\tilde{f}_2(\mu, y)(t)=f_1(\mu(t), y), \quad f_2=\tilde{f}_2 | \Omega F\times E_1,$$

where PF (resp. PC) denotes the path space of F (resp. C) and $\nu \in F$, $y \in E_1$, $\mu \in PF$, $t \in I$ [3, §4]. By the definition of f_2, it follows that

$$f_2(\xi^{\vee}\zeta, y)=f_2(\xi, y)^{\vee}f_2(\zeta, y) \qquad \text{for} \quad \xi, \zeta \in \Omega F, \quad y \in E_1.$$

Let η be the homotopy class of a cross section $s\colon SZ\longrightarrow E_1$ of $p_1\colon E_1\longrightarrow SZ$ and let θ be the homotopy class of $f=k^2$. Define

$$\Delta(\theta, \eta)\colon [SZ, \Omega F]\longrightarrow[SZ, \Omega C]$$

as follows; for a map $a\colon SZ\longrightarrow\Omega F$, let $b\colon SZ\longrightarrow\Omega C$ be the map given by

(1) $$b(x)=f_2(a(x), s(x)) \qquad \text{for} \quad x \in SZ\,.$$

Put $\Delta(\theta, \eta)[a]=[b]$ in $[SZ, \Omega C]$. Then $\Delta(\theta, \eta)\colon [SZ, \Omega F]\longrightarrow[SZ, \Omega C]$ is well-defined and a homomorphism. Since $[SZ, \Omega C]$ is isomorphic to $H^{4n-2}(SZ; Z_2)$, we regard $\Delta(\theta, \eta)$ as $\Delta(\theta, \eta)\colon [SZ, \Omega F]\longrightarrow H^{4n-2}(SZ; Z_2)$. For the determination of $\Delta(\theta, \eta)$, we prepare some results.

Let σ denote the suspension homomorphism of the path fibration $\Omega A\longrightarrow PA\xrightarrow{p}A$ and $H^i(A)$ stand for $H^i(A; Z_2)$ unless otherwise stated. Consider the following diagram:

$$\begin{array}{ccccc}
H^{4n-2}(\Omega C) & \xleftarrow[\approx]{} & H^{4n-2}(\Omega C, *) & \xrightarrow[\approx]{\delta} & H^{4n-1}(PC, \Omega C) \\
\downarrow f_2^* & & \downarrow f_2^* & & \downarrow \tilde{f}_2^* \\
H^{4n-2}(\Omega F\times E_1) & \longleftarrow & H^{4n-2}((\Omega F, *)\times E_1) & \xrightarrow[\approx]{\delta\times id} & H^{4n-1}((PF, \Omega F)\times E_1)
\end{array}$$

$$\begin{array}{ccccc}
\xleftarrow[\approx]{p^*} & H^{4n-1}(C, *) & \xrightarrow[\approx]{} & H^{4n-1}(C) \\
 & \downarrow f_1^* & & \downarrow f_1^* \\
\xleftarrow[\approx]{(p\times id)^*} & H^{4n-1}((F, *)\times E_1) & \longrightarrow & H^{4n-1}(F\times E_1)
\end{array}$$

The commutativity of this diagram implies that

(2) $$(\sigma\times id)f_1^*=f_2^*\sigma\,.$$

Let ι and $\bar{\iota}$ denote the mod 2 reductions of the characteristic classes of $F=K(Z, 4n-3)$ and ΩF, and let ι' and $\bar{\iota}'$ denote the characteristic classes of $C=K(Z_2, 4n-1)$ and ΩC, respectively. Then

(3) $$\sigma(\iota)=\bar{\iota}, \qquad \sigma(\iota')=\bar{\iota}'\,.$$

By the definition of $f_1\colon F\times E_1\longrightarrow C$, we have

(4) $$f_1^*(\iota')=m^*f^*(\iota')-1\times f^*(\iota') \quad \text{in} \quad H^{4n-1}(F\times E_1).$$

Now $f^*(\iota')$ is the element of $H^{4n-1}(F\times SZ)\cap \mathrm{Ker} h_1^*$ and $H^{4n-1}(F\times SZ)\cap \mathrm{Ker} h_1^* = H^{4n-1}(F)\otimes H^0(SZ)+H^{4n-3}(F)\otimes H^2(SZ)$ has $\{Sq^2\iota\otimes 1, \iota\otimes v^2, \iota\otimes c_1\}$ as basis by [6, Th. 4,9]. Hence $f^*(\iota')$ has the form $f^*(\iota')=\varepsilon_1 Sq^2\iota\otimes 1+\varepsilon_2\iota\otimes v^2+\varepsilon_3\iota\otimes c_1$, where $\varepsilon_i=0$ or 1 $(i=1, 2, 3)$. Referring to [5, IV], we have $\varepsilon_1=\varepsilon_2=1$. $\varepsilon_3=0$ and so

(5) $$f^*(\iota')=Sq^2\iota\otimes 1+\iota\otimes v^2\,.$$

By the definition of m: $F\times(F\times SZ)\longrightarrow F\times SZ$, m^*: $H^*(F)\otimes H^*(SZ)\longrightarrow H^*(F)\otimes H^*(F)\otimes H^*(SZ)$ is given by

(6) $m^*(x\otimes y)=x\otimes 1\otimes y+1\otimes x\otimes y$ for the primitive element $x\in H^*(F)$.

Using the above preparation, we now compute $\Delta(\theta, \eta)$.

$$\begin{aligned}
\Delta(\theta,\eta)[a] &= b^*(\bar{\iota}') \\
&= d^*(a\times s)^* f_2^*(\bar{\iota}') \quad \text{by (1), where } d \text{ is the diagonal map of } SZ \\
&= d^*(a\times s)^* f_2^*\sigma(\iota') \quad \text{by (3)} \\
&= d^*(a\times s)^*(\sigma\times id) f_1^*(\iota') \quad \text{by (2)} \\
&= d^*(a\times s)^*(\sigma\times id)(m^*f^*(\iota')-1\otimes f^*(\iota')) \quad \text{by (4)} \\
&= d^*(a\times s)^*(\sigma\times id)\{m^*(Sq^2\iota\otimes 1+\iota\otimes v^2)-1\otimes(Sq^2\iota\otimes 1+\iota\otimes v^2)\} \quad \text{by (5)} \\
&= d^*(a\times s)^*(\sigma\times id)(Sq^2\iota\otimes 1\otimes 1+\iota\otimes 1\otimes v^2) \quad \text{by (6)} \\
&= d^*(a\times s)^*(Sq^2\bar{\iota}\otimes 1\otimes 1+\bar{\iota}\otimes 1\otimes v^2) \quad \text{by (3)} \\
&= Sq^2a^*(\bar{\iota})+a^*(\bar{\iota})v^2 \quad \text{in} \quad H^{4n-2}(SZ)\,.
\end{aligned}$$

The element $c_1^{2^{r+1}-2}c_2^s$ of $H^{4n-4}(SZ)$ ($n=2^r+s$, $0\leq s<2^r$ is contained in the iamge of the mod 2 reduction and so there exists a: $SZ\longrightarrow \Omega F$ such that $a^*(\bar{\iota})=c_1^{2^{r+1}-2}c_2^s$. For such a map a, $\Delta(\theta, \eta)[a]=Sq^2(c_1^{2^{r+1}-2}c_2^s)+c_1^{2^{r+1}-2}c_2^s v^2\neq 0$, because $c_1^{2^{r+1}-2}c_2^s v^2\neq 0$ and $c_1^{2^{r+1}-1}=0$ by [6, Prop. 4.14]. Thus $\Delta(\theta, \eta)$: $[SZ, \Omega F]\longrightarrow[SZ, \Omega C]$ is an epimorphism. While $[SZ, F]=H^{4n-3}(SZ; Z)=0$ by [6, Th. 4.10]. Using [3, Th. 4.3], $[SZ, E_2; id]$ consists of one element and so there exists a unique isotopy class of embeddings of CP^n in R^{4n-2} for $n\neq 2^r$.

REMARK. Theorem 5.5. (2) is a special case of A. Haefliger's theorem of [1, 1. 3. e] for $V=CP^n$, $k=1$,

Refernces

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