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The Admissibility of Tests for the Equality of Mean Vectors and Covariance Matrices

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0. Summary

In the previous paper [3], the admissibility of certain tests and classifications in multivariate normal analysis was obtained using the method in Kiefer and Schwartz [2]. In this paper we consider the problem of testing the equality of mean vectors and covariance matrices. Two cases are considered. One case is that of testing the equality of mean vectors and covariance matrices in k normal populations, and the other case is testing the equality of a mean vector and a covariance matrix to given vector and matrix in a normal population. We shall prove admissibility of certain test procedures for the problems by modifying the Kiefer-Schwartz's method. The test procedures include the likelihood ratio test for each problem.

1. Preliminaries

Throughout this paper we consider random matrices whose columns are independently distributed, each p-variate normal. The parameter space in each problem will be denoted by $\Omega=\{\theta\}=H_0+H_1$. The Lebesgue density function of X for given θ will be denoted by $f_X(x; \theta)$. *A priori* probability measure or its constant multiples will be denoted by Π and $\Pi=\Pi_0+\Pi_1$ with Π_i a finite measure on H_i.

Let $V=(X, U)$ be a random matrix whose columns, under H_1, have common unknown covariance matrix Σ and $\mathrm{E}U=\nu(p\times 1)$ (unspecified). Let θ^* be the parameter of the distribution of X, i.e., $\theta=(\theta^*, \nu)$. Let H_1^* be the domain of θ^* under H_1, and consider the case where the domain of ν is E^p and $H_1=H_1^*\times \mathrm{E}^p$, i.e., $\theta\in H_1$ if and only if $\theta^*\in H_1^*$. Let H_1^{**} be a subset of H_1^* for which Σ can be written as $\Sigma=(C_0+D)^{-1}$ where C_0 is a given positive definite matrix and D is nonnegative definite matrix. And consider a finite measure Π_1^* on H_1^* which assigns whole measure to H_1^{**}. Then the following lemma holds:

LEMMA 1.1. *There exists a finite measure* Π_1 *on* H_1 *which satisfies*

$$(1.1)\qquad \int f_V(v; \theta)\,\Pi_1(d\theta)=c\cdot \mathrm{etr}\left\{-\frac{1}{2}C_0(U-\nu_0)(U-\nu_0)'\right\}\cdot\int f_X(x; \theta^*)\Pi_1^*(d\theta^*)$$

for any fixed vector ν_0.

PROOF. As in the proof of Lemma 3.1 in Kiefer and Schwartz [2] let the marginal distribution of θ^* be given by Π_1^*. Then we define the conditional distribution of ν, given θ^*, as follows: Let rank $D=r$, then we can write $D=\eta\eta'$ for $\eta(p\times r)$. Let $\gamma(r\times 1)$ be distributed according to $N(0, (I_r-\eta'\Sigma\eta)^{-1})$ and define the conditional distribution of ν by

$$\Sigma^{-1}(\nu-\nu_0)=\eta\gamma.$$

For this *a priori* distribution we obtain the lemma.

Lemma 1,1. is of course available for $i=0$. Moreover it can be extended to the case when $V=(V^{(1)}, V^{(2)})$, where the parameters of $V^{(1)}$ and $V^{(2)}$ are independent and each $V^{(i)}$ is of the above form. Using this lemma and Lemma 1.1 in Nishida [3] we derive Bayes critical regions for 0–1 loss, which are admissible.

2. k sample problem

Let $p\times N_i$ matrix $X^{(i)}=(X_1^{(i)},\cdots, X_{N_i}^{(i)})$ be a random sample from a multivariate normal distribution with unknown mean vector μ_i and unknown covariance matrix $\Sigma_i (i=1,\cdots, k)$. Then we want to test the hypothesis $H_0: \mu_1=\cdots=\mu_k, \Sigma_1=\cdots=\Sigma_k$ against the alternatives $H_1: \mu_i\neq\mu_j$ or $\Sigma_i\neq\Sigma_j$ for some i and j. We write

$$\bar{X}^{(i)}=\frac{1}{N_i}\sum_{t=1}^{N_i} X_t^{(i)},\quad \bar{X}=\frac{1}{N}\sum_{i=1}^{k}\sum_{t=1}^{N_i} X_t^{(i)},$$

$$S_i=\sum_{t=1}^{N_i}(X_t^{(i)}-\bar{X}^{(i)})(X_t^{(i)}-\bar{X}^{(i)})', \tag{2.1}$$

$$S=\sum_{i=1}^{k}\sum_{t=1}^{N_i}(X_t^{(i)}-\bar{X})(X_t^{(i)}-\bar{X})',$$

where $N=\sum_{i=1}^{k} N_i$. Then the following theorem holds where $n_i=N_i-1$, $n=N-1$:

THEOREM 2.1. *Suppose* $p-1<r<n-p+1$, $p-1<r_i<n_i-p+1 (i=1,\cdots, k)$, *then a test with the critical region*

$$\frac{|S|^r}{\prod_{i=1}^{k}|S_i|^{r_i}}\geq c \tag{2.2}$$

is admissible Bayes.

PROOF. We prove this theorem in the case when $k=2$. The proof to the case when $k>2$ is a straightfoward extension. Consider an orthogonal matrix $Q^{(0)}$ under H_0 such that $(X^{(1)}, X^{(2)})Q^{(0)}=XQ^{(0)}=(Y_1,\cdots, Y_n, \sqrt{N}\bar{X})$ where each column vector of $XQ^{(0)}$ has common unknown covariance matrix Σ and $\mathrm{E}\,Y_t=0$. Similarly consider an orthogonal matrix $Q^{(1)}$ under H_1 such that $XQ^{(1)}=(Y_1^{(1)},\cdots, Y_{n_1}^{(1)}, \sqrt{N_1}\,\bar{X}^{(1)}, Y_1^{(2)},\cdots, Y_{n_2}^{(2)}, \sqrt{N_2}\,\bar{X}^{(2)})$, where column vectors $Y_1^{(i)},\cdots, Y_{n_i}^{(i)}$ and $\sqrt{N_i}\,\bar{X}^{(i)}$ have common unknown covariance matrix Σ_i and $\mathrm{E}\,Y_t^{(i)}=0$. We apply Lemma 1.1 to $\sqrt{N}\,\bar{X}, \sqrt{N_1}\,\bar{X}^{(1)}$ and $\sqrt{N_2}\,\bar{X}^{(2)}$ respectively, by setting $\nu_0=0$. Let $\Sigma^{-1}=I_p+\eta\eta'$ under H_0 for $\eta(p\times q)$ where $q\geq p$ and let $\Sigma_i^{-1}=I_p+\eta_i\eta_i'$ under H_1 for $\eta_i(p\times q_i)$ where $q_i\geq p$. We set

$$
\begin{aligned}
&d\Pi_0(\eta)/d\eta=|\eta\eta'|^{(r-q)/2}|I_p+\eta\eta'|^{-n/2} \\
&d\Pi_1(\eta)/d\eta=\prod_{i=1}^{2}\left[|\eta_i\eta_i'|^{(r_i-q_i)/2}|I_p+\eta_i\eta_i'|^{-n_i/2}\right]
\end{aligned}
\tag{2.3}
$$

for the Lebesgue density function of $\Pi_0(\eta)$ and $\Pi_1(\eta)$ respectively. The integrability of (2.3) under the condition of Theorem 2.1 is proved in [2]. For this *a priori* distribution, the statistic giving the Bayes critical region (1,1) in [3], is calculated as follows:

$$
\frac{\operatorname{etr}\left\{-\frac{1}{2}(N_1\bar{X}^{(1)}\bar{X}^{(1)\prime}+N_2\bar{X}^{(2)}\bar{X}^{(2)\prime})\right\}}{\operatorname{etr}\left\{-\frac{1}{2}N\bar{X}\bar{X}'\right\}}
\cdot\frac{\int\prod_{i=1}^{2}\left[|\eta_i\eta_i'|^{(r_i-q_i)/2}\operatorname{etr}\left\{-\frac{1}{2}(I_p+\eta_i\eta_i')S_i\right\}\right]d\eta_1 d\eta_2}{\int|\eta\eta'|^{(r-q)/2}\operatorname{etr}\left\{-\frac{1}{2}(I_p+\eta\eta')S\right\}d\eta}
$$

$$
=\frac{|S|^{r/2}}{|S_1|^{r_1/2}|S_1|^{r_2/2}}\cdot\frac{\int\prod_{i=1}^{2}\left[|\eta_i^*\eta_i^{*\prime}|^{(r_i-q_i)/2}\operatorname{etr}\left\{-\frac{1}{2}\eta_i^*\eta_i^{*\prime}\right\}\right]d\eta_1^* d\eta_2^*}{\int|\eta^*\eta^{*\prime}|^{(r-q)/2}\operatorname{etr}\left\{-\frac{1}{2}\eta^*\eta^{*\prime}\right\}d\eta^*}
$$

where $\eta^*=S^{\frac{1}{2}}\eta$, $\eta_i^*=S_i^{\frac{1}{2}}\eta_i$. Since the integral in the last line is constant, we obtain the theorem by Lemma 1.2 in [3].

COROLLARY 2.1. *The likelihood ratio test*

$$
\frac{|S|^N}{\prod_{i=1}^{k}|S_i|^{N_i}}\geq c
\tag{2.4}
$$

is admissible Bayes, when $\min_i n_i>2(p-1)$.

PROOF. put $r=c_1N$, $r_i=c_1N_i$ in Theorem 2.1 where c_1 is slightly larger than $(p-1)/\min N_i$, and we obtain Corollary 2.1. To satisfy the integrability condition, it is required that $\min_i n_i>2(p-1)$.

If we replace N and N_i's by n and n_i's respectively in (2.4) and in the proof of Corollary 2.1, we obtain the same statement. This is a modified version of the likelihood ratio test by the degrees of freedom. It is easy to obtain tests with more complicated critical regions by using *a priori* distribution like (3.2) in Section 3.

3. One sample problem

Let $p\times N$ matrix $X=(X_1,\cdots, X_N)$ be a random sample from a multivariate normal distribution with unknown mean vector μ and unknown covariance matrix Σ. From this sample we want to test the hypothesis $H_0: \mu=\mu_0$, $\Sigma=\Sigma_0$ against the alternatives $H_1: \mu\neq\mu_0$ or $\Sigma\neq\Sigma_0$ where μ_0 is a given vector and Σ_0 is a given positive definite matrix. This problem can be reduced to the following form: $Y=(Y_1,\cdots, Y_n, \sqrt{N}\bar{X})(N=n+1)$ with $\mathrm{E}Y_i=0(p\times 1)$, $\bar{X}=\frac{1}{N}\sum_{i=1}^{N} X_i$ and all the columns of Y are independently distributed with common unknown covariance matrix Σ. We treat the problem in this form. If we write $S=\sum_{i=1}^{n} Y_iY_i'$, then the following theorem holds:

Theorem 3.1. *For given $p\times p$ positive definite matrix B_0 and nonnegative definite matrices $B_1,\cdots, B_{m_1+m_2}$, a test with the critical region*

$$\frac{\operatorname{etr}\left[(\Sigma_0^{-1}-B_0)\{S+N(\bar{X}-\mu_0)(\bar{X}-\mu_0)'\}\right]}{\left[\prod_{i=1}^{m_1}|B_i+S|^{q_i}\right]\left[\prod_{i=m_1+1}^{m_1+m_2}|B_i+S|^{q_i+t_i}\right]}\geq c \tag{3.1}$$

is admissible Bayes, provided that (i) *$q_i\geq p$ for $i=m_1+1,\cdots, m_1+m_2$ where $q_1,\cdots, q_{m_1+m_2}$ are positive integers,* (ii) *$p-1<q_i+t_i$ for $i=m_1+1,\cdots, m_1+m_2$ and* (iii) *$\sum_{i=1}^{m_1+m_2} q_i+\sum_{i=m_1+1}^{m_1+m_2}\max(0, t_i)<n-p+1$. When $m_1=0$ and $m_2=1$, the condition* (iii) *is improved to $q_1+t_1<n-p+1$.*

Proof. We apply Lemma 1.1 to $\sqrt{N}\bar{X}$ by setting $\nu_0=\sqrt{N}\mu_0$ under H_1. Let $\Sigma^{-1}=B_0+\sum_{i=1}^{m_1+m_2}\eta_i\eta_i'$ under H_1, for $\eta_i(p\times q_i)$. We set

$$d\Pi_1(\eta)/d\eta=\left[\prod_{i=m_1+1}^{m_1+m_2}|\eta_i\eta_i'|^{t_i/2}\right]\left|B_0+\sum_{i=1}^{m_1+m_2}\eta_i\eta_i'\right|^{-n/2} \cdot\operatorname{etr}\left\{-\frac{1}{2}\sum_{i=1}^{m_1+m_2}B_i\eta_i\eta_i'\right\} \tag{3.2}$$

for the Lebesgue density function of $\Pi_1(\eta)$. The integrability of (3.2) under the assumptions is shown in [3]. After the same calculations as for Theorem 2.1 in [3], we obtain the theorem.

Corollary 3.1. *The likelihood ratio test*

$$\frac{\operatorname{etr}\left[\Sigma_0^{-1}\{S+N(\bar{X}-\mu_0)(\bar{X}-\mu_0)'\}\right]}{|S|^{N}}\geq c \tag{3.3}$$

is admissible Bayes when $n>p$.

PROOF. Let $m_1=1$, $m_2=0$, $q_1=1$, $B_1=0$ and $B_0=[(N-1)/N]\Sigma_0^{-1}$ in Theorem 3.1. Then we obtain the corollary.

We can generalize these results to the k sample case. Let $p\times N_i$ matrix $X^{(i)}=(X_1^{(i)},\cdots,X_{N_i}^{(i)})$ be a random sample from a multivariate normal distribution with unknown mean vector μ_i and unknown covariance matrix $\Sigma_i (i=1,\cdots,k)$. We consider the problem of testing $H_0:\mu_i=\mu_{0i}$, $\Sigma_i=\Sigma_{0i}$ $(i=1,\cdots,k)$ against $H_1:\mu_j\neq\mu_{0j}$ or $\Sigma_j\neq\Sigma_{0j}$ for some j. We write $\bar{X}^{(i)}=\frac{1}{N_i}\sum_{t=1}^{N_i}X_t^{(i)}$, $S_i=\sum_{t=1}^{N_i}(X_t^{(i)}-\bar{X}^{(i)})(X_t^{(i)}-\bar{X}^{(i)})'$ and $n_i=N_i-1$.

THEOREM 3.2. *For given* $p\times p$ *positive definite matrix* B_{0j} *and nonnegative definite matrices* $B_{1j},\cdots,B_{m_{1j}+m_{2j}j}$, *a test with the critical region*

$$\prod_{j=1}^{k}\left\{\frac{\operatorname{etr}(\Sigma_{0j}^{-1}-B_{0j})[S_j+N_j(\bar{X}^{(j)}-\mu_{0j})(\bar{X}^{(j)}-\mu_{0j})']}{\left[\prod_{i=1}^{m_{1j}}|B_{ij}+S_j|^{q_{ij}}\right]\left[\prod_{i=m_{1j}+1}^{m_{1j}+m_{2j}}|B_{ij}+S_j|^{q_{ij}+t_{ij}}\right]}\right\}\geq c \tag{3.4}$$

is admissible Bayes, provided that (i) $q_{ij}\geq p$ *for* $i=m_{1j}+1,\cdots,m_{1j}+m_{2j}$ *where* $q_{1j},\cdots,q_{m_{1j}+m_{2j}j}$ *are positive integers,* (ii) $p-1<q_{ij}+t_{ij}$ *for* $i=m_{1j}+1,\cdots,m_{1j}+m_{2j}$ *and* (iii) $\sum_{i=1}^{m_{1j}+m_{2j}}q_{ij}+\sum_{i=m_{1j}+1}^{m_{1j}+m_{2j}}\max(0,t_{ij})<n_j-p+1$ *hold for all* $j=1,\cdots,k$. *When* $m_{1j}=0$ *and* $m_{2j}=1$, *the condition* (iii) *is improved to* $q_{1j}+t_{1j}<n_j-p+1$.

The admissibility of the likelihood ratio test in this case is obtained as follows: Put $m_{1j}=0$, $m_{2j}=1$, $B_{1j}=0(p\times p)$, $q_{1j}+t_{1j}=c_1N_j$ and $B_{0j}=(1-c_1)\Sigma_{0j}^{-1}$ for $j=1,\cdots,k$ in Theorem 3.2, where c_1 is slightly larger than $(p-1)/\min_j N_j$. To satisfy the integrability condition, it is required that $\min_j n_j>2(p-1)$.

COROLLARY 3.2. *The likelihood ratio test*

$$\prod_{j=1}^{k}\frac{\operatorname{etr}\Sigma_{0j}^{-1}[S_j+N_j(\bar{X}^{(j)}-\mu_{0j})(\bar{X}^{(j)}-\mu_{0j})']}{|S_j|^{N_j}}\geq c \tag{3.5}$$

is admissible Bayes, when $\min_j n_j>2(p-1)$.

Birnbaum [1] proved that the Bayes acceptance region for the exponential family is convex when null hypothesis is simple. In this case, he also proved that the set of convex acceptance regions is the minimal complete class under some additional assumptions. It can be shown by his method that our Bayes rules in this section have convex acceptance regions. But the minimal complete class for our problem seems to be unknown yet, even in the simplest case where $p=1$ and $k=1$.

References

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