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On a Characterization of Almost Dedekind Domains

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Introduction.

Throughout this discussion R will be a commutative ring with unit. The purpose of this paper is to characterize the rings over which each module with $D.C.C.$ decomposes to a direct sum of cocyclic modules.

We shall introduce a homomorphism ϕ_A, related with an ideal of R of an indecomposable injective module over R. The ϕ_A plays an important role for our purpose; we shall discuss in §3 some basic properties of the ϕ_A and also of the image of ϕ_A, which enable us to show that a locally noetherian domain with the property mentioned above must be an almost Dedekind domain.

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Notation and terminology.

Let M be an R-module and $x \neq 0 \in M$. We denote by $E(M)$ the *injective envelope of* M and by $0(x)=\{r \in R \mid rx=0\}$ the *order ideal of* x. Let E be an injective R-module, N a submodule of E and A an ideal of R; we put $A^*=\{x \in E \mid ax=0$ for every $a \in A\}$ and $N^*=\{r \in R \mid rx=0$ for every $x \in N\}$. We shall say that a module M has $D.C.C.$ if M satisfies the descending chain condition for submodules. We shall say that an R-module M is a P-primary module, where P is a prime ideal of R, if for every non-zero element x of M, $0(x)$ is a P-primary ideal. A ring R is called a *quasi-local ring* if it has only one maximal ideal and a noetherian quasi-local ring is called a *local ring*. If R is a local ring, $\bar{R}$ denotes the completion of R.

§1. Cocyclic modules

Let B be an R-module. If B has a non-zero element c with the following condition (*), then we shall B a *cocyclic R-module.*

(*) For every R-module C, every R-homomorphism ϕ: $B \rightarrow C$ with $c \notin Ker\phi$ is *monic.*

Then we shall call c a *cogenerator* of B. (We borrow this definition from Fuchs [3].) It is easily seen that B is a cocyclic R-module if and only if every non-zero submodule of B contains c; *i.e.* Rc is the smallest submodule of B.

PROPOSITION 1. *B is a cocyclic R-module if and only if B is an essential extension of R/P for some maximal ideal P of R.*

PROOF. Let B be a cocyclic R-module and c a cogenerator of B. It is clear that B is an essential extension of Rc. Since Rc is isomorphic to R/P for some ideal P of R and Rc is the smallest submodule of B, R/P is a field. The assertion follows immediately from the fact that Rc is simple.

LEMMA 1. *Let P be a maximal ideal of R and $E=E(R/P)$. Then we have the following:*

(1) *For every non-zero element x of E, $0(x)\subseteqq P$.*

(2) *For every element s of $R-P$, the homothety $s: E \ni x \rightarrow sx \in E$ is an automorphism.*

(3) *E has the structure of an R_P-module.*

PROOF. (1) and (2) are obvious.

(3) Let $s \in R-P$, $r \in R$ and $x \in E$. Then by (2), there exists a unique element y of E such that $x=sy$. If we define $(r/s)x=ry$, it is easily verified that this definition is well-defined and makes E into an R_P-module.

LEMMA 2. *With the notation of Lemma 1,*

(1) *The order ideal of x in R_P is $0(x)R_P$.*

(2) $0(x)=0(x)R_P\cap R$.

PROOF. Trivial.

PROPOSITION 2. *Let R be a locally noetherian ring and B a cocyclic R-module; i.e. B is an essentail extension of R/P for a maximal ideal P of R. Then B has the structure of an R_P-module.*

PROOF. By Proposition 1, we may assume $B\subseteqq E(R/P)$. Let $s \in R-P$ and $x \in B$. Then by Lemma 1 (2), there exists a unique element y of $E(R/P)$ such that $x=sy$. Since R_P is a noetherian ring, then by Lemma 2 and by Lemma 3.2 of E. Matlis [4], $0(y)R_P$ is a PR_P-primary. Therefore $P^nR_P\subseteqq 0(y)R_P$ for some positive integer n. By Lemma 2, $P^n=P^nR_P\cap R\subseteqq 0(y)R_P\cap R=0(y)$; this implies that, combining the fact $P^n+Rs=R$, $y \in Rx$. Thus the proof is completed.

COROLLARY. *With the notation of Proposition 2, B has the structure of an $\bar{R}_P$-module.*

PROOF. This follows from Proposition 2 and Theorem 3.6 of E. Matlis [4].

THEOREM 1. *For a ring R, the following statements are equivalent:*
(1) *R is a locally noetherian ring.*
(2) *Every cocyclic R-module has D.C.C.*
(3) *For every maximal ideal P of R, E(R/P) has D.C.C.*

PROOE. (1)⇒(2): Let B be a cocyclic R-module. Then there exists a maximal ideal P of R such that $R/P\subseteqq B\subseteqq E(R/P)=E$. By Proposition 2, B has the structure of an R_P-module. Since R_P is a noetherian ring, by Proposition 4.1 of E. Matlis [4], B has D.C.C. as an R_P-module; therefore B has D.C.C. as an R-module.
(2) ⇒ (3): Trivial.
(3) ⇒ (1): Let P be a maximal ideal of R and $E=E(R/P)$. By Lemma 1, E has the structure of an R_P-module and therefore E has D.C.C. as an R_P-module. Then by Theorem 4.1 of Ishikawa [1], R_P has the ascending chain condition for ideals; i.e. R_P is a noetherian ring.

§2. Modules with D.C.C.

We show in §2 that every module with D.C.C. over an almost Dedekind domain has a decomposition into a direct sum of cocyclic modules. Throughout this section the ring will be a locally noetherian ring. Now we give a generalization of the result in Proposition 3 of E. Matlis [5] in the following

PROPOSITION 3. *Let B be an R-module. Then B has D.C.C. if and only if* $E(B)=E(R/P_1)\oplus\cdots\oplus E(R/P_n)$ *for a finite number of maximal ideals* P_i *of R.* $(i=1,2,\cdots,n)$

PROOF. By virtue of Theorem 1, the proof can be done quite similarly as in Proposition 3 of E. Matlis [5].

COROLLARY. *If B is an R-module with D.C.C., then* $B=B_1\oplus\cdots\oplus B_n$, *where every* B_i *is the* P_i*-component of B for distinct maximal ideals* $P_1,\cdots,P_n$ *of R.*

PROOF. By Proposition 3, there exist a finite number of distinct maximal ideals $P_1,\cdots,P_n$ of R such that $E(B)=E_1\oplus\cdots\oplus E_n$, where every E_i is the P_i-component of $E(B)$. Let $B_i=B\cap E_i$ for each i. Then B_i is the P_i-component of B. Let $x\neq 0 \in B$. Then $x=x_1+\cdots+x_n$, where $x_i\in E_i$. It is sufficient to prove that $x_i\in B$. Clearly $x_1\in B$, if $x=x_1$. If $x\neq x_1$, then $P_1 \supsetneqq \bigcap_{i=2}^{n} 0(x_i)$, hence there exists an element r of $\bigcap_{i=2}^{n} 0(x_i)$ such that $r\notin P_1$. Hence $rx=rx_1$. Since $0(x_1)\supseteqq P_1^s$ for some integer s and there exist two elements $p\in P_1^s$ and $a\in R$ such that $1=p+ra$, $x_1=rax$. Thus $x_1\in B$ and so on.

PROPOSITION 4. *Suppose that R is a locally noetherian domain. Let B be a P-primary R-module, where P is a maximal ideal of R.*

Then:
(1) *B has the structure of an R_P-module.*
(2) *B has D.C.C. as an R-module if and only if B has D.C.C. as an R_P-module.*

PROOF. (1) This can be done similarly as Proposition 2.
(2) Trivial.

The next result is a generalization of the result in Theorem 17.1 of Fuchs [3].

THEOREM 2. *Let R be a local ring with the maximal ideal (p), generated by an element p of R, and B a (p)-primary R-module. We assume that there exist an ascending chain $\{B_n\}$ of R-submodules of B such that $B=\bigcup_{n=1}^{\infty} B_n$ and $B_n \cap P^n B=0$ for each integer n. Then B is a direct sum of cocyclic R-modules; more precisely, $B=\bigoplus_\alpha B_\alpha$, where $B_\alpha=B\cap E_\alpha$ for some decomposition $\bigoplus_\alpha E_\alpha$ of $E(B)$ with $E_\alpha \cong E(R/(p))$.*

PROOF. We can prove this by the method analogous to the proof of Theorem 17.1 of Fuchs [3].

COROLLARY 1. *Let R be a discrete valuation ring with the maximal ideal (p) and B a (p)-primary R-module with D.C.C. Then $B=B\cap E_1\oplus\cdots\oplus B\cap E_n$ for some decomposition $E_1\oplus\cdots\oplus E_n$ of $E(B)$, where $E_i \cong E(R/(p))$.*

PROOF. Since R is a discrete valuation ring, each divisible R-module is injective. Therefore we may assume that B has no divisible submodule. Since B has *D.C.C.*, there exists some integer m such that $p^m B=0$. Then this Corollary holds by Theorem 2.

COROLLARY 2. *Let R be an almost Dedekind domain. Then every R-module with D.C.C. is a direct sum of a finite number of cocyclic R-modules.*

PROOF. This follows from Proposition 3, Proposition 4 and Corollary 1 of Theorem 2.

§3. On a homomorphism ϕ_A

In this section, we shall prove the converse of Corollary 2 of Theorem 2. Hereafter let R be a quasi-local ring, P the maximal ideal of R and $E=E(R/P)$.

DEFINITION. For an ideal $A=(a_1\cdots, a_n)$ of R, we define a homorphism ϕ_A: $E\to\overset{n}{\oplus}E$ by $\phi_A(x)=(a_1x\cdots, a_nx)$. ($\overset{n}{\oplus}E$ denotes a direct sum of n copies of E.) We denote $Im\phi_A$ by E_A. Since $Ker\phi_A=A^*$, E_A is independent of the choice of the ideal basis of A up to isomorphisms.

PROPOSITION 5. *With the above notation, E_A contains $\overset{n}{\oplus}R/P$ if and only if A is generated by n elements at least.*

PROOF. Let $E_A \supseteqq \overset{n}{\oplus} R/P$. If A is generated by k elements with $k \leq n-1$, $E(E_A) \subseteqq \overset{n-1}{\oplus} E$. Clearly this leads to a contradiction.

Conversely, let $(a_1, \cdots, a_n)$ be a minimal basis of A and $A_i = (a_1, \cdots, \check{a}_i, \cdots, a_n)$. Then $A_i \supsetneqq (a_i)$ for each i, therefore there exists $x_i \in E$ such that $0(x_i) \supseteqq A_i$ and $0(x_i) \not\ni a_i$ for each i by noting the annihilator relation of ideals. Then $E_A \supseteqq \bigoplus_{i=1}^{n} R(a_1 x_i, \cdots, a_n x_i) = \bigoplus_{i=1}^{n} R(0, \cdots, a_i x_i, \cdots, 0) \supseteqq \overset{n}{\oplus} R/P$.

COROLLARY. *Let R be a quasi-local domain. Then E_A is isomorphic to E if and only if A is principal.*

PROOF. This follows immediately from the fact that E is divisible.

N.B. If R is a complete local domain, indecomposable submodules in each direct sum of a finite number of E are only E_A, where A is principal.

LEMMA 3. *Let A be an ideal of R with a minimal basis $(a_1, \cdots, a_n)$, where $n \geqq 2$. Then we have $E_A \not\cong \overset{n}{\oplus} E$.*

PROOF. Assume that $E_A = \overset{n}{\oplus} E$. Then $(a_1)^* = \cdots = (a_n)^*$. In fact let $x \in (a_i)^*$. Since $(x, \cdots, x) \in E_A$, there exists an element y of E such that $x = a_j y$ for $j = 1, 2, \cdots, n$. Then $a_j x = a_j a_i y = a_i x = 0$. Hence $x \in (a_j)^*$. Thus $(a_1)^* = (a_2)^* = \cdots = (a_n)^*$. Then by noting the annihilator relation for ideals, $(a_1) = (a_2) = \cdots = (a_n)$. This is a contradiction.

PROPOSITION 6. *Let R be a complete local domain and let A_1 and A_2 be ideals of R. Then E_{A_1} is isomorphic to E_{A_2} if and only if there exist two non-zero elements a, b of R such that $aA_1 = bA_2$.*

PROOF. $\Rightarrow$. Trivial.
$\Leftarrow$. If E_{A_1} is isomorphic to E_{A_2}, then E/A_1^* is isomorphic to E/A_2^*. Then by Theorem 4.2 of E. Matlis [4], $A_1 \cong A_2$. Since R is an integral domain, there exist two non-zero elements a, b of R such that $aA_1 = bA_2$.

PROPOSITION 7. *Let R be a complete local domain. Then E can not have a decomposition into a sum of two non-zero proper submodules. Moreover every non-zero homomorphic image of E has this property.*

PROOF. Assume that $E = M + N$, where M and N are two submodules of E. Then $M^* \cap N^* = E^* = 0$. Since R is an integral domain $M^* = 0$ or $N^* = 0$. Say $M^* = 0$; then by Theorem 4.2 of E. Matlis [4], $M = E$. Thus the first assertion is proved. The latter is obvious.

COROLLARY. *Let R be a complete local domain and A a non-zero ideal of R. Then E_A is indecomposable.*

PROOF. This follows immediately from Proposition 7.

THEOREM 3. *Let R be a locally noetherian domain. Then R is an almost Dedekind domain if and only if every R-module with D.C.C. decomposes into a direct sum of cocyclic R-modules.*

PROOF. $\Rightarrow$. This follows from Corollary 2 of Theorem 2.
$\Leftarrow$. Suppose R be not an almost Dedekind domain. Then there exists a maximal ideal P of R such that R_P is not a principal ideal domain but a noetherian ring. Let $E=E(R/P)$. Then E has the structure of an $\bar{R}_P$-module by Corollary of Proposition 2 and by Theorem 2 of I.S. Cohen [2] $\bar{R}_P$ is not a principal ideal domain; therefore there exists an ideal A of $\bar{R}_P$ such that A is not principal. Then E_A has *D.C.C.* and is not a direct sum of cocyclic R-modules by Theorem 1 and Corollary of Proposition 7. This is a contradiction.

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