HIROSHIMA MATH. J.
2 (1972), 263–270

Oscillation Theorems for Delay Equations of Arbitrary Order

Takaŝi KUSANO and Hiroshi ONOSE

(Received September 14, 1972)

1. Introduction

We consider the n-th order delay equations

$$x^{(n)}(t)+p(t)f(x(t),\ x(\delta(t)))=0, \tag{1}$$

$$x^{(n)}(t)+p(t)g(x(\delta(t)))=0, \tag{2}$$

where $p(t)$ is continuous and eventually positive on $R_+=[0, \infty)$ and $\delta(t)$ is continuous on R_+ with $\delta(t)\leqq t$, $\lim_{t\to\infty} \delta(t)=\infty$. (These assumptions on $p(t)$ and $\delta(t)$ will be assumed without further mention.) We restrict attention to solutions of (1) or (2) which exist on some positive half-line. A nontrivial solution $x(t)$ is called oscillatory if there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k=\infty$ and $x(t_k)=0$ for all k. Otherwise, a solution is called nonoscillatory. A nonoscillatory solution is said to be strongly monotone if it tends monotonically to zero as $t\to\infty$ together with its first $n-1$ derivatives.

In [2] we established an oscillation theorem for (2) under the assumption that the retarded argument $\delta(t)$ is continuously differentiable and nondecreasing on R_+. The purpose here is to give oscillation criteria for (1) and (2) by avoiding this assumption and requiring that $\delta(t)$ has a continuously differentiable and nondecreasing minorant $\delta_*(t)$. The use of a differentiable minorant was suggested by Travis [4]. This will allow our theorems to be applied to delay equations of the form $x^{(n)}(t)+p(t)g(x(t-\tau(t)))=0$, $0\leqq\tau(t)\leqq M$, where $\tau(t)$ is not assumed differentiable.

2. Main Theorems

We now state our major results.

THEOREM 1. *With regard to equation* (1) *assume that*:

(i) *there exists a continuously differentiable and nondecreasing function on* R_+, $\delta_*(t)$, *such that* $\delta_*(t)\leqq\delta(t)$ *and* $\lim_{t\to\infty}\delta_*(t)=\infty$;

(ii) $f(x, y)$ *is continuous on* $R\times R$, $R=(-\infty, \infty)$, *is nondecreasing in* y, *and has the sign of* x *and* y *when they have the same sign;*

(iii) *there exist positive numbers* M *and* $\alpha\neq1$ *such that*

$$\liminf_{|y|\to\infty}\frac{|f(x,y)|}{|y|^{\alpha}}>0 \quad \text{if} \quad |x|\geqq M.$$

Then if

$$\int^{\infty}[\delta_*(t)]^{\alpha^*(n-1)}p(t)\,dt=\infty, \quad \alpha^*=\min(\alpha,1), \tag{3}$$

every solution of (1) *is oscillatory in the case* n *is even, and every solution is either oscillatory or strongly monotone in the case* n *is odd.*

THEOREM 2. *With regard to equation* (2) *assume that:*

(i) *there exists a continuously differentiable and nondecreasing function on* R_+, $\delta_*(t)$, *such that* $\delta_*(t)\leqq\delta(t)$ *and* $\lim_{t\to\infty}\delta_*(t)=\infty$;

(ii) $g(x)$ *is continuous and nondecreasing on* R, $xg(x)>0$ *for* $x\neq0$;

(iii) *for some* $\varepsilon>0$

$$\int_{\varepsilon}^{\infty}\frac{dx}{g(x)}<\infty \quad \text{and} \quad \int_{-\varepsilon}^{-\infty}\frac{dx}{g(x)}<\infty.$$

Let

$$\int^{\infty}[\delta_*(t)]^{n-1}p(t)\,dt=\infty. \tag{4}$$

Then if n *is even, every solution of* (2) *is oscillatory, and if* n *is odd, every solution is either oscillatory or strongly monotone.*

THEOREM 3. *Let equation* (2) *be subject to* (i), (ii) *of Theorem* 2 *and*

(iii′) *there exist positive numbers* M, λ_0, $\alpha<1$ *such that for* $\lambda\geqq\lambda_0$

$$g(\lambda x)\geqq M\lambda^{\alpha}g(x) \text{ if } x>0 \text{ and } g(\lambda x)\leqq M\lambda^{\alpha}g(x) \text{ if } x<0.$$

Then if

$$\int^{\infty}[\delta_*(t)]^{\alpha(n-1)}p(t)\,dt=\infty, \tag{5}$$

the conclusion of Theorem 2 *holds.*

Remark 1. If, in Theorem 1, $\delta(t)$ is of the form $\delta(t)=t-\tau(t)$, $0\leqq\tau(t)\leqq M$, then we can take $\delta_*(t)=t-M$, and condition (3) is equivalent to the following

$$\int^{\infty}t^{\alpha^*(n-1)}p(t)\,dt=\infty, \quad \alpha^*=\min(\alpha,1).$$

The same remark also applies to Theorems 2 and 3.

Remark 2. Theorem 1 is an extension of our previous result [2, Theorem 1] and includes as special cases (sufficiency parts of) the theorems of Gollwitzer [1]. Theorems 2 and 3 also extends Gollwitzer's Theorems 1 and 2, respectively.

3. Proofs of Theorems

The following lemma is neeeded (see Ryder and Wend [3]).

LEMMA. *If* $x(t)\in C^n[a,\infty)$, $x(t)\geqq 0$ *and* $x^{(n)}(t)\leqq 0$ *on* $[a,\infty)$, *then exactly one of the following cases occurs*:

(I) $x'(t),\cdots, x^{(n-1)}(t)$ *tend monotonically to zero as* $t\to\infty$;

(II) *there exists an odd integer* k, $1\leqq k\leqq n-1$, *such that*

$$\lim_{t\to\infty} x^{(n-j)}(t)=0 \text{ for } 1\leqq j\leqq k-1,\ \lim_{t\to\infty} x^{(n-k)}(t)\geqq 0,\ \lim_{t\to\infty} x^{(n-k-1)}(t)>0,$$

and $x(t), x'(t),\cdots, x^{(n-k-2)}(t)$ *tend to* ∞ *as* $t\to\infty$.

PROOF OF THEOREM 1. The proof is patterned on that contained in our previous paper [2]. Let $x(t)$ be a nonoscillatory solution of (1). We may assume that $x(t)>0$ for large t. The case $x(t)<0$ can be treated similarly. Since $\lim_{t\to\infty}\delta_*(t)=\infty$, there exists a T such that $x(\delta_*(t))>0$ for $t\geqq T$. In view of (1),

$$x^{(n)}(t)=-p(t)f(x(t), x(\delta(t)))<0,\ t\geqq T. \tag{6}$$

Therefore, $x^{(n-1)}(t)$ decreases to a nonnegative limit as t increases to ∞. Integrationg (6) from t to ∞, we obtain

$$x^{(n-1)}(t)\geqq\int_t^\infty p(u)f(x(u), x(\delta(u)))du.$$

Since $x^{(n-1)}(t)$ is decreasing and $\delta_*(t)\leqq t$, we have

$$x^{(n-1)}(\delta_*(t))\geqq\int_t^\infty p(u)f(x(u), x(\delta(u)))du,\ t\geqq T. \tag{7}$$

Suppose case (I) of Lemma holds. Multiply both sides of (7) by $\delta_*'(t)$, integrate from t to s with $T<t<s$, and then let s tend to ∞ in the resulting inequality. Then we have for $t\geqq T$

$$-x^{(n-2)}(\delta_*(t))\geqq\int_t^\infty[\delta_*(u)-\delta_*(t)]p(u)f(x(u), x(\delta(u)))du.$$

Repeating the above procedure we have

(8) $$(-1)^n x'(\delta_*(t)) \geqq \int_t^\infty \frac{[\delta_*(u)-\delta_*(t)]^{n-2}}{(n-2)!} p(u)f(x(u), x(\delta(u)))du.$$

Let n be even. Then, from (8), we see that $x'(t) \geqq 0$ for $t \geqq T$, i.e., $x(t)$ is nondecreasing for $t \geqq T$. It follows that

(9) $$x'(\delta_*(t)) \geqq \int_t^\infty \frac{[\delta_*(u)-\delta_*(t)]^{n-2}}{(n-2)!} p(u)f(x(u), x(\delta_*(u)))du,$$

since $\delta_*(t) \leqq \delta(t)$ and $f(x, y)$ is nondecreasing in y. Multiplying both sides of (9) by $\delta_*'(t)$ and integrating from T to t, $T<t$, we have

(10) $$\begin{aligned} x(\delta_*(t)) \geqq & \int_T^t \frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!} p(u)f(x(u), x(\delta_*(u)))du \\ & + \frac{[\delta_*(t)-\delta_*(T)]^{n-1}}{(n-1)!} \int_t^\infty p(u)f(x(u), x(\delta_*(u)))du. \end{aligned}$$

If $\alpha>1$, from (10) with the second term on the right side removed, we have

(11) $$[x(\delta_*(t))]^{-\alpha} \leqq \left\{ \int_T^t \frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!} p(u)f(x(u), x(\delta_*(u)))du \right\}^{-\alpha}.$$

Multiplication of both sides of (11) by $\frac{[\delta_*(t)-\delta_*(T)]^{n-1}}{(n-1)!} p(t)f(x(t), x(\delta_*(t)))$ and integration from t_1 to t_2, $T<t_1<t_2$, give

(12) $$\begin{aligned} & \int_{t_1}^{t_2} \frac{[\delta_*(t)-\delta_*(T)]^{n-1}}{(n-1)!} p(t)f(x(t), x(\delta_*(t)))[x(\delta_*(t))]^{-\alpha} dt \\ & \leqq \frac{1}{1-\alpha} \left\{ \int_T^t \frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!} p(u)f(x(u), x(\delta_*(u)))du \right\}^{1-\alpha} \Bigg|_{t_1}^{t_2}. \end{aligned}$$

Since the right side remains finite for all $t_2>t_1$, the integral on the left converges as $t_2 \to \infty$. There are two possible cases: Either $\lim_{t\to\infty} x(t)=c$ (finite) or $\lim_{t\to\infty} x(t)=\infty$. In the former case we can choose a $\tau>T$ such that

$$f(x(t), x(\delta_*(t)))[x(\delta_*(t))]^{-\alpha} \geqq \frac{1}{2} f(c, c)c^{-\alpha} \text{ for } t \geqq \tau.$$

Then from (12) we obtain

$$\begin{aligned} & \int_\tau^\infty [\delta_*(t)-\delta_*(T)]^{n-1} p(t)dt \\ & \leqq \frac{2c^\alpha}{f(c, c)} \int_\tau^\infty [\delta_*(t)-\delta_*(T)]^{n-1} p(t)f(x(t), x(\delta_*(t)))[x(\delta_*(t))]^{-\alpha} dt. \end{aligned}$$

But this is in contradiction to (3). In the latter case, by (iii), there exists a positive constant K such that

$$f(x(t),\ x(\delta_*(t)))[x(\delta_*(t))]^{-\alpha} \geqq K \quad \text{for} \quad t \geqq \tau,$$

provided τ is sufficiently large. Consequently, from (12) we conclude that

$$\int_\tau^\infty [\delta_*(t)-\delta_*(T)]^{n-1} p(t)\,dt < \infty$$

which again contradicts (3).

If $\alpha<1$, from (10) with the second term on the right removed we have

$$(13) \quad [x(\delta_*(t))]^{-\alpha}[\delta_*(t)-\delta_*(T)]^{\alpha(n-1)} \leqq \left\{\int_t^\infty \frac{p(u)f(x(u),\ x(\delta_*(u)))}{(n-1)!}\,du\right\}^{-\alpha}.$$

Multiplying both sides of (13) by $p(t)f(x(t),\ x(\delta_*(t)))/(n-1)!$ and integrating from t_1 to t_2, $T<t_1<t_2$, we obtain

$$(14) \quad \begin{aligned} &\int_{t_1}^{t_2} \frac{[\delta_*(t)-\delta_*(T)]^{\alpha(n-1)}}{(n-1)!} p(t)f(x(t),\ x(\delta_*(t)))[x(\delta_*(t))]^{-\alpha}\,dt \\ &\qquad \leqq -\frac{1}{1-\alpha}\left\{\int_t^\infty \frac{p(u)f(x(u),\ x(\delta_*(u)))}{(n-1)!}\,du\right\}^{1-\alpha}\Bigg|_{t_1}^{t_2}, \end{aligned}$$

from which we can derive the contradiction

$$\int^\infty [\delta_*(t)-\delta_*(T)]^{\alpha(n-1)} p(t)\,dt < \infty$$

exactly as in the case $\alpha>1$.

Let n be odd. Then (8) reduces to

$$(8') \qquad -x'(\delta_*(t)) \geqq \int_t^\infty \frac{[\delta_*(u)-\delta_*(t)]^{n-2}}{(n-2)!} p(u)f(x(u),\ x(\delta(u)))\,du,$$

and this implies that $x(t)$ is nonincreasing for $t \geqq T$. Let $\lim_{t\to\infty} x(t)=L$. We shall prove that $L=0$. Suppose $L>0$. We take T so large that $f(x(t),\ x(\delta(t))) \geqq \frac{1}{2} f(L,L)$ for $t \geqq T$. Integration of (8′) multiplied by $\delta_*'(t)$ from T to t yields

$$\begin{aligned} x(\delta_*(T))-x(\delta_*(t)) &\geqq \int_T^t \frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!} p(u)f(x(u),\ x(\delta(u)))\,du \\ &\quad + \frac{[\delta_*(t)-\delta_*(T)]^{n-1}}{(n-1)!}\int_t^\infty p(u)f(x(u),\ x(\delta(u)))\,du. \end{aligned}$$

Letting $t\to\infty$, we have the following contradiction:

$$\begin{aligned} x(\delta_*(T)) > x(\delta_*(t))-L &\geqq \int_T^\infty \frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!} p(u)f(x(u),\ x(\delta(u)))\,du \\ &\geqq \frac{f(L,L)}{2(n-1)!}\int_T^\infty [\delta_*(u)-\delta_*(T)]^{n-1} p(u)\,du. \end{aligned}$$

Therefore, if n is odd, a nonoscillatory solution of (1) must be strongly monotone.

Suppose now case (II) of Lemma holds. We observe that there exists a $t_0 \geqq T$ such that $x^{(j)}(\delta_*(t))>0$ for $t \geqq t_0$, $j=0, 1, \cdots, n-k-1$. Proceeding as in case (I), we obtain

$$x^{(n-k)}(\delta_*(t)) \geqq \int_t^\infty \frac{[\delta_*(u)-\delta_*(t)]^{k-1}}{(k-1)!} p(u) f(x(u), x(\delta_*(u)))du.$$

Multiplying both sides of the above inequality by $\delta_*'(t)$ and integrating from t_0 to t,

$$x^{(n-k-1)}(\delta_*(t)) \geqq \frac{[\delta_*(t)-\delta_*(t_0)]^k}{k!} \int_t^\infty p(u) f(x(u), x(\delta_*(u)))du.$$

Repeating the above procedure we otain

$$x'(\delta_*(t)) \geqq \frac{[\delta_*(t)-\delta_*(t_0)]^{n-2}}{(n-2)!} \int_t^\infty p(u) f(x(u), x(\delta_*(u)))du,$$

from which we can easily deribve the following inequality analogous to (10):

$$\begin{aligned} x(\delta_*(t)) \geqq & \int_{t_0}^t \frac{[\delta_*(u)-\delta_*(t_0)]^{n-1}}{(n-1)!} p(u) f(x(u), x(\delta_*(u)))du \\ & + \frac{[\delta_*(t)-\delta_*(t_0)]^{n-1}}{(n-1)!} \int_t^\infty p(u) f(x(u), x(\delta_*(u)))du. \end{aligned} \tag{15}$$

The proof now proceeds exactly as in case (I). The proof is therefore complete.

PROOF OF THEOREM 2. Let $x(t)$ be a nonoscillatory solution of (2) which may be assumed positive for large t.

Let case (I) of Lemma hold and let n be even. Then, proceeding as in the proof of Theorem 1, we obtain an inequality corresponding to (10) which yields

$$x(\delta_*(t)) \geqq \int_T^t \frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!} p(u) g(x(\delta_*(u)))du.$$

Since $g(x)$ is nondecreasing,

$$g(x(\delta_*(t)))/ g\left[\int_T^t \frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!} p(u) g(x(\delta_*(u)))du\right] \geqq 1. \tag{16}$$

Following [3], we multiply both sides of (16) by $[\delta_*(t)-\delta_*(T)]^{n-1}p(t)/(n-1)!$, integrate from t_1 to t_2, $T<t_1<t_2$, to obtain

$$\int_{t_1}^{t_2} \frac{[\delta_*(t)-\delta_*(T)]^{n-1}}{(n-1)!} p(t)dt \leqq \int_{x_1}^{x_2} \frac{dx}{g(x)}, \tag{17}$$

where

$$x_i=\int_T^{t_i}\frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!}p(u)g(x(\delta_*(u)))du, \quad i=1, 2.$$

If $x_1\geqq\varepsilon$ for some $t_1\geqq T$, then, in view of condition (iii), (17) gives a contradiction to (4). If $x_1<\varepsilon$ for all $t_1\geqq T$, then

$$\varepsilon>x_1\geqq g(x(\delta_*(T)))\int_T^{t_1}\frac{[\delta_*(u)-\delta_*(T)]^{n-1}}{(n-1)!}p(u)du$$

which again contradicts (4).

If n is odd, then, as in the proof of Theorem 1, we are led to the contradiction: $\int^\infty[\delta_*(t)]^{n-1}p(t)dt<\infty$.

When case (II) of Lemma holds, an inequality corresponding to (15) enables us to preceed entirely as in case (I). This completes the proof.

Proof of Theorem 3. Let $x(t)$ be a nonoscillatory solution of (2) which is positive for large t.

Suppose case (I) of Lemma holds. If n is even, from the inequality

$$x(\delta_*(t))\geqq\frac{[\delta_*(t)-\delta_*(T)]^{n-1}}{(n-1)!}\int_t^\infty p(u)g(x(\delta_*(u)))du$$

which follows from an inequality corresponding to (10), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\frac{[\delta_*(t)-\delta_*(T)]^{\alpha(n-1)}}{(n-1)!}p(t)g(x(\delta_*(t)))[x(\delta_*(t))]^{-\alpha}dt\\ &\qquad\leqq-\frac{1}{1-\alpha}\left\{\int_t^\infty\frac{p(u)g(x(\delta_*(u)))}{(n-1)!}du\right\}^{1-\alpha}\Bigg|_{t_1}^{t_2}\end{aligned}\tag{18}$$

which corresponds to (14), where $t_2>t_1>T$. In view of (iii′), the integral on the left side of (18) exceeds

$$Mg(1)\int_{t_1}^{t_2}\frac{[\delta_*(t)-\delta_*(T)]^{\alpha(n-1)}}{(n-1)!}p(t)dt.$$

But this contradicts (5), since the right side remains bounded as $t_2\to\infty$. Let n be odd and assume the existence of a nonoscillatory solution $x(t)$. If $\lim_{t\to\infty}x(t)=L>0$, then it is not hard to show that $\int^\infty[\delta_*(t)]^{n-1}p(t)dt<\infty$, and a fortiori $\int^\infty[\delta_*(t)]^{\alpha(n-1)}p(t)dt<\infty$, in contradiction to (5).

When case (II) of Lemma occurs, we can derive a contradiction on the basis of an inequality corresponding to (15). The proof is thus complete.

References

[1] H. E. Gollwitzer, On nonlinear oscillation for a second-order delay equation, J. Math. Anal. Appl. **26** (1969), 385–389.
[2] T. Kusano and H. Onose, Oscillation of solutions of nonlinear differential delay equations of arbitrary order, Hiroshima Math. J. **2** (1972), 1–13.
[3] G. H. Ryder and D. V. V. Wend, Oscillation of solutions of certain ordinary differential equations of n th order, Proc. Amer. Math. Soc. **25** (1970), 463–469.
[4] C. C. Travis, Oscillation theorems for second-order differential equations with functional arguments, Proc. Amer. Math. Soc. **31** (1972), 199–202.

Department of Mathematics,
Faculty of Science,
Hiroshima University
and
Department of Mathematics,
College of General Education,
Ibaraki University