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*A Note on the Space of Dirichlet-Finite Solutions of $\Delta u = Pu$ on a Riemann Surface**

Joel L. SCHIFF

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1. The space $PD(R)$ was initially investigated by Royden [10] and Nakai [7], with more recent contributions also by Nakai [8, 9] and Glasner-Nakai [2]. It was shown in [2] that the set Δ_P of P-energy nondensity points determines the space $PD(R)$ in some sense. In this note we give further evidence along these lines.

2. Let R be an open Riemann surface and $P \geq 0$, $P \not\equiv 0$ a density on R. Denote by $PD(R)$ the space of Dirichlet-finite C^2 solutions on R of the equation $\Delta u = Pu$. Let $\tilde{M}(R)$ be the class of all Dirichlet-finite Tonelli functions on R, and $\tilde{M}_\Delta(R)$ the set of functions $f \in \tilde{M}(R)$ such that $f = 0$ on the Royden harmonic boundary Δ, of the Royden compactification R^*. Since $PD(R) \subset \tilde{M}(R)$, the orthogonal decomposition of $\tilde{M}(R)$ (cf. eg. Sario-Nakai [11]) yields a vector space isomorphism T: $PD(R) \to HD(R)$ which preserves the sup norm. The distribution of $PD(R)\,|\,\Delta$ in $HD(R)\,|\,\Delta$ is still an important subject for investigation (cf. Singer [12]).

We shall make essential use of the operator T_Ω given by

$$T_\Omega \phi = \frac{1}{2\pi} \int_\Omega G_\Omega(\cdot, z)\, \phi(z) P(z)\, dv(z),$$

where Ω is an open subset of R having a smooth relative boundary and $G_\Omega(\cdot, z)$ is the harmonic Green's function on Ω, $dv(z) = dx\,dy$. It is known that the Dirichlet integral of $T_\Omega u$ for $u \in PD(R)$ is given by

$$D_\Omega(T_\Omega u) = \frac{1}{2\pi} \int_{\Omega \times \Omega} G_\Omega(z, w) u(z) u(w) P(z) P(w)\, dv(z)\, dv(w).$$

For a comprehensive discussion of the operator T_Ω see Nakai [9]. A *P-energy nondensity point* z^* is a point of R^* with the property that there exists an open neighborhood U^* of z^* in R^* such that

$$\text{(1)} \qquad \int_{U \times U} G_U(z, w) P(z) P(w)\, dv(z)\, dv(w) < \infty,$$

* Similar results have been obtained independently by Professor Wellington H. Ow, "*PD*-minimal solutions of $\Delta u = Pu$ on open Riemann surfaces", to appear in the Proc. Amer. Math. Soc.

where $U=U^*\cap R$.

3. We observe that the following maximum principle holds for PD-functions (Glasner-Nakai [2]):

THEOREM 1. *If* $u \in PD(R)$, *then* $\sup_R |u| = \sup_{\Delta_P} |u|$. *Moreover*, $u|\Delta_P \geq 0$ *implies* $u \geq 0$ *on* R.

PROOF. For $u \in PD(R) \subset \tilde{M}(R)$, we have $u = Tu + g$, where $Tu \in HD(R)$ and $g|\Delta = 0$. Since $PD|\Delta - \Delta_P = 0$, the HD-maximum principle implies

$$\sup_R |u| = \sup_R |Tu| = \sup_\Delta |Tu| = \sup_\Delta |u| = \sup_{\Delta_P} |u|.$$

Furthermore, from Glasner-Katz [1], $u \geq 0$ on Δ gives $u \geq 0$ on R for $u \in PD(R)$.

COROLLARY. *If* $p \in \Delta_P$ *is isolated, then for any* $u \in PD(R)$, $u(p) \neq \pm\infty$.

PROOF. Since $u \in PD(R)$ has the decomposition $u = u_1 - u_2$, $u_1, u_2 \in PD(R)$, $u_1, u_2 \geq 0$ (Nakai [7]), it suffices to consider $u \geq 0$. Suppose $u(p) = \infty$. From the proof of the next theorem, there exists a function $v \in PBD(R)$ such that $v(p) = 1$, $v|\Delta_P - \{p\} = 0$. Then for each n, $u - nv \geq 0$ on Δ_P, and by the maximum principle, $u - nv \geq 0$ on R. This leads to the contradiction $u(z) = \infty$, $z \in R$.

As a result we have the following characterization of $PD(R)$, which is analogous to that for HD-functions (Kusunoki-Mori [3]) and for PE-functions (Kwon-Sario-Schiff [4]).

THEOREM 2. $\dim PD(R) = n$ *if and only if* Δ_P *consists of exactly* n *points.*

PROOF. Assume $\Delta_P = \{z_1^*, z_2^*, \cdots, z_n^*\}$. We can find neighborhoods U_i^* of z_i^* with smooth relative boundary such that $U_i^* \cap U_j^* = \phi$ for $i \neq j$ and (1) is valid for U_i, $i = 1, 2, \cdots, n$. Construct a function $h_i \in HBD(U_i)$ such that $h_i|\partial U_i = 0$, $0 \leq h_i \leq 1$ on U_i, and $h_i\,(z_i^*) = 1$. Then the Fredholm equation $(I - T_{U_i})u_i = h_i$ has a solution u_i on U_i such that $u_i \in PBD(U_i)$, $u_i|\partial U_i = 0$, $0 \leq u_i \leq h_i \leq 1$ on $\bar{U}_i$, and $u_i(z_i^*) = 1$. Extending u_i such that $u_i|R - U_i = 0$, the extended function, again denoted by u_i, is a bounded Dirichlet-finite subsolution, and $u_i|\Delta_P - U_i^* = 0$.

Let $\{\Omega_n\}_{n=1}^\infty$ be a regular exhaustion of R, and let $P_{u_i}^{\Omega_n}$ be a solution on Ω_n such that $P_{u_i}^{\Omega_n}|\partial\Omega_n = u_i|\partial\Omega_n$. Then $P_{u_i}^{\Omega_n} \geq 0$ and the function

$$w_i^n = \begin{cases} P_{u_i}^{\Omega_n} - u_i & \text{on } \Omega_n \\ 0 & \text{on } R - \Omega_n, \end{cases}$$

by the weak Dirichlet principle satisfies $D(w_i^n) \leq 4D_{U_i}(u_i) < \infty$. Therefore,

$$w_i = \lim_{n\to\infty} w_i^n = \lambda_P u_i - u_i$$

exists, where $\lambda_P u_i$ is the *canonical extension* of u_i (cf. Nakai [9]). Since $w_i^n \in \tilde{M}_\Delta(R)$, the potential subalgebra (cf. eg. Sario-Nakai [11]), $w_i \in \tilde{M}_\Delta(R)$, i.e. $v_i = \lambda_P u_i = u_i$ on Δ, and $v_i \in PBD(R)$, $i=1, 2, \cdots, n$. Hence $v_i(z_j^*) = \delta_{ij}$. At this stage it is not difficult to see that the functions $\{v_1, v_2, \cdots, v_n\}$ form a basis for $PD(R)$. Conversely, if $\dim PD(R) = n$, similarly as in the case of $HD(R)$ and $PE(R)$, one shows that Δ_P consists of exactly n points.

As an immediate consequence we have:

COROLLARY. *If Δ_P consists of n points*, $\dim PBD(R) = \dim PD(R) = n$.

A positive function $u \in PD(R)$ is a *PD-minimal* function if for $v \in PD(R)$, $0 \leq v \leq u$, there exists a constant c_v such that $v = c_v u$. Our next result also has an analog for *HD*-minimal functions (Nakai [6]) and for *PE*-minimal functions (Kwon-Sario-Schiff [5]).

THEOREM 3. *If u is a PD-minimal function, then there exists an isolated point $p \in \Delta_P$ such that $0 < u(p) < \infty$ and $u \mid \Delta_P - \{p\} = 0$. Conversely, if $p \in \Delta_P$ is isolated in Δ_P, then there exists a PD-minimal function u such that $u(p) = 1$ and $u \mid \Delta_P - \{p\} = 0$.*

PROOF. Let u be a *PD*-minimal function on R. Then $\Delta_P \neq \phi$ and $u \geq 0$ on Δ_P. Thus there is a point $p \in \Delta_P$ such that $u(p) > 0$. Assume there exists another point $q \in \Delta_P$ such that $u(q) > 0$. Choose disjoint neighborhoods U_p, U_q such that $u > \delta > 0$ on U_p, and construct a function $h \in HBD(U_p)$ with $h \mid \partial U_p = 0$, $0 \leq h \leq \delta$ on U_p, and $h(p) = \delta$. As before, there exists a function $w \in PBD(U_p)$ such that $0 \leq w \leq h \leq \delta$ on $\bar{U}_p$, and $w(p) = \delta$. Extending w to $w \mid R - U_p = 0$, the canonical extension $v = \lambda_p w$ belongs to $PBD(R)$, with $v \mid \Delta = w \mid \Delta$. Therefore $v(q) = 0$. However, $0 \leq v \leq \delta < u$ on $\bar{U}_p$, whence $0 \leq v \leq u$ on Δ_P, and by the maximum principle $0 \leq v \leq u$ on R. Thus there exists a constant c_v with $v = c_v u$, and $v(q) > 0$, a contradiction. Then $u \mid \Delta_P - \{p\} = 0$ implies p is isolated.

On the other hand, suppose p is isolated in Δ_P. As above, there exists a function $u \in PBD(R)$, $0 \leq u \leq 1$ on R, $u(p) = 1$, and $u \mid \Delta_P - \{p\} = 0$. If $v \in PD(R)$ is a function satisfying $0 \leq v \leq u$ on R, then $v \mid \Delta_P - \{p\} = 0$ and $0 \leq v(p) \leq 1$. Thus there is a constant c_v such that $v = c_v u$ on Δ_P, with the equality holding on R by the maximum principle. This proves the theorem.

Denote by $\mathcal{U}_{PD}$ the class of Riemann surfaces on which there exists a *PD*-minimal function.

COROLLARY. *$R \in \mathcal{U}_{PD}$ if and only if there exists an isolated point of Δ_P.*

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Department of Mathematics,
University of Auckland,
Auckland, New Zealand