HIROSHIMA MATH. J.
3 (1973), 121-151

# *Harmonic Functions on Real Hyperbolic Spaces*

Katsuhiro MINEMURA

(Received January 25, 1973)

## Introduction

The present paper deals with the problem of an integral representation of harmonic functions of the laplacian on real hyperbolic spaces.

There are some types of theorems on Dirichlet problem, which show that certain classes of harmonic functions on the unit disc are given by the Poisson integral (cf. [1]). However, in order to obtain arbitrary harmonic functions we should consider the ,,Poisson transform of hyperfunctions", as S. Helgason showed in [6]. He proved there that any eigenfunction of the laplacian on the unit disc (with respect to the Poincaré metric) is given as the Poisson transform of a hyperfunction on the unit circle. In the case of Euclidean space [3], the situation is different. In this case we should consider a space which properly contains the hyperfunctions on the unit sphere.

It is our object to prove that, in the case of real hyperbolic spaces, the Poisson transform is an isomorphism of the space of hyperfunctions on the boundary onto the space of harmonic functions of the laplacian (Theorem 4.6 in § 4).

This paper consists of four sections. In § 1, we characterize the hyperfunctions on a compact real analytic riemannian manifold by their Fourier coefficients with respect to the eigenfunctions of the laplacian. In § 2, we show that any harmonic function on a symmetric space of rank one can be expanded in an absolutely convergent series of $K$-finite harmonic functions. In § 3 we restrict our argument to the case of real hyperbolic spaces and determine the $K$-finite harmonic functions by solving differential equations. In the final section we define the Poisson transform of hyperfunctions, and making use of the characterization of hyperfunctions, we prove Theorem 4.6.

## § 1. Hyperfunctions on a compact riemannian manifold.

In this section we characterize the hyperfunctions on a compact real analytic riemannian manifold by their Fourier coefficients with respect to the eigenfunctions of the laplacian on the manifold.

Let $B$ be a compact real analytic manifold with a riemannian metric $g$, $\omega$ the laplacian corresponding to $g$ and $L^2(B)$ the space of square integrable functions on $B$ with respect to the measure induced by $g$. We denote the unitary inner product and the norm of $L^2(B)$ by ( , ) and $\| \|$ respectively. We denote by $\mathscr{A}(B)$ the space of analytic functions on $B$ equipped with the usual topology.

As is well-known, the eigenvalues of $\omega$ are non-negative and countable, and the space of eigenfunctions of each eigenvalue is finite-dimensional. Let $\boldsymbol{N}^0$ be the set of non-negative integers. We denote the eigenvalues of $\omega$ by $\lambda_n$ ($n \in \boldsymbol{N}^0$) and order them so that $\lambda_n < \lambda_m$ if $n < m$. Let $E_n = E_{\lambda_n}$ be the space of eigenfunctions of $\omega$ with eigenvalue $\lambda_n$ and $d(n) = d(\lambda_n)$ be the dimension of $E_n$. Then as an orthonormal base of $E_n$, we can choose analytic functions $\phi_i^n$ on $B$ ($n \in \boldsymbol{N}^0$, $1 \leqq i \leqq d(n)$), and

$$\{\phi_i^n \mid n \in \boldsymbol{N}^0,\ 1 \leqq i \leqq d(n)\}$$

makes a complete orthonormal base of $L^2(B)$.

LEMMA 1.1. *For $s \in \boldsymbol{C}$ such that $\mathscr{R}e(s) > \frac{1}{2}\dim B$, the series*

$$\sum_{n \in \boldsymbol{N}^0} d(n)(1+\lambda_n)^{-s}$$

*is convergent and holomorphic in s.*

For a proof, see [11].

LEMMA 1.2. *For $t > 0$, the series*

$$\sum_{n \in \boldsymbol{N}^0} d(n)\exp(-t\lambda_n^{1/2})$$

*is convergent.*

PROOF. Fix an arbitrary $t > 0$ and put $s = \dim B + 1$. Since there exists an $M \in \boldsymbol{R}$ such that

$$(1+\lambda_n)^s \exp(-t\lambda_n^{1/2}) \leqq M$$

for $n \in \boldsymbol{N}^0$, we have

$$\sum_{n \in \boldsymbol{N}^0} d(n)\exp(-t\lambda_n^{1/2})$$
$$\leqq M \sum_{n \in \boldsymbol{N}^0} d(n)(1+\lambda_n)^{-s},$$

which is convergent by Lemma 1.1. This finishes the proof.

Let $C^\infty(B)$ denote the space of $C^\infty$-functions on $B$. It is well-known (cf. [11]) that any $\phi \in C^\infty(B)$ can be expanded in an absolutely and uniformly convergent Fourier series

$$\phi = \sum_{n \in \boldsymbol{N}^0} \sum_{i=1}^{d(n)} (\phi, \phi_i^n)\, \phi_i^n.$$

In the following we write this as

$$\phi=\sum_{n\in N^0}\sum_{i=1}^{d(n)} a_i^n\phi_i^n$$

for $\phi\in C^\infty(B)$, where $a_i^n=(\phi, \phi_i^n)$. Since

$$\omega\phi=\sum_{n\in N^0}\lambda_n\sum_{i=1}^{d(n)} a_i^n\phi_i^n,$$

the series

$$\sum_{n\in N^0}\lambda_n^{1/2}\sum_{i=1}^{d(n)} a_i^n\phi_i^n$$

converges also absolutely and uniformly on $B$ and defines an element of $C^\infty(B)$. We denote it by $\omega^{1/2}\phi$. It is easy to show

LEMMA 1.3. *Let $\phi$ and $\psi\in C^\infty(B)$. Then*

$$(\omega^{1/2}\phi, \psi)=(\phi, \omega^{1/2}\psi).$$

Analogously for any $t\geqq 0$ we can define a mapping $\exp(-t\omega^{1/2})$ by

$$\exp(-t\omega^{1/2})\phi=\sum_{n\in N^0}\exp(-t\lambda_n^{1/2})\sum_{i=1}^{d(n)} a_i^n\phi_i^n$$

for $\phi\in C^\infty(B)$. We have

LEMMA 1.4. *Let $\phi, \psi\in C^\infty(B)$ and $t\geqq 0$. Then*

$$(\exp(-t\omega^{1/2})\phi, \psi)=(\phi, \exp(-t\omega^{1/2})\psi).$$

Now we define two systems of semi-norms $|\ |_h$ and $\|\ \|_h$ $(h>0)$ on $C^\infty(B)$. They are defined by

$$|\phi|_h=\sup_{m\in N^0}\frac{1}{(2m)!h^{u}}\|\omega^m\phi\|,$$

$$\|\phi\|_h=\sup_{m\in N^0}\frac{1}{m!h^m}\|\omega^{m/2}\phi\|.$$

For $h>0$, we put

$$\mathscr{A}_{0,h}(B)=\{\phi\in C^\infty(B)|\ |\phi|_h<\infty\},$$

$$\mathscr{A}_h(B)=\{\phi\in C^\infty(B)|\ \|\phi\|_h<\infty\}.$$

It is easy to check that $|\ |_h$ and $\|\ \|_h$ are norms on $\mathscr{A}_{0,h}(B)$ and $\mathscr{A}_h(B)$ respectively. About these semi-norms, we have

LEMMA 1.5. *Let* $\phi\in C^{\infty}(B)$. *Then*

$$(1) \qquad |\phi|_h \leqq \|\phi\|_{\sqrt{h}},$$

$$(2) \qquad \|\phi\|_h \leqq \sqrt{2}\,|\phi|_{h^2}.$$

PROOF. (1) For $m\in N^0$, the equality

$$\frac{1}{(2m)!h^m}\|\omega^m\phi\|=\frac{1}{(2m)!(h^{1/2})^{2m}}\|(\omega^{1/2})^{2m}\phi\|$$

means that $|\phi|_h\leqq\|\phi\|_{\sqrt{h}}$.
(2) When $m=2l$ ($l\in N^0$), we have

$$\frac{1}{m!h^m}\|\omega^{m/2}\phi\|$$

$$=\frac{1}{(2l)!(h^2)^l}\|\omega^l\phi\|,$$

which means that

$$\frac{1}{m!h^m}\|\omega^{m/2}\phi\|\leqq|\phi|_{h^2}. \tag{1.1}$$

When $m=2l+1$ ($l\in N^0$),

$$\left(\frac{1}{m!h^m}\|\omega^{m/2}\phi\|\right)^2$$

$$=\frac{((\omega^{1/2})^{2l+1}\phi,\ (\omega^{1/2})^{2l+1}\phi)}{\{(2l+1)!\}^2\ h^{2(2l+1)}}\ .$$

By Lemma 1.3 and Schwarz's inequality, we have

$$\left(\frac{1}{m!h^m}\|\omega^{m/2}\phi\|\right)^2$$

$$=\frac{(\omega^{l+1}\phi,\ \omega^l\phi)}{\{(2l+1)!\}^2\ h^{2(2l+1)}}$$

$$\leqq\frac{\|\omega^{l+1}\phi\|\ \|\omega^l\phi\|}{\{(2l+1)!\}^2\ h^{2(2l+1)}}$$

$$=\frac{\|\omega^{l+1}\phi\|}{\{2(l+1)\}!(h^2)^{l+1}}\cdot\frac{\|\omega^l\phi\|}{(2l)!(h^2)^l}\cdot\frac{\{2(l+1)\}!}{(2l+1)!}\cdot\frac{(2l)!}{(2l+1)!}$$

$$\leqq\frac{2(l+1)}{2l+1}(|\phi|_{h^2})^2\ .$$

Since $0<\dfrac{2(l+1)}{2l+1}\leqq 2$ for $l\in N^0$, we have

$$\frac{1}{m!h^m}\|\omega^{m/2}\phi\| \leqq \sqrt{2}|\phi|_{h^2}. \tag{1.2}$$

Therefore we have from (1.1) and (1.2),

$$\frac{1}{m!h^m}\|\omega^{m/2}\phi\| \leqq \sqrt{2}|\phi|_{h^2}$$

for $m \in N^0$. Taking supremum we obtain the required inequality

$$\|\phi\|_h \leqq \sqrt{2}|\phi|_{h^2},$$

which completes the proof.

The above lemma implies that the inductive limit of $\mathscr{A}_{0,h}(B)$, denoted by $\varinjlim_{h\to\infty} \mathscr{A}_{0,h}(B)$, coincides with that of $\mathscr{A}_h(B)$, denoted by $\varinjlim_{h\to\infty} \mathscr{A}_h(B)$, as linear topological spaces. On the other hand, $\mathscr{A}(B)$ is equal to $\varinjlim_{h\to\infty} \mathscr{A}_{0,h}(B)$ (see [10]). Therefore we have the following

PROPOSITION 1.6. $\mathscr{A}(B) = \varinjlim_{h\to\infty} \mathscr{A}_h(B)$.

We define a subset $\mathscr{F}_a = \mathscr{F}_a(B)$ of $C^N = \prod_{n\in N^0} C^{d(n)}$ by

$$\mathscr{F}_a = \{(a_i^n)_{1\leqq i\leqq d(n)}^{n\in N^0} \mid a_i^n \in C, \sum_{n\in N^0}\sum_{i=1}^{d(n)} |a_i^n| \exp(t\lambda_n^{1/2}) < \infty \text{ for some } t>0\}$$

and a mapping $\Phi$ of $\mathscr{A}(B)$ into $C^N$ by

$$\Phi(\phi) = (a_i^n),$$

where $\phi \in \mathscr{A}(B)$ and $a_i^n = (\phi, \phi_i^n)$. As is easily seen, $\mathscr{F}_a$ is a vector subspace of $C^N$ and $\Phi$ is a $C$-linear mapping of $\mathscr{A}(B)$ into $C^N$. On this mapping we have

PROPOSITION 1.7. *$\Phi$ is an isomorphism of $\mathscr{A}(B)$ onto $\mathscr{F}_a(B)$. Let $(a_i^n) \in \mathscr{F}_a$. Then the inverse mapping $\Phi^{-1}$ of $\Phi$ is given by*

$$\Phi^{-1}(a_i^n) = \sum_{n\in N^0}\sum_{i=1}^{d(n)} a_i^n \phi_i^n$$

*which converges in $\mathscr{A}(B)$ and defines a unique element of $\mathscr{A}(B)$.*

PROOF. Let $\phi \in \mathscr{A}(B)$. Then by Proposition 1.6, there exists an $h_0 > 0$ such that

$$\sup_{m\in N^0} \frac{1}{m!h_0^m}\|\omega^{m/2}\phi\| = \|\phi\|_{h_0} < \infty.$$

On the other hand, since $\phi$ has an absolutely and uniformly convergent Fourier expansion

$$\phi = \sum_{n \in N^0} \sum_{i=1}^{d(n)} a_i^n \phi_i^n,$$

we have

$$\|\phi\|_{h_0} = \sup_{m \in N^0} \frac{1}{m! h_0^m} \| \sum_{n \in N^0} \sum_{i=1}^{d(n)} a_i^n \lambda_n^{m/2} \phi_i^n \|$$

$$\geqq \frac{1}{m! h_0^m} \| \sum_{n \in N^0} \sum_{i=1}^{d(n)} a_i^n \lambda_n^{m/2} \phi_i^n \|$$

$$\geqq \frac{1}{m! h_0^m} \lambda_n^{m/2} |a_i^n|$$

for $n, m \in N^0$. Multiplying $2^{-m}$ and summing the above inequality with respect to $m$, we have

$$|a_i^n| \exp\left(\frac{1}{2h_0} \lambda_n^{1/2}\right) \leqq 2\|\phi\|_{h_0} \tag{1.3}$$

for $n \in N^0$. Putting $t_0 = \frac{1}{4h_0}$, we obtain

$$\sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n| \exp(t_0 \lambda_n^{1/2})$$

$$\leqq 2\|\phi\|_{h_0} \sum_{n \in N^0} d(n) \exp\left(\frac{1}{4h_0} \lambda_n^{1/2}\right) \exp\left(-\frac{1}{2h_0} \lambda_n^{1/2}\right)$$

$$= 2\|\phi\|_{h_0} \sum_{n \in N^0} d(n) \exp\left(-\frac{1}{4h_0} \lambda_n^{1/2}\right),$$

which is finite by Lemma 1.2. This implies that $\Phi(\phi) \in \mathscr{F}_a$.

Next, let $(a_i^n) \in \mathscr{F}_a$. By the definition of $\mathscr{F}_a$, there exists a $t_0 > 0$ such that

$$\sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n| \exp(t_0^{-1} \lambda_n^{1/2}) < \infty. \tag{1.4}$$

On the other hand, for $n \in N^0$, $1 \leqq i \leqq d(n)$ and $h > 0$,

$$\|\phi_i^n\|_h = \sup_{m \in N^0} \frac{1}{m! h^m} \lambda_n^{m/2}$$

$$\leqq \sum_{m \in N^0} \frac{1}{m! h^m} \lambda_n^{m/2}$$

$$= \exp\left(\frac{1}{h} \lambda_n^{1/2}\right),$$

which means that $\phi_i^n \in \mathscr{A}_h(B)$. Therefore putting $h = t_0$ we obtain

$$\left\| \sum_{n=N}^{N+l} \sum_{i=1}^{d(n)} a_i^n \phi_i^n \right\|_{t_0} \leqq \sum_{n=N}^{N+l} \sum_{i=1}^{d(n)} |a_i^n| \, \|\phi_i^n\|_{t_0}$$

$$\leqq \sum_{n=N}^{N+l} \sum_{i=1}^{d(n)} |a_i^n| \exp(t_0^{-1} \lambda_n^{1/2}),$$

which implies by (1.4) that the sequence

$$\left( \sum_{n=0}^{N} \sum_{i=1}^{d(n)} a_i^n \phi_i^n \right)_{N \geqq 0}$$

is a Cauchy sequence in the Banach space $\mathscr{A}_{t_0}(B)$. Therefore there exists a unique element $\phi$ in $\mathscr{A}(B)$ such that

$$\sum_{n \in N^0} \sum_{i=1}^{d(n)} a_i^n \phi_i^n$$

converges to $\phi$ in $\mathscr{A}(B)$. Since the above series also converges absolutely and uniformly on $B$, we conclude immediately that $\Phi(\phi) = (a_i^n)$. This shows the surjectivity of $\Phi$.

Finally we prove the injectivity of $\Phi$. Assume $\Phi(\phi) = 0$ for $\phi \in \mathscr{A}(B)$. Then $(\phi, \phi_i^n) = 0$ for $n \in N^0$ and $1 \leqq i \leqq d(n)$. On the other hand

$$\phi = \sum_{n \in N^0} \sum_{i=1}^{d(n)} (\phi, \phi_i^n) \phi_i^n,$$

which means $\phi = 0$. This completes the proof.

In the proof of the above proposition we have proved the following

Corollary 1. *For $\phi \in \mathscr{A}(B)$, the Fourier series*

$$\sum_{n \in N^0} \sum_{i=1}^{d(n)} (\phi, \phi_i^n) \phi_i^n$$

*converges to $\phi$ in $\mathscr{A}(B)$.*

Corollary 2. *Let $h_0 > 0$ and $1/(2h_0) > t \geqq 0$. Then for $\phi \in \mathscr{A}_{h_0}(B)$, the series*

$$\sum_{n \in N^0} \exp(\lambda_n^{1/2}) \sum_{i=1}^{d(n)} (\phi, \phi_i^n) \phi_i^n$$

*converges in $\mathscr{A}(B)$ and defines an element of $\mathscr{A}(B)$, which we denote by $\exp(t\omega^{1/2})\,\phi$. Furthermore we have*

$$\|\exp(t\omega^{1/2})\phi\| \leqq 2\|\phi\|_{h_0}.$$

Proof. From (1.3) in the proof of Proposition 1.7,

$$|(\phi, \phi_i^n)| \leqq 2\|\phi\|_{h_0} \exp\left(-\frac{1}{2h_0} \lambda_n^{1/2}\right)$$

for $n\in\boldsymbol{N}^0$ and $\phi\in\mathscr{A}_{h_0}(B)$. Putting $t_0=1/(4h_0)-t/2$, we have

$$\sum_{n\in\boldsymbol{N}^0}\sum_{i=1}^{d(n)}|(\phi,\ \phi_i^n)\exp(t\lambda_n^{1/2})|\exp(t_0\lambda_n^{1/2})$$

$$\leqq\sum_{n\in\boldsymbol{N}^0}\sum_{i=1}^{d(n)}\left(2\|\phi\|_{h_0}\exp\left(-\frac{1}{2h_0}\lambda_n^{1/2}\right)\right)\exp((t+t_0)\lambda_n^{1/2})$$

$$=2\|\phi\|_{h_0}\sum_{n\in\boldsymbol{N}^0}d(n)\exp\left(\left(t+t_0-\frac{1}{2h_0}\right)\lambda_n^{1/2}\right).$$

Since $t+t_0-1/2h_0=-t_0<0$, it follows from Lemma 1.2 that the sequence $((\phi,\ \phi_i^n)\exp(t\lambda_n^{1/2}))\in\mathscr{F}_a$. By Proposition 1.7, we conclude that the series

$$\sum_{n\in\boldsymbol{N}^0}\sum_{i=1}^{d(n)}(\phi,\ \phi_i^n)\exp(t\lambda_n^{1/2})\phi_n$$

converges in $\mathscr{A}(B)$ and defines an element $\exp(t\omega^{1/2})\phi$ of $\mathscr{A}(B)$ since $\mathscr{A}(B)$ is complete. Therefore

$$\|\exp(t\omega^{1/2})\phi\|\leqq\sum_{m\in\boldsymbol{N}^0}\frac{t^m}{m!}\|\omega^{m/2}\phi\|$$

$$=\sum_{m=0}^{\infty}\frac{(th_0)^m}{m!h_0^m}\|\omega^{m/2}\phi\|$$

$$\leqq\left(\sup_{m\in\boldsymbol{N}^0}\frac{1}{m!h_0^m}\|\omega^{m/2}\phi\|\right)\left(\sum_{m\in\boldsymbol{N}^0}(th_0)^m\right)$$

$$\leqq\frac{1}{1-th_0}\|\phi\|_{h_0}$$

$$<2\|\phi\|_{h_0},$$

which completes the proof.

We denote by $\mathscr{B}=\mathscr{B}(B)$ the space of continuous linear functionals of $\mathscr{A}(B)$ into $\boldsymbol{C}$. It is known [12] that on a compact real analytic manifold, $\mathscr{B}$ coincides with the space of Sato's hyperfunctions. Henceforth we call the elements of $\mathscr{B}$ the hyperfunctions on $B$.

We define a subset $\mathscr{F}_b=\mathscr{F}_b(B)$ of $\boldsymbol{C}^{\boldsymbol{N}}$ by

$$\mathscr{F}_b=\{(a_i^n)_{1\leqq i\leqq d(n)}^{n\in\boldsymbol{N}^0}|a_i^n\in\boldsymbol{C},\ \sum_{n\in\boldsymbol{N}^0}\sum_{i=1}^{d(n)}|a_i^n|\exp(-t\lambda_n^{1/2})<\infty\ \text{ for any } t>0\}$$

and a mapping $\Psi$ of $\mathscr{B}(B)$ into $\boldsymbol{C}^{\boldsymbol{N}}$ by

$$\Psi(T)=(a_i^n),$$

where $T\in\mathscr{B}(B)$ and $a_i^n=T(\bar{\phi}_i^n)$, $\bar{\phi}_i^n$ denoting the complex conjugate of $\phi_i^n$. Then

$\mathscr{F}_b$ is a vector subspace of $\boldsymbol{C}^{\boldsymbol{N}}$ and $\Psi$ is a $\boldsymbol{C}$-linear mapping of $\mathscr{B}(B)$ into $\boldsymbol{C}^{\boldsymbol{N}}$. On this mapping we have the following

THEOREM 1.8. *$\Psi$ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{F}_b(B)$. Let $(a_i^n)\in \mathscr{F}_b(B)$. Then the invesrse mapping $\Psi^{-1}$ of $\Psi$ is given by*

$$\Psi^{-1}(a_i^n)(\phi)=\sum_{n\in \boldsymbol{N}^0}\sum_{i=1}^{d(n)} a_i^n(\phi_i^n, \bar{\phi})$$

*which converges absolutely and defines an element $\Psi^{-1}(a_i^n)$ of $\mathscr{B}(B)$.*

PROOF. At first we prove that the image of $T\in\mathscr{B}(B)$ by $\Psi$ lies in $\mathscr{F}_b$. It is enough to show that

$$\sum_{n\in \boldsymbol{N}^0}\sum_{i=1}^{d(n)} |a_i^n|\exp(-t\lambda_n^{1/2})<\infty$$

for any $t>0$, where $a_i^n=T(\bar{\phi}_i^n)$. Take an $h_0>0$ such that $t>1/h_0$. As is shown in Proposition 1.7, $\bar{\phi}_i^n\in\mathscr{A}_{h_0}(B)$. Since $T$ is continuous on $\mathscr{A}_{h_0}(B)$, there exists a constant $c$ such that

$$\begin{aligned}|a_i^n| &= |T(\bar{\phi}_i^n)|\\ &\leqq c\ \|\bar{\phi}_i^n\|_{h_0}\\ &= c\sup_{m\in \boldsymbol{N}^0}\frac{1}{m!h_0^m}\|\omega^{m/2}\bar{\phi}_i^n\|\\ &= c\sup_{m\in \boldsymbol{N}^0}\frac{\lambda_n^{m/2}}{m!h_0^m}\\ &\leqq c\exp\left(\frac{1}{h_0}\lambda_n^{1/2}\right)\end{aligned}$$

for $n\in\boldsymbol{N}^0$, $1\leqq i\leqq d(n)$. Hence

$$\begin{aligned}&\sum_{i=1}^{d(l)}|a_i^n|\exp(-t\lambda_n^{1/2})\\ &\leqq c\ d(n)\exp\left(\frac{1}{h_0}\lambda_n^{1/2}\right)\exp(-t\lambda_n^{1/2})\\ &= c\ d(n)\exp\left(\left(\frac{1}{h_0}-t\right)\lambda_n^{1/2}\right).\end{aligned}$$

Since $1/h_0-t<0$, we have by Lemma 1.2 that

$$\begin{aligned}&\sum_{n\in \boldsymbol{N}^0}\sum_{i=1}^{d(n)}|a_i^n|\exp(-t\lambda_n^{1/2})\\ &\leqq c\sum_{n\in \boldsymbol{N}^0} d(n)\exp\left(\left(\frac{1}{h_0}-t\right)\lambda_n^{1/2}\right)\\ &<\infty,\end{aligned}$$

which proves that $\Psi(T)\in\mathscr{F}_b$.

Let $(a_i^n)\in\mathscr{F}_b$ and take an arbitrary $h>0$. Then by Corollary 2 to Proposition 1.7, $\exp\left(\frac{1}{4h}\omega^{1/2}\right)\phi\in\mathscr{A}(B)$ for $\phi\in\mathscr{A}_h(B)$. From Lemma 1.4, we have

$$\begin{aligned}(\phi_i^n, \bar{\phi})&=\left(\phi_i^n, \exp\left(-\frac{1}{4h}\omega^{1/2}\right)\exp\left(\frac{1}{4h}\omega^{1/2}\right)\bar{\phi}\right)\\&=\left(\exp\left(-\frac{1}{4h}\omega^{1/2}\right)\phi_i^n, \exp\left(\frac{1}{4h}\omega^{1/2}\right)\bar{\phi}\right)\\&=\left(\exp\left(-\frac{1}{4h}\omega^{1/2}\right)\phi_i^n, \left(\exp\left(\frac{1}{4h}\omega^{1/2}\right)\phi\right)^{-}\right).\end{aligned}$$

Therefore by Corollary 2 to Proposition 1.7, we have

$$\begin{aligned}&\sum_{n\in N^0}\sum_{i=1}^{d(n)}|a_i^n|\,|(\phi_i^n, \bar{\phi})|\\&=\sum_{n\in N^0}\sum_{i=1}^{d(n)}|a_i^n|\,\left|\left(\exp\left(-\frac{1}{4h}\omega^{1/2}\right)\phi_i^n, \left(\exp\left(\frac{1}{4h}\omega^{1/2}\right)\phi\right)^{-}\right)\right|\\&\leqq\sum_{n\in N^0}\sum_{i=1}^{d(n)}|a_i^n|\,\left\|\exp\left(-\frac{1}{4h}\omega^{1/2}\right)\phi_i^n\right\|\,\left\|\exp\left(\frac{1}{4h}\omega^{1/2}\right)\phi\right\|\\&\leqq 2\|\phi\|_h\sum_{n\in N^0}\sum_{i=1}^{d(n)}|a_i^n|\exp\left(-\frac{1}{4h}\lambda_n^{1/2}\right)\qquad(1.5)\\&<\infty,\end{aligned}$$

since $(a_i^n)\in\mathscr{F}_b$. Therefore the series

$$\sum_{n\in N^0}\sum_{i=1}^{d(n)}a_i^n(\phi_i^n, \bar{\phi})$$

converges aboslutely. We put

$$T(\phi)=\sum_{n\in N^0}\sum_{i=1}^{d(n)}a_i^n(\phi_i^n, \bar{\phi}).$$

It is clear that $T$ defines a $C$-linear mapping of $\mathscr{A}(B)$ into $C$ and $T(\bar{\phi}_i^n)=a_i^n$. Since $h$ is arbitrary and (1.5) implies the continuity of $T$ on $\mathscr{A}_h(B)$, we can deduce by Proposition 1.6 that $T$ is continuous on $\mathscr{A}(B)$, which proves the surjectivity of $\Psi$.

Finally we prove the injectivity of $\Psi$. Assume that $\Psi(T)=0$ for $T\in\mathscr{B}$. That is, $T(\bar{\phi}_i^n)=0$ for $n\in N^0$, $1\leqq i\leqq d(n)$. On the other hand, by Corollary 1 to Proposition 1.7, the series

$$\sum_{n\in N^0}\sum_{i=1}^{d(n)}(\bar{\phi}, \phi_i^n)\phi_i^n$$

converges to $\bar{\phi}$ in $\mathscr{A}(B)$ for $\phi \in \mathscr{A}(B)$. Therefore

$$\begin{aligned} T(\phi) &= \sum_{n \in N^0} \sum_{i=1}^{d(n)} (\bar{\phi}, \phi_i^n)^- T(\bar{\phi}_i^n) \\ &= 0, \end{aligned}$$

which means that $T=0$. This completes the proof of the theorem.

REMARK. The following two conditions are equivalent.

(1) $$\sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n| \exp(-t\lambda_n^{1/2}) < \infty \quad \text{for any} \quad t>0.$$

(2) $$\sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n|^2 \exp(-s\lambda_n^{1/2}) < \infty \quad \text{for any} \quad s>0.$$

Assume that (1) is satisfied. Since

$$\sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n| \exp\left(-\frac{s}{2}\lambda_n^{1/2}\right) < \infty,$$

there exists an integer $N$ such that

$$|a_i^n| \exp\left(-\frac{s}{2}\lambda_n^{1/2}\right) < 1$$

for $n>N$ and $1 \leqq i \leqq d(n)$. Then we have

$$|a_i^n|^2 \exp(-s\lambda_n^{1/2}) < |a_i^n| \exp\left(-\frac{s}{2}\lambda_n^{1/2}\right),$$

which means that

$$\begin{aligned} &\sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n|^2 \exp(-s\lambda_n^{1/2}) \\ &\leqq \sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n| \exp\left(-\frac{s}{2}\lambda_n^{1/2}\right) \\ &< \infty. \end{aligned}$$

Conversely, using Schwarz's inequality, we have

$$\begin{aligned} &\sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n| \exp(-t\lambda_n^{1/2}) = \sum_{n \in N^0} \left( \sum_{i=1}^{d(n)} |a_i^n| \exp\left(-\frac{t}{2}\lambda_n^{1/2}\right)\right) \exp\left(-\frac{t}{2}\lambda_n^{1/2}\right) \\ &\leqq \left( \sum_{n \in N^0} \sum_{i=1}^{d(n)} |a_i^n|^2 \exp(-t\lambda_n^{1/2})\right)\left( \sum_{n \in N^0} d(n) \exp(-t\lambda_n^{1/2})\right), \end{aligned}$$

which is finite by Lemma 1.2.

Therefore $\mathscr{F}_b$ is also given by

$$\{(a_i^n)|a_i^n\in \boldsymbol{C},\ \sum_{n\in N^0}\sum_{i=1}^{d(n)}|a_i^n|^2\exp(-t\lambda_n^{1/2})<\infty \quad \text{for any} \quad t>0\}.$$

## §2. Poisson transform of K-finite functions.

In this section we assume that $G$ is a connected real semisimple Lie group with finite center and of real rank one. Let $\mathfrak{g}_0$ be the Lie algebra of $G$, $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$ and $\theta$ the corresponding Cartan involution of $\mathfrak{g}_0$. We denote the complexification of $\mathfrak{g}_0$ by $\mathfrak{g}$. Let $\mathfrak{a}_+$ be a maximal abelian subspace of $\mathfrak{p}_0$. Since we assume that the real rank of $G$ is one, $\mathfrak{a}_+$ is one-dimensional. Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{a}_+$. Then $\mathfrak{a}_0=\mathfrak{a}_-+\mathfrak{a}_+$ (direct sum) and $\mathfrak{a}_+=\mathfrak{a}_0\cap\mathfrak{p}_0$, where $\mathfrak{a}_-=\mathfrak{a}_0\cap\mathfrak{k}_0$. Complexify $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{a}_0$, $\mathfrak{a}_+$ and $\mathfrak{a}_-$ to $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{k}$ in $\mathfrak{g}$ respectively and introduce compatible orders in the spaces of real-valued linear functions on $\mathfrak{a}_++\sqrt{-1}\,\mathfrak{a}_-$ and $\mathfrak{a}_+$. Let $P$ be the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ under this ordering. For a root $\alpha$, we denote the root subspace corresponding to $\alpha$ by $\mathfrak{g}^\alpha$. Put $P_+$ be the set of $\alpha$ with $\alpha\circ\theta\neq\alpha$, $\mathfrak{n}=\sum_{\alpha\in P_+}\mathfrak{g}^\alpha$, $\mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_0$ and $\rho=\frac{1}{2}\sum_{\alpha\in P_+}\alpha$. Let $K$, $A$ and $N$ denote the analytic subgroups of $G$ with Lie algebras $\mathfrak{k}_0$, $\mathfrak{a}_+$ and $\mathfrak{n}_0$ respectively. Then $G=KAN$ is an Iwasawa decomposition. Since this decomposition is unique, we can define an element $H(x)$ in $\mathfrak{a}_+$ for $x\in G$ by $x\in K(\exp H(x))N$. Let $X=G/K$ and $B=K/M$, where $M$ is the centralizer of $A$ in $K$. We define a real analytic function $P(xK, kM)$ on $X\times B$, called the Poisson kernel, by

$$P(xK, kM)=\exp(-2\rho H(x^{-1}k)).$$

We denote by $R$ the set of the equivalence classes of irreducible unitary representations of $K$ and by $R^0$ the subset of $R$ which consists of the representations of class one with respect to $M$. For each $\gamma\in R$, we take and fix a representative $(\tau^\gamma, W^\gamma)\in\gamma$ and choose a base $\{w_1^\gamma, \ldots, w_{d(\gamma)}^\gamma\}$ of $W^\gamma$ orthonormal with respect to the unitary inner product $(\ ,\ )$ of $W^\gamma$ so that $w_1^\gamma$ is an $M$-fixed vector for $\gamma\in R^0$, where $d(\gamma)$ is the dimension of $W^\gamma$. Since $\operatorname{rank}(G/K)=1$, $w_1^\gamma$ is unique up to a scalar for $\gamma\in R^0$. Put $\tau_{ij}^\gamma(k)=(\tau^\gamma(k)w_j^\gamma, w_i^\gamma)$, $\phi_{ij}^\gamma=d(\gamma)^{1/2}\bar{\tau}_{ij}^\gamma$ for $\gamma\in R$ and $\phi_i^\gamma=\phi_{i1}^\gamma$ for $\gamma\in R^0$. We identify the functions on $B$ with those on $K$ which are right $M$-invariant, and define the representation $\pi$ of $K$ on $C^\infty(K)$ and $C^\infty(X)$ by

$$\pi(k)\phi(k_0)=\phi(k^{-1}k_0),$$

$$\pi(k)f(z)=f(k^{-1}z).$$

We denote by $V_\gamma$ the space of the elements in $C^\infty(K)$ which transform according to $\gamma$ by the representation $\pi$. It is easy to see that

$$\pi(k)\phi_{ij}^\gamma=\sum_{l=1}^{d(\gamma)}\tau_{li}^\gamma\phi_{lj}^\gamma$$

for $\gamma \in R$, and

$$\phi_i^\gamma(km) = \phi_i^\gamma(k)$$

for $\gamma \in R^0$, $k \in K$ and $m \in M$. Therefore for $\gamma \in R$, $\phi_{ij}^\gamma \in V_\gamma$ and in particular $V_\gamma \subset C^\infty(B)$ for $\gamma \in R^0$. As is well-known, $\{\phi_{ij}^\gamma \mid 1 \leqq i, j \leqq d(\gamma)\}$ is an orthonormal base of $V_\gamma$ ($\gamma \in R$) and $\{\phi_i^\gamma \mid \gamma \in R^0, 1 \leqq i \leqq d(\gamma)\}$ is a complete orthonormal base of $L^2(B)$.

Let $g$ be the $G$-invariant riemannian metric on $X$ induced by the Killing form of $\mathfrak{g}_0$ and $\Delta$ be the laplacian corresponding to $g$. We identify the functions on $X$ with those on $G$ which are right $K$-invariant. We denote by $\mathfrak{B}$ the universal enveloping algebra of $\mathfrak{g}$ and regard the elements of $\mathfrak{B}$ as left $G$-invariant differential operators on $G$. Then $\Delta$ can be identified with the Casimir operator $\Omega$ on $G$ by

$$(\Delta f)(z) = (\Omega f)(x),$$

where $z = xK \in X$. We put

$$\mathscr{H}(X) = \{f \in C^\infty(X) \mid \Delta f = 0\},$$

$$\mathscr{H}_\gamma(X) = \{f \in \mathscr{H}(X) \mid f \text{ transforms according to } \gamma \text{ by } \pi\}.$$

For simplicity we write often $\mathscr{H}(X) = \mathscr{H}$ and $\mathscr{H}_\gamma(X) = \mathscr{H}_\gamma$.

Now we define the Poisson transform $\mathscr{P}\phi$ of $\phi \in C^\infty(B)$. Put

$$(\mathscr{P}\phi)(x) = \int_K P(xK, kM)\phi(k)dk,$$

where $x \in G$, $k \in K$ and $dk$ is the normalized Haar measure on $K$. Clearly $\mathscr{P}\phi$ is a function on $X$, and the following results hold.

PROPOSITION 2.1. (1) *$\mathscr{P}$ maps $\phi \in C^\infty(B)$ into $\mathscr{H}(X)$. When $\gamma \in R^0$, the restriction of $\mathscr{P}$ on $V_\gamma$ is an isomorphism onto $\mathscr{H}_\gamma(X)$.*
(2) *If $\mathscr{H}_\gamma(X) \neq \{0\}$, then $\gamma \in R^0$.*

For the proof of the proposition, see Lemma 1.2 and Theorem 1.4 in Chap. IV in [6], where more precise results are found.

We put $f_i^\gamma = \mathscr{P}\phi_i^\gamma$. Then we have

PROPOSITION 2.2. (1) *For $f \in \mathscr{H}(X)$, there exists a unique complex number $a_i^\gamma$ for $\gamma \in R^0$ and $1 \leqq i \leqq d(\gamma)$ such that*

$$f(z) = \sum_{\gamma \in R^0} \sum_{i=1}^{d(\gamma)} a_i^\gamma f_i^\gamma(z),$$

*which converges absolutely for any $z$ in $X$.*
(2) *Put $\phi_f^z(k) = f(kz)$. Then*

$$\phi_f^z = \sum_{\gamma \in R^0} d(\gamma)^{-1/2} \sum_{i,j=1}^{d(\gamma)} a_i^\gamma f_j^\gamma(z)\phi_{ij}^\gamma,$$

*which converges absolutely and uniformly on K.*
(3) *Let* $\|\ \|$ *denote the norm of* $L^2(K)$. *Then*

$$\|\phi_f^z\|^2=\sum_{\gamma\in R^0} d(\gamma)^{-1}\left(\sum_{i=1}^{d(\gamma)}|a_i^\gamma|^2\right)\left(\sum_{j=1}^{d(\gamma)}|f_j^\gamma(z)|^2\right).$$

PROOF. By the theory of Fourier expansion of $C^\infty$-functions on compact Lie groups (cf. [14]),

$$\phi_f^z=\sum_{\gamma\in R}\sum_{i,j=1}^{d(\gamma)} b_{ij}^\gamma(z)\phi_{ij}^\gamma, \tag{2.1}$$

which converges absolutely and uniformly on $K$ and $b_{ij}^\gamma(z)$ is given by

$$b_{ij}^\gamma(z)=\int_K f(kz)\bar{\phi}_{ij}^\gamma(k)dk.$$

Since

$$\pi(k)b_{ij}^\gamma(z)=\sum_{l=1}^{d(\gamma)}\tau_{lj}^\gamma(k)b_{il}^\gamma(z),$$

$b_{ij}^\gamma$ lies in $\mathscr{H}_\gamma$. From (2.1), putting $k=1$, we have an absolutely convergent series

$$f(z)=\sum_{\gamma\in R} d(\gamma)^{1/2}\sum_{i=1}^{d(\gamma)} b_{ii}^\gamma(z), \tag{2.2}$$

since $\phi_{ij}^\gamma(1)=d(\gamma)^{1/2}\delta_{ij}$. If $\sum_{i=1}^{d(\gamma)} b_{ii}^\gamma\neq 0$, we can deduce by Proposition 2.1 that $\gamma\in R^0$ and that there exist complex numbers $a_i^\gamma$ ($\gamma\in R^0$, $1\leqq i\leqq d(\gamma)$) such that

$$d(\gamma)^{1/2}\sum_{i=1}^{d(\gamma)} b_{ii}^\gamma=\sum_{i=1}^{d(\gamma)} a_i^\gamma f_i^\gamma. \tag{2.3}$$

Since $z$ is arbitrary, replacing $z$ by $kz$ in (2.3) we have

$$\begin{aligned} d(\gamma)^{1/2}\sum_{i=1}^{d(\gamma)} b_{ii}^\gamma(kz)&=d(\gamma)^{1/2}\sum_{i,l=1}^{d(\gamma)}\tau_{li}^\gamma(k^{-1})b_{il}^\gamma(z)\\ &=\sum_{i,l=1}^{d(\gamma)} b_{il}^\gamma(z)\phi_{il}^\gamma(k) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} f_i^\gamma(kz)&=\int_K P(kz, k_0M)\phi_i^\gamma(k_0)dk_0\\ &=\int_K P(z, k^{-1}k_0M)\phi_i^\gamma(k_0)dk_0\\ &=\int_K P(z, k_0M)\phi_i^\gamma(kk_0)dk_0 \end{aligned}$$

$$=d(\gamma)^{-1/2}\sum_{l=1}^{d(\gamma)}\int_K P(z,\, k_0 M)\phi_{il}^{\gamma}(k)\phi_i^{\gamma}(k_0)dk_0$$

$$=d(\gamma)^{-1/2}\sum_{l=1}^{d(\gamma)} f_l^{\gamma}(z)\phi_{il}^{\gamma}(k), \tag{2.5}$$

for $1\leqq i\leqq d(\gamma)$. From (2.3), (2.4) and (2.5) we have

$$\sum_{i,l=1}^{d(\gamma)} b_{il}^{\gamma}(z)\phi_{il}^{\gamma}=d(\gamma)^{-1/2}\sum_{i,l=1}^{d(\gamma)} a_i^{\gamma}f_l^{\gamma}\phi_{il}^{\gamma}.$$

Since $\phi_{il}^{\gamma}$ are linearly independent we can deduce that

$$b_{il}^{\gamma}=d(\gamma)^{-1/2}a_i^{\gamma}f_l^{\gamma}.$$

Putting $i=l$ in the above equality, we obtain from (2.2) an absolutely convergent series

$$f(z)=\sum_{\gamma\in R^0}\sum_{i=1}^{d(\gamma)} a_i^{\gamma}f_i^{\gamma}(z),$$

which proves (1) in the proposition.

Next, from (1) and (2.5) we have

$$\phi_f^z(k)=f(kz)=\sum_{\gamma\in R^0}\sum_{i=1}^{d(\gamma)} a_i^{\gamma}f_i^{\gamma}(kz)$$

$$=\sum_{\gamma\in R^0} d(\gamma)^{-1/2}\sum_{i,j=1}^{d(\gamma)} a_i^{\gamma}f_j^{\gamma}(z)\phi_{ij}^{\gamma}(k),$$

which proves (2) and (3) immediately. This completes the proof.

Let $u\in\mathfrak{B}\mathfrak{k}$. Then as is easily seen, $uf=0$ on $X$ for $f\in C^\infty(X)$. On the other hand, as is stated before, $(\Delta f)(xk)=(\Omega f)(x)$. Therefore we may transform $\Omega$ modulo $\mathfrak{B}\mathfrak{k}$. In the following we transform $\Omega$ modulo $\mathfrak{B}\mathfrak{k}$ to obtain the differential equation on $A$ which the elements in $\mathscr{H}$ satisfy.

For an element $\lambda$ of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, let $\bar{\lambda}$ denote the restriction of $\lambda$ on $\mathfrak{a}_\mathfrak{p}$. Let $P_+$ be the set of $\alpha\in P$ such that $\bar{\alpha}\neq 0$. For every root $\alpha$, we select $X_\alpha\in\mathfrak{g}^\alpha$ so that $<X_\alpha, X_{-\alpha}>=1$ where $<\,,\,>$ is the Killing form of $\mathfrak{g}$. Then $[X_\alpha, X_{-\alpha}]=H_\alpha$ where $H_\alpha$ is the unique element such that $<H, H_\alpha>=\alpha(H)$ for any $H\in\mathfrak{a}$. Choose bases $H_1$ and $H_2, \dots, H_m$ of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{k}$ respectively so that $<H_i, H_j>=\delta_{ij}$ for $1\leqq i, j\leqq m$. Then $H_1, \dots, H_m$ together with $X_\alpha$, $X_{-\alpha}$ $(\alpha\in P)$ form a base of $\mathfrak{g}$. Put $P_-=P-P_+$. Since $X_\alpha$, $X_{-\alpha}(\alpha\in P_-)$ and $H_i$ $(2\leqq i\leqq m)$ lie in $\mathfrak{k}$,

$$\Omega=H_1^2+H_2^2+\dots+H_m^2+\sum_{\alpha\in P_+}(X_\alpha X_{-\alpha}+X_{-\alpha}X_\alpha)$$

$$\equiv H_1^2+\sum_{\alpha\in P_+}(X_\alpha X_{-\alpha}+X_{-\alpha}X_\alpha)\quad \mathrm{mod}\ \mathfrak{B}\mathfrak{k}. \tag{2.6}$$

For $\alpha\in P_+$, let $X_\alpha=Z_\alpha+Y_\alpha$ where $Z_\alpha\in\mathfrak{k}$ and $Y_\alpha\in\mathfrak{p}$, and put $X^a=Ad(a)X$ where $a=\exp H$ and $H\in\mathfrak{a}_+-\{0\}$. Then

$$X_\alpha^a=Z_\alpha^a+Y_\alpha^a$$

and

$$X_\alpha^a=e^{\alpha(H)}Z_\alpha+e^{\alpha(H)}Y_\alpha.$$

Therefore

$$Z_\alpha^a+Y_\alpha^a=e^{\alpha(H)}Z_\alpha+e^{\alpha(H)}Y_\alpha. \tag{2.7}$$

Since $\theta(Z_\alpha^a+Y_\alpha^a)=Z_\alpha^{a^{-1}}-Y_\alpha^{a^{-1}}$, we have also

$$Z_\alpha^{a^{-1}}-Y_\alpha^{a^{-1}}=e^{\alpha(H)}Z_\alpha-e^{\alpha(H)}Y_\alpha. \tag{2.8}$$

In (2.7), replacing $H$ by $-H$, we have

$$Z_\alpha^{a^{-1}}+Y_\alpha^{a^{-1}}=e^{-\alpha(H)}Z_\alpha+e^{-\alpha(H)}Y_\alpha. \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$Y_\alpha=(\coth\alpha(H))Z_\alpha-(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}. \tag{2.10}$$

On the other hand, since

$$\begin{aligned} X_\alpha X_{-\alpha} &= X_\alpha(Z_{-\alpha}+Y_{-\alpha}) \\ &= X_\alpha Z_{-\alpha}+X_\alpha Y_{-\alpha} \\ &\equiv X_\alpha Y_{-\alpha} \\ &= (Z_\alpha+Y_\alpha)Y_{-\alpha}, \end{aligned}$$

we get from (2.10) that

$$\begin{aligned} X_\alpha X_{-\alpha} &\equiv \{(1+\coth\alpha(H))Z_\alpha-(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}\}Y_{-\alpha} \\ &= (1+\coth\alpha(H))[Z_\alpha, Y_{-\alpha}]+(1+\coth\alpha(H))Y_{-\alpha}Z_\alpha \\ &\quad -(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}Y_{-\alpha} \\ &\equiv (1+\coth\alpha(H))[Z_\alpha, Y_{-\alpha}]-(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}Y_{-\alpha}. \end{aligned} \tag{2.11}$$

Thus we have

$$X_{-\alpha}X_\alpha\equiv(1-\coth\alpha(H))[Z_{-\alpha}, Y_\alpha]+(\sinh\alpha(H))^{-1}Z_{-\alpha}^{a^{-1}}Y_\alpha.$$

Repalcing $H$ by $-H$ in the above expression, we have

$$X_{-\alpha}X_\alpha\equiv(1+\coth\alpha(H))[Z_{-\alpha}, Y_\alpha]-(\sinh\alpha(H))^{-1}Z_{-\alpha}^{a}Y_\alpha.$$

Therefore

$$\theta(X_{-\alpha}X_\alpha)\equiv -(1+\coth\alpha(H))[Z_{-\alpha}, Y_\alpha] + (\sinh\alpha(H))^{-1}Z_{-\alpha}^{a^{-1}}Y_\alpha, \tag{2.12}$$

since $[Z_{-\alpha}, Y_\alpha]\in\mathfrak{p}$. (2.11) together with (2.12) gives

$$X_\alpha X_{-\alpha}+\theta(X_{-\alpha}X_\alpha) \equiv(1+\coth\alpha(H))([Z_\alpha, Y_{-\alpha}]+[Y_\alpha, Z_{-\alpha}]) -(\sinh\alpha(H))^{-1}(Z_\alpha^{a^{-1}}Y_{-\alpha}-Z_{-\alpha}^{a^{-1}}Y_\alpha).$$

Since

$$[Z_\alpha, Y_{-\alpha}]+[Y_\alpha, Z_{-\alpha}]=\frac{1}{2}\{[X_\alpha, X_{-\alpha}]-[\theta X_\alpha, \theta X_{-\alpha}]\}$$

$$=\frac{1}{2}\{H_\alpha-\theta H_\alpha\}=H_{\bar{\alpha}},$$

we obtain that

$$X_\alpha X_{-\alpha}+\theta(X_{-\alpha}X_\alpha) \equiv(1+\coth\alpha(H))H_{\bar{\alpha}}-(\sinh\alpha(H))^{-1}(Z_\alpha^{a^{-1}}Y_{-\alpha}-Z_{-\alpha}^{a^{-1}}Y_\alpha). \tag{2.13}$$

As is easily seen, $\theta\Omega=\Omega$. Therefore from (2.6) and (2.13) we find that

$$\Omega=\frac{1}{2}(\Omega+\theta\Omega)$$

$$\equiv H_1{}^2+\frac{1}{2}\sum_{\alpha\in P_+}\{(X_\alpha X_{-\alpha}+\theta(X_{-\alpha}X_\alpha))+(X_{-\alpha}X_\alpha+\theta(X_\alpha X_{-\alpha}))\}$$

$$=H_1{}^2+\frac{1}{2}\sum_{\alpha\in P_+}\{(1+\coth\alpha(H))H_{\bar{\alpha}}-(\sinh\alpha(H))^{-1}(Z_\alpha^{a^{-1}}Y_{-\alpha}-Z_{-\alpha}^{a^{-1}}Y_\alpha) +(1-\coth\alpha(H))H_{-\bar{\alpha}}+(\sinh\alpha(H))^{-1}(Z_{-\alpha}^{a^{-1}}Y_\alpha-Z_\alpha^{a^{-1}}Y_{-\alpha})\}.$$

Taking $H_{-\bar{\alpha}}=-H_{\bar{\alpha}}$ into account, we get

$$\Omega\equiv H_1{}^2+\sum_{\alpha\in P_+}\{(\coth\alpha(H))H_{\bar{\alpha}}+(\sinh\alpha(H))^{-1}(Z_{-\alpha}^{a^{-1}}Y_\alpha-Z_\alpha^{a^{-1}}Y_{-\alpha})\}.$$

Since $Y_\alpha=(\coth\alpha(H))Z_\alpha-(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}$ from (2.10), we find that

$$\Omega\equiv H_1{}^2+\sum_{\alpha\in P_+}(\coth\alpha(H))H_{\bar{\alpha}}-\sum_{\alpha\in P_+}(\sinh\alpha(H))^{-2}(Z_\alpha^{a^{-1}}Z_{-\alpha}^{a^{-1}}+Z_{-\alpha}^{a^{-1}}Z_\alpha^{a^{-1}}).$$

Now, let $L_X(X\in\mathfrak{g})$ be the differential of the left regular representation of $G$ on $C^\infty(G)$ and extend it to the representation of $\mathfrak{B}$. Then

$$(X^{x^{-1}}f)(x)=(L_{-X}f)(x)$$

for $x \in G, f \in {}^{\infty}C(G)$ and $X \in \mathfrak{g}$. Therefore for $a \in A$,

$$(\Omega f)(a) = [\{H_1{}^2 + \sum_{\alpha \in P_+} (\coth\alpha(H))H_{\bar{\alpha}} - \sum_{\alpha \in P_+} (\sinh\alpha(H))^{-2} L_{(Z_\alpha Z_{-\alpha} + Z_{-\alpha} Z_\alpha)}\} f](a). \tag{2.14}$$

Let $\mu_0$ be the restriction of an $\alpha \in P_+$ on $\mathfrak{a}_+$ such that $\frac{1}{2}\mu_0$ is not equal to the restriction of any root $\alpha \in P_+$ on $\mathfrak{a}_+$. Then $\bar{\alpha} = \mu_0$ or $2\mu_0$ for $\alpha \in P_+$. Let $P_{\mu_0}$ (resp. $P_{2\mu_0}$) be the set of $\alpha \in P_+$ such that $\bar{\alpha} = \mu_0$ (resp. $2\mu_0$), and let $p$(resp. $q$) denote the number of roots in $P_{\mu_0}$ (resp. $P_{2\mu_0}$). We normalize $H_0$ in $\mathfrak{a}_+$ so that $\mu_0(H_0) = 1$. Then $<H_0, H_0> = 2p + 8q$ and $H_1 = (2p+8q)^{-1/2} H_0$. For $t \in \boldsymbol{R}$, put $a_t = \exp tH_0$. Then $t$ can be regarded as the coordinate function on the one-dimensional Lie group $A$, and we write often $f(t)$ for $f(a_t)$. It is clear that $H_{\mu_0} = (2p+8q)^{-1} H_0$ and

$$(\Omega f)(t) = \left\{\frac{1}{2p+8q} \frac{d^2}{dt^2} + \frac{p \coth t}{2p+8q} \frac{d}{dt} + \frac{q \coth 2t}{2p+8q} \cdot 2\frac{d}{dt}\right\} f(t) - \left[\left\{\frac{1}{(\sinh t)^2} \sum_{\alpha \in P_{\mu_0}} L_{(Z_\alpha Z_{-\alpha} + Z_{-\alpha} Z_\alpha)} + \frac{1}{(\sinh 2t)^2} \sum_{\alpha \in P_{2\mu_0}} L_{(Z_\alpha Z_{-\alpha} + Z_{-\alpha} Z_\alpha)}\right\} f\right](t).$$

We define $D$, $\omega_1$ and $\omega_2$ by

$$D = \frac{1}{2p+8q}\left\{\frac{d^2}{dt^2} + (p \coth t + 2q \coth 2t)\frac{d}{dt}\right\},$$

$$\omega_1 = \sum_{\alpha \in P_+} (Z_\alpha Z_{-\alpha} + Z_{-\alpha} Z_\alpha),$$

$$\omega_2 = \sum_{\alpha \in P_{2\mu_0}} (Z_\alpha Z_{-\alpha} + Z_{-\alpha} Z_\alpha).$$

Then we have

PROPOSITION 2.3. *Let $f \in C^\infty(X)$. Then*

$$\Omega f(t) = Df(t) - \frac{1}{(\sinh t)^2} L_{\omega_1} f(t) - \left\{\frac{1}{(\sinh 2t)^2} - \frac{1}{(\sinh t)^2}\right\} L_{\omega_2} f(t).$$

COROLLARY. *Let $f \in \mathscr{H}(X)$. Then*

$$Df(t) - \frac{1}{(\sinh t)^2} L_{\omega_1} f(t) - \left\{\frac{1}{(\sinh 2t)^2} - \frac{1}{(\sinh t)^2}\right\} L_{\omega_2} f(t) = 0.$$

## § 3. K-finite harmonic functions on real hyperbolic spaces.

From now on, we assume that $G=SO_0(n, 1)$ $(n\geqq 3)$, the generalized Lorentz group. That is, we deal with the real hyperbolic space $X=G/K$. We keep to the notation in § 2.

At first we review the structure of the Lie algebra $\mathfrak{g}_0=\mathfrak{so}(n, 1)$ (cf. [15]). The Lie algebra $\mathfrak{g}_0$ consists of real matrices $X$ of order $n+1$ such that

$$ {}^tXJ+JX=0, $$

where

$$ J=\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{pmatrix}. $$

As a base of $\mathfrak{g}_0$, we can take the matrices

$$ Y_i \qquad (1\leqq i\leqq n) $$

and

$$ X_{ij} \qquad (1\leqq i<j\leqq n), $$

where

$$ Y_i=E_{0i}+E_{i0}, \qquad X_{ij}=E_{ij}-E_{ji}, $$

$E_{ij}$ denoting the matrix of order $n+1$ whose $(i, j)$ entry is 1 and others are 0. By this base the Killing from $<\ ,\ >$ of $\mathfrak{g}_0$ is given by

$$ \begin{aligned} <X, Y> &=(\mathrm{n}-1)\mathrm{Tr}(XY) \\ &=2(n-1)\Big(\sum_{1\leqq i\leqq n} c_i d_i-\sum_{1\leqq i<j\leqq n} c_{ij}d_{ij}\Big), \end{aligned} $$

where

$$ X=\sum_{1\leqq i\leqq n} c_iY_i+\sum_{1\leqq i<j\leqq n} c_{ij}X_{ij}, $$

$$ Y=\sum_{1\leqq i\leqq n} d_iY_i+\sum_{1\leqq i<j\leqq n} d_{ij}X_{ij}. $$

Put

$$ \mathfrak{k}_0=\sum_{1\leqq i<j\leqq n} \boldsymbol{R}X_{ij}, \quad \mathfrak{p}_0=\sum_{1\leqq i\leqq n} \boldsymbol{R}Y_i. $$

Then $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ is a Cartan decomposition and the corresponding Cartan involution is given by

$$ \theta(Y_i)=-Y_i, \qquad \theta(X_{ij})=X_{ij}. $$

Let $m$ be the integer determined by $n=2m$ or $n=2m-1$ according to the parity of $n$, and put

$$E_1=Y_1, \quad E_2=\sqrt{-1}\ X_{23}, \dots, \quad E_m=\sqrt{-1}\ X_{2m-2,\,2m-1},$$

and

$$\mathfrak{a}_+=\boldsymbol{R}E_1, \quad \mathfrak{a}_-=\sqrt{-1}\sum_{2\leqq i\leqq m}\boldsymbol{R}E_i, \quad \mathfrak{a}_0=\mathfrak{a}_++\mathfrak{a}_-.$$

Then $\mathfrak{a}_+$ is a maximal abelian subspace of $\mathfrak{p}_0$ and $\mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Put $\mathfrak{g}=\mathfrak{g}_0{}^c$, $\mathfrak{a}=\mathfrak{a}_0{}^c$ and introduce a lexicographic order in the space of real-valued linear forms on $\mathfrak{a}_++\sqrt{-1}\mathfrak{a}_-$ with respect to the base $(E_1, E_2, \dots, E_m)$. Let $\lambda_i(1\leqq i\leqq m)$ be the elements of $\mathfrak{a}^*$ difined by $\lambda_i(E_j)=\delta_{ij}$ $(1\leqq i, j\leqq m)$. Then the set $P$ of positive roots of $(\mathfrak{g}, \mathfrak{a})$ is given by

$$P=\{\lambda_i\ (1\leqq i\leqq m), \quad \lambda_i\pm\lambda_j\ (1\leqq i, j\leqq m)\} \qquad (n=2m),$$

$$P=\{\lambda_i\pm\lambda_j\ (1\leqq i, j\leqq m)\} \qquad (n=2m-1),$$

and $P_+$ is given by

$$P_+=\{\lambda_1, \quad \lambda_1\pm\lambda_j\ (2\leqq j\leqq m)\} \qquad (n=2m), \tag{3.1}$$

$$P_+=\{\lambda_1\pm\lambda_j\ (2\leqq j\leqq m)\} \qquad (n=2m-1). \tag{3.2}$$

Put

$$X_{\lambda_1}=X_n,$$

$$X_{\lambda_1+\lambda_j}=\frac{1}{2}(\sqrt{-1}\ X_{2j-2}+X_{2j-1}),$$

$$X_{\lambda_1-\lambda_j}=\frac{1}{2}(\sqrt{-1}\ X_{2j-2}-X_{2j-1}),$$

$$X_{-\lambda_1+\lambda_j}=\theta X_{\lambda_1+\lambda_j},$$

$$X_{-\lambda_1-\lambda_j}=\theta X_{\lambda_1-\lambda_j}\ (2\leqq j\leqq m),$$

where $X_i=Y_i+X_{1i}\ (2\leqq i\leqq n)$. Then they are the root vectors of $\mathfrak{g}=\mathfrak{o}(n+1, \boldsymbol{C})$. Since

$$<X_{\lambda_1}, X_{-\lambda_1}>=-4(n-1)$$

and

$$<X_{\lambda_1+\lambda_j}, X_{-\lambda_1-\lambda_j}>=<X_{\lambda_1-\lambda_j}, X_{-\lambda_1+\lambda_j}>=2(n-1),$$

it is easy to see that $\mathfrak{k}$-components $Z_{\pm\alpha}$ $(\alpha\in P_+)$ are given by the following

LEMMA 3.1. $Z_{\lambda_1}=Z_{-\lambda_1}=-\frac{\sqrt{-1}}{2\sqrt{n-1}}X_{1n}$,

$$Z_{\lambda_1+\lambda_j}=Z_{-\lambda_1+\lambda_j}=\frac{1}{2\sqrt{2(n-1)}}(\sqrt{-1}\ X_{1,2j-2}+X_{1,2j-1}),$$

$$Z_{\lambda_1-\lambda_j}=Z_{-\lambda_1-\lambda_j}=\frac{1}{2\sqrt{2(n-1)}}(\sqrt{-1}X_{1,2j-2}-X_{1,2j-1}).$$

As is easily seen, $\mu_0=\lambda_1$, $P_{\mu_0}=P_+$ and $P_{2\mu_0}=\emptyset$. Therefore $\omega_2=0$ and we have only to compute $\omega_1$. When $n=2m$, from (3.1) and Lemma 3.1 we have

$$\begin{aligned}\omega_1&=\sum_{\alpha\in P_+}(Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha)\\&=2\left(\frac{-\sqrt{-1}}{2\sqrt{n-1}}X_{1n}\right)^2\\&\quad+2\left(\frac{1}{2\sqrt{2(n-1)}}\right)^2\sum_{2\leq j\leq m}\{(\sqrt{-1}\ X_{1,2j-2}+X_{1,2j-1})(\sqrt{-1}\ X_{1,2j-2}-X_{1,2j-1})\\&\qquad+(\sqrt{-1}\ X_{1,2j-2}-X_{1,2j-1})(\sqrt{-1}\ X_{1,2j-2}+X_{1,2j-1})\}\\&=-\frac{1}{2(n-1)}X_{1n}^2+\frac{1}{4(n-1)}\sum_{2\leq j\leq m}(-2X_{1,2j-2}^2-2X_{1,2j-1}^2)\\&=-\frac{1}{2(n-1)}X_{1n}^2-\frac{1}{2(n-1)}\sum_{2\leq i\leq n-1}X_{1i}^2\\&=-\frac{1}{2(n-1)}\sum_{2\leq i\leq n}X_{1i}^2\,.\end{aligned}$$

Similarly, when $n=2m-1$, we have from (3.2) and Lemma 3.1,

$$\begin{aligned}\omega_1&=-\frac{1}{2(n-1)}\sum_{2\leq j\leq m}(X_{1,2j-2}^2+X_{1,2j-1}^2)\\&=-\frac{1}{2(n-1)}\sum_{2\leq i\leq n}X_{1i}^2\,.\end{aligned}$$

Consequently for any $n$, we have

$$\omega_1=-\frac{1}{2(n-1)}\sum_{2\leq i\leq n}X_{1i}^2\,. \tag{3.3}$$

Let $<\ ,\ >_k$ be the Killing form of $\mathfrak{k}_0$ and $\omega_k$ be the Casimir operator of $\mathfrak{k}_0$. Since $\mathfrak{k}_0=\mathfrak{so}(n)$,

$$\begin{aligned}<X, Y>_k&=(n-2)\mathrm{Tr}(XY)\\&=\frac{n-2}{n-1}<X, Y>\end{aligned}$$

for $X, Y \in \mathfrak{k}_0$. Therefore

$$\omega_k = -\frac{1}{2(n-2)} \sum_{1 \leqq i < j \leqq n} X_{ij}^2, \tag{3.4}$$

since $\mathfrak{k}_0 = \sum_{1 \leqq i, j \leqq n} \boldsymbol{R} X_{ij}$ and $<X_{ij}, X_{rs}>_k = -2(n-2)\delta_{ir}\delta_{js}$.

Let $\mathfrak{m}_0$ be the Lie algebra of $M$. It is easy to see that

$$\mathfrak{m}_0 = \sum_{2 \leqq i < j \leqq n} \boldsymbol{R} X_{ij}.$$

Therefore, from (3.4) we have

$$\omega_k \equiv -\frac{1}{2(n-2)} \sum_{2 \leqq i \leqq n} X_{1i}^2 \quad \text{mod} \quad \mathfrak{m}_0 \mathfrak{B}. \tag{3.5}$$

From (3.3) and (3.5) we obtain the following

LEMMA 3.2. $\omega_1 \equiv \frac{n-2}{n-1} \omega_k \quad mod \quad \mathfrak{m}_0 \mathfrak{B}.$

Let $m_0$ be the integer determined by $n = 2m_0$ or $n = 2m_0 + 1$ according to the parity of $n$. Put

$$H_j = \sqrt{-1} X_{2j-1, 2j} \qquad (1 \leqq j \leqq m_0)$$

and

$$\mathfrak{h}_0 = \sqrt{-1} \sum_{1 \leqq j \leqq m_0} \boldsymbol{R}\, H_j.$$

Then $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. Put $\mathfrak{h} = \mathfrak{h}_0{}^c$ and introduce a lexicographic order in the space of real linear forms on $\sqrt{-1}\,\mathfrak{h}_0$ with respect to the base $(H_1, \ldots, H_{m_0})$. Let $e_i$ be the elements of $\mathfrak{h}^*$ defined by $e_i(H_j) = \delta_{ij}$ $(1 \leqq i, j \leqq m_0)$. For $\gamma \in R$, let $\Lambda_\gamma$ be the highest weight of $\gamma$ with respect to $\mathfrak{h}$, where $R$ is the set of equivalence classes of irreducible unitary representations of $K$. From the representation theory of compact Lie groups, the mapping $R \ni \gamma \to \Lambda_\gamma \in \mathfrak{h}^*$ is injective. We denote by $L^0$ the image of $R^0$ by this mapping. Then $L_0$ is given by

$$L^0 = \{\Lambda_l | \Lambda_l = l e_1,\ l \in \boldsymbol{N}^0\}$$

(cf. [13]). From now on we identify $L^0$ with $\boldsymbol{N}^0$ and write $\tau_l$, $\Lambda_l$, $V_l$, $\mathscr{H}_l$, $\phi_i^l$, $f_i^l$ and $d(l)$ instead of $\tau_\gamma$, $\Lambda_\gamma$, $V_\gamma$, $\mathscr{H}_\gamma$, $\phi_i^\gamma$, $f_i^\gamma$ and $d(\gamma)$. Put $\rho_k = \frac{1}{2} \sum_{\beta \in Q} \beta$, where $Q$ is the set of positive roots of $(\mathfrak{k}, \mathfrak{h})$. Then from Proposition 2.1 and Schur's lemma,

$$L_{\omega_k} f = <\Lambda_l + 2\rho_k, \Lambda_l>_k f \tag{3.6}$$

for $f \in \mathscr{H}_l$. By a simple computation we see that

$$<\Lambda_l + 2\rho_k, \Lambda_l>_k = \frac{l(l+n-2)}{2(n-2)} \tag{3.7}$$

By the way, since $M$ normalizes $A$, $f((\exp tY)a)=f(a\exp tY)$ for $a\in A$, $t\in\boldsymbol{R}$ and $Y\in\mathfrak{m}_0$. Therefore

$$(L_u f)(a)=0$$

for $a\in A$, $f\in C^\infty(X)$ and $u\in\mathfrak{m}_0\mathfrak{B}$. Using Lemma 3.2 and (3.6), (3.7) we have

LEMMA 3.3. *Let $f\in\mathscr{H}_l$. Then, for $a\in A$*

$$(L_{\omega_1}f)(a)=\frac{l(l+n-2)}{2(n-1)}f(a).$$

PROPOSITION 3.4. *Let $l\in L^0$ and $f\in\mathscr{H}_l$. Then $f(t)$ satisfies the differential equation*

$$\frac{d^2f}{dt^2}+(n-1)\coth t\frac{df}{dt}-\frac{l(l+n-2)f}{(\sinh t)^2}=0.$$

PROOF. Since $\Omega f=\Delta f=0$ and $p=(n-1)$, $q=0$ in case of $\mathfrak{g}_0=\mathfrak{so}(n,\ 1)$, we have this proposition immediately from Proposition 2.3 and Lemma 3.3. This completes the proof.

We introduce a new parameter $z=\left(\tanh\frac{t}{2}\right)^2$. Then the differential equation in Proposition 3.4 turns into

$$4z(1-z)^2\frac{d^2f}{dz^2}+2(1-z)(nz-4z+n)\frac{df}{dz}-l(l+n-2)\frac{(1-z)^2}{z}f=0.$$

A fundamental system of solutions of this differential equation is given by

$$z^{l/2}F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;\ z\right)$$

and

$$z^{-\frac{l+n-2}{2}}F\left(-l-n+2,\ 1-\frac{n}{2},\ -l-\frac{n}{2}+2\ ;\ z\right),$$

where $F$ is the hypergeometric function. Since $f(t)$ is a $C^\infty$-function in $t$, there exists a complex number $c$ such that

$$f(t)=c\ z^{l/2}F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;\ z\right).$$

Thus we have

PROPOSITION 3.5. *For $f\in\mathscr{H}_l$, there exists a complex number $c$ such that*

$$f(a_t)=c\left(\tanh\frac{t}{2}\right)^l F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;\ \left(\tanh\frac{t}{2}\right)^2\right).$$

By the above proposition, there exist $c_i^l$ for $l\in L^0$, $1\leqq i\leqq d(l)$ such that

$$f_i^l(a_t)=c_i^l\left(\tanh\frac{t}{2}\right)^l F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;\ \left(\tanh\frac{t}{2}\right)^2\right).$$

On the other hand, A.W. Knapp proved ([7], Theorem 1.1) that in case of rank$(X)=1$,

$$\lim_{t\to\infty}(\mathscr{P}\phi)(ka_t)=\phi(k) \qquad a.e.\quad k\in K,$$

where $\phi$ is an integrable function on $B=K/M$. Since $F\left(l, 1-\frac{n}{2},\ l+\frac{n}{2}\ ;1\right)$ exists and

$$\lim_{t\to\infty} F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;\ \left(\tanh\frac{t}{2}\right)^2\right)$$

$$=F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;1\right)$$

$$=\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\ \frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma(l+n-1)},$$

where $\Gamma$ denotes the gamma function, we have

$$d(l)^{1/2}\,\delta_{i1}=c_i^l\,\frac{\Gamma(n-1)}{\Gamma\left(\frac{n}{2}\right)}\ \frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma(l+n-1)}.$$

We put

$$F_l(z)=\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n-1)}\ \frac{\Gamma(l+n-1)}{\Gamma\left(l+\frac{n}{2}\right)}F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;\ z\right).$$

Then we have

PROPOSITION 3.6. *Let $l\in L^0$. Then*

$$f_1^l(a_t)=d(l)^{1/2}\left(\tanh\frac{t}{2}\right)^l F_l\left(\left(\tanh\frac{t}{2}\right)^2\right)$$

$$f_i^l(a_t)=0\ (2\leqq i\leqq d(l)).$$

## §4. Poisson transform of hyperfunctions.

In this section we keep to the notation in the previous sections. We can take the Casimir operator $\omega_K$ as the laplacian $\omega$ on $B$ introduced in §1. Then there exists an isomorphism $\Psi$ of the space of hyperfunctions $\mathscr{B}(B)$ onto $\mathscr{F}_b(B)$, where

$$\mathscr{F}_b(B)=\{(a_i^l)_{1\leq i\leq d(l)}^{l\in L^0}|a_i^l\in \boldsymbol{C},\ \sum_{l\in L^0}\sum_{i=1}^{d(l)}|a_i^l|\exp(-t\lambda_l^{1/2})<\infty \quad \text{for any} \quad t>0\}$$

and

$$\lambda_l=\frac{l(l+n-2)}{2(n-2)}\ .$$

At first we prove two lemmas. It is easy to see

LEMMA 4.1. *For integers* $l\geq 0$ and $n\geq 3$, we have

$$\frac{l}{\sqrt{2(n-2)}}\leq \lambda_l^{1/2}\leq l.$$

LEMMA 4.2. *For* $0\leq r<1$, *we have*

$$(1)\quad |F_l(r^2)|\leq \frac{\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l+1)}\,,$$

$$(2)\quad \frac{\Gamma(n/2)\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l+n/2)}(1-r^2)^{n-1}\leq F_l(r^2).$$

PROOF. (1) From the definition of hypergeometric function, we have

$$F\Big(l,\ 1-\frac{n}{2},\ l+\frac{n}{2};\ r^2\Big)=\sum_{p=0}^{\infty}\frac{(l)_p\Big(1-\frac{n}{2}\Big)_p}{\Big(l+\frac{n}{2}\Big)_p}\ \frac{r^{2p}}{p!}\,,$$

where $(\alpha)_p=\alpha(\alpha+1)\dots(\alpha+p-1)$ for $\alpha\in\boldsymbol{C}$. Since we have

$$\begin{aligned}|(\alpha)_p|&\leq|\alpha|\ |\alpha+1|\dots|\alpha+p-1|\\ &\leq|\alpha|\ (|\alpha|+1)\dots(|\alpha|+p-1)\\ &=(|\alpha|)_p\end{aligned}$$

by triangle inequality, it follows that

$$\begin{aligned}\Big|F\Big(l,\ 1-\frac{n}{2},\ l+\frac{n}{2};\ r^2\Big)\Big|&\leq\sum_{p=0}^{\infty}\frac{(l)_p\Big(\frac{n}{2}-1\Big)_p}{\Big(l+\frac{n}{2}\Big)_p}\ \frac{r^{2p}}{p!}\\ &=F\Big(l,\ \frac{n}{2}-1,\ l+\frac{n}{2};\ r^2\Big).\end{aligned}\qquad(4.1)$$

On the other hand

$$F\left(l,\ \frac{n}{2}-1,\ l+\frac{n}{2};\ 1\right)$$

exists ([9]) and is equal to

$$\frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(l+1)}\ .$$

Therefore, from (4.1) we have

$$\left|F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2};\ r^2\right)\right| \leqq \frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(l+1)}\ ,$$

since $F(l,\ n/2-1,\ l+n/2;\ r^2)$ is a positive term series of $r^2$. From the definition of $F_l$ (§ 3), we find that

$$\begin{aligned}|F_l(r^2)| &\leqq \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(l+n-1)}{\Gamma(n-1)\Gamma\left(l+\frac{n}{2}\right)}\ \frac{\Gamma\left(l+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(l+1)} \\ &= \frac{\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l+1)}\ .\end{aligned}$$

(2) We notice the equality ([9], p. 248)

$$F\left(l,\ 1-\frac{n}{2},\ l+\frac{n}{2}\ ;\ z\right)=(1-z)^{n-1}\ F\left(\frac{n}{2},\ l+n-1,\ l+\frac{n}{2}\ ;\ z\right).$$

Since $F(n/2,\ l+n-1,\ l+n/2;\ r^2)$ is also a positive term series of $r^2$, it follows that

$$F\left(\frac{n}{2},\ l+n-1,\ l+\frac{n}{2}\ ;\ r^2\right)\geqq 1.$$

Therefore we have

$$\begin{aligned}F_l(r^2) &= \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(l+n-1)}{\Gamma(n-1)\Gamma\left(l+\frac{n}{2}\right)}(1-r^2)^{n-1}F\left(\frac{n}{2},\ l+n-1,\ l+\frac{n}{2}\ ;\ r^2\right) \\ &\geqq \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(l+n-1)}{\Gamma(n-1)\Gamma\left(l+\frac{n}{2}\right)}(1-r^2)^{n-1},\end{aligned}$$

which completes the proof.

For $s>0$, put

$$U_s=\{z=ka_tK\in X|k\in K,\ |\tanh\frac{t}{2}|\leqq \exp(-2s)\}.$$

Let $(a_i^l)\in\mathscr{F}_b$. We consider the series

$$S(z)=\sum_{l\in L^0}\sum_{i=1}^{d(l)}|a_i^l||f_i^l(z)|$$

for $z=ka_tK\in U_s$. Since

$$f_i^l(ka_t)=\sum_{j=1}^{d(l)}f_j^l(a_t)\bar{\tau}_{ij}^l(k),\quad |\bar{\tau}_{ij}^l(k)|\leqq 1,$$

we have

$$S(z)\leqq\sum_{l\in L^0}\sum_{i,j=1}^{d(l)}|a_i^l|\ |f_j^l(a_t)|.$$

From Proposition 3.6 and Lemma 4.2, we obtain

$$S(z)\leqq\sum_{l\in L^0}\sum_{i=1}^{d(l)}|a_i^l|\frac{d(l)^{1/2}\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l+1)}r^l, \tag{4.2}$$

where $r=\left|\tanh\frac{t}{2}\right|$. Put

$$c(l)=\frac{d(l)^{1/2}\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l+1)}\ .$$

Since $d(l)$ is a polynomial function in $l$ (Weyl's dimension formula), $c(l)$ is also a polynomial function in $l$. It is easy to see that

$$\lim_{l\to\infty}c(l)^{1/l}=1.$$

Therefore there exists an integer $l_0$ such that

$$c(l)^{1/l}\exp(-2s)\leqq\exp(-s)$$

for $l>l_0$. Then from (4.2) we have

$$\begin{aligned}S(z)&\leqq\sum_{l=0}^{l_0}\sum_{i=1}^{d(l)}|a_i^l|c(l)r^l\\&\quad+\sum_{l=l_0+1}^{\infty}\sum_{i=1}^{d(l)}|a_i^l|(c(l)^{1/l}r)^l\\&\leqq\sum_{l=0}^{l_0}\sum_{i=1}^{d(l)}|a_i^l|c(l)\\&\quad+\sum_{l=l_0+1}^{\infty}\sum_{i=1}^{d(l)}|a_i^l|\exp(-sl),\end{aligned}$$

since $z=ka_tK\in U_s$. On the other hand, from Lemma 4.1, we have

$$\exp(-sl)\leqq\exp(-s\lambda_l^{1/2}).$$

Therefore we find that

$$S(z)\leqq\sum_{l=0}^{l_0}\sum_{i=1}^{d(l)}|a_i^l|c(l)$$
$$+\sum_{l=l_0+1}^{\infty}\sum_{i=1}^{d(l)}|a_i^l|\exp(-s\lambda_l^{1/2}),$$

which is finite uniformly for $z\in U_s$, since $(a_i^l)\in\mathscr{F}_b$. This implies that the series

$$\sum_{l\in L^0}\sum_{i=1}^{d(l)}a_i^l f_i^l(z)$$

converges absolutely and uniformly in $U_s$. Since $f_i^l\in\mathscr{H}$ $(l\in L^0, 1\leqq i\leqq d(l))$ and every compact set is contained in $U_s$ for some $s>0$, it follows that

$$\sum_{l\in L^0}\sum_{i=1}^{d(l)}a_i^l f_i^l(z)$$

defines an element of $\mathscr{H}$. Thus we have

LEMMA 4.3. *Let $(a_i^l)\in\mathscr{F}_b$. Then the series*

$$\sum_{l\in L^0}\sum_{i=1}^{d(l)}a_i^l f_i^l(z)$$

*converges absolutely and uniformly in every compact subset of $X$ and defines a harmonic function on $X$.*

Conversely, if $f\in\mathscr{H}$, by Proposition 2.2, we have an expansion

$$f(z)=\sum_{l\in L^0}\sum_{i=1}^{d(l)}a_i^l f_i^l(z).$$

About this expansion we obtain the following

LEMMA 4.4. *The sequence $(a_i^l)$ in the above expansion lies in $\mathscr{F}_b$.*

PROOF. From Proposition 2.2 in § 2, we have

$$\|\phi_f^z\|^2=\sum_{l\in L^0}d(l)^{-1}\Big(\sum_{i=1}^{d(l)}|a_i^l|^2\Big)\Big(\sum_{j=1}^{d(l)}|f_j^l(z)|^2\Big)$$
$$\geqq\sum_{l\in L^0}\Big(\sum_{i=1}^{d(l)}|a_i^l|^2\Big)d(l)^{-1}|f_1^l(z)|^2.$$

Put $z=a_t$ and $r=\left|\tanh\frac{t}{2}\right|$. Then from Proposition 3.6 and Lemma 4.2 we have

$$\|\phi_f^z\|^2 \geqq (1-r^2)^{2(n-1)}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n-1)}\right)^2 \sum_{l\in L^0} \sum_{i=1}^{d(l)} \left(\frac{\Gamma(l+n-1)}{\Gamma\left(l+\frac{n}{2}\right)}\right)^2 |a_i^l|^2 r^{2l}.$$

Since

$$\left(\frac{\Gamma(l+n-1)}{\Gamma\left(l+\frac{n}{2}\right)}\right)^2 \geqq 1 \quad \text{for} \quad l \geqq 0 \quad \text{and} \quad n \geqq 3,$$

we have

$$\|\phi_f^z\|^2 \geqq \varepsilon_n (1-r^2)^{2(n-1)} \sum_{l\in L^0} \sum_{i=1}^{d(l)} |a_i^l|^2 r^{2l}, \tag{4.3}$$

where $\varepsilon_n = \Gamma(n/2)^2 \Gamma(n-1)^{-2}$. Take and fix an arbitrary $s>0$. Then we can find a $t \in \boldsymbol{R}$ such that

$$r = \left|\tanh\frac{t}{2}\right| = \exp\left(-\frac{s}{2\sqrt{2(n-2)}}\right).$$

Then from (4.3) we obtain

$$\|\phi_f^z\|^2 \geqq \varepsilon_n (1-r^2)^{2(n-1)} \sum_{l\in L^0} \sum_{i=1}^{d(l)} |a_i^l|^2 \exp\left(-\frac{sl}{\sqrt{2(n-2)}}\right).$$

Using Lemma 4.1, we find that

$$\|\phi_f^z\|^2 \geqq \varepsilon_n (1-r^2)^{2(n-1)} \sum_{l\in L^0} \sum_{i=1}^{d(l)} |a_i^l|^2 \exp(-s\lambda_l^{1/2})$$

for $z = a_t$, which implies that $(a_i^l) \in \mathscr{F}_b$ by the remark following Theorem 1.8 in § 1. This completes the proof.

We define the Poisson transform of a hyperfunction on $B$. Let $T \in \mathscr{B}$. Since $P(z, b)$ is a real analytic function in $b$, we can operate $T$ on $P(z, b)$. Then $T(P(z, b))$ is a function on $X$. We denote this function by $\mathscr{P}(T)$ and call it the Poisson transform of $T$. By Theorem 1.8, there exists an isomorphism $\Psi$ of $\mathscr{B}$ onto $\mathscr{F}_b$.

LEMMA 4.5. *Let $T \in \mathscr{B}$ and $(a_i^l) = \Psi(T)$. Then for any $z \in X$,*

$$\mathscr{P}(T)(z) = \sum_{l\in L^0} \sum_{i=1}^{d(l)} a_i^l f_i^l(z),$$

*which is absolutely convergent.*

PROOF. Fix an arbitrary $z$ in $X$. Then from Corollary 1 to Proposition 1.7, $P(z, b)$ has an expansion

$$P(z, b)=\sum_{l\in L^0}\sum_{i=1}^{d(l)}\phi_i^l(b)\int_K P(z, kM)\bar{\phi}_i^l(k)dk, \tag{4.4}$$

which converges in $\mathscr{A}(B)$. Since $P(z, b)$ is real-valued and

$$f_i^l(z)=\int_K P(z, kM)\phi_i^l(k)dk,$$

taking complex conjugate of (4.4), we have

$$P(z, b)=\sum_{l\in L^0}\sum_{i=1}^{d(l)} f_i^l(z)\bar{\phi}_i^l(b),$$

which also converges in $\mathscr{A}(B)$. Therefore

$$T(P(z, b))=\sum_{l\in L^0}\sum_{i=1}^{d(l)} f_i^l(z)T(\bar{\phi}_i^l),$$

since $T$ is continuous on $\mathscr{A}(B)$. From the definition of $\Psi(T)$, $a_i^l$ is equal to $T(\bar{\phi}_i^l)$, which finishes the proof.

Now we are in position to state the main

THEOREM 4.6. *Poisson transform $\mathscr{P}$ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{H}(X)$, where $X$ is a real hyperbolic space.*

PROOF. From Lemma 4.3 and Lemma 4.5, we can see that the image of hyperfunctions by $\mathscr{P}$ is contained in $\mathscr{H}$. Lemma 4.4 together with Lemma 4.5 implies that the mapping $\mathscr{P}$ is surjective. Let $\mathscr{P}(T)=0$. Then, putting $\Psi(T)=(a_i^l)$, we have

$$\sum_{l\in L^0}\sum_{i=1}^{d(l)} a_i^l f_i^l(z)=0$$

for any $z\in X$. Replacing $z$ by $ka_t$, we have from (2.5) and Proposition 3.6,

$$\sum_{l\in L^0}\left(\tanh\frac{t}{2}\right)^l F_l\left(\left(\tanh\frac{t}{2}\right)^2\right)\sum_{i=1}^{d(l)} a_i^l\phi_i^l(k)=0$$

for $k\in K$. Since $\phi_i^l$ are linearly independent, we can deduce that $a_i^l=0$ for $l\in L^0$ and $1\leqq i\leqq d(l)$. Hence $T=0$, which completes the proof of the theorem.

REMARK. We can indentify a $C^\infty$-function $\phi$ on $B$ with the hyperfunction defined by

$$\mathscr{A}(B)\ni\psi\mapsto\int_K\psi(k)\phi(k)dk.$$

Then the Poisson transform of a hyperfunction $\phi$ coincides with the Poisson transform of a $C^\infty$-function $\phi$ defined in § 2.

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*Department of Mathematics,*
*Faculty of Science,*
*Hiroshima University*