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An Independence Condition in Semi-Infinite Programs

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§1. Introduction

An independence condition was introduced by M. Ohtsuka [2] in relation with the condititonal Gauss variational problem. This notion was generalized by the author [3] and applied to the study of semi-infinite programs. It was shown in [3] that a decomposition theorem (Lemma 4 in [3]), i.e., the existence of a full system of components, plays an important role in the study of the conditional Gauss variational problem. One of our aims is to further generalize the independence condition and the decomposition theorem. By making use of our decomposition theorem, we shall study a change of values of semi-infinite programs.

§2. An independence condition

Denote by R^n the n-dimensional Euclidean space, by R_0^n the positive orthant of R^n and by e_k the vector in R^n whose j-th coordinate is equal to 0 if $j \neq k$ and 1 if $j=k$. We set $R=R^1$ and $R_0=R_0^1$. For a subset B of R^n, we denote by B° the interior of B in R^n. Denote by $((v, w))$ and $\|w\|$ the usual inner product of $v, w \in R^n$ and the usual distance from 0 to $w \in R^n$ respectively, i.e.,

$$((v, w)) = \sum_{j=1}^n r_j s_j \quad \text{and} \quad \|w\| = [((w, w))]^{1/2}$$

for $v=(r_1, \ldots, r_n)$ and $w=(s_1, \ldots, s_n)$.

Let X be a real linear space and P be a convex subset of X such that $0 \in P$. Let $f_i(x)$ $(i=1, \ldots, n)$ be a real-valued function defined on P satisfying the following conditions:

(a) $f_i(tx)=tf_i(x)$ for all $t \in R_0$ and $x \in P$ such that $tx \in P$,

(b) $f_i(x+y)=f_i(x)+f_i(y)$ for all $x, y \in P$ such that $x+y \in P$.

In case P is a convex cone, conditions (a) and (b) imply that $f_i(x)$ is positively homogeneous and additive.

Now we introduce an independence condition which coincides with the one in [3] in the case where P is a convex cone.

DEFINITION. Let $x \in P$. We say that $\{f_i\}=\{f_i; i=1, \ldots, n\}$ is x-independent if there exists a set $\{x_j\}=\{x_j;\ j=1, \ldots, n\}$ in P called *a system of components* of x such that $x-x_j \in P$ for each j and $\det(f_i(x_j)) \neq 0$, where $\det(a_{ij})$ denotes the determinant of a matrix (a_{ij}).

Let A be the transformation from P into R^n defined by

$$Ax=(f_1(x), \ldots, f_n(x)).$$

The condition $\det(f_i(x_j)) \neq 0$ in the above definition is equivalent to that $\{Ax_j;\ j=1, \ldots, n\}$ is linearly independent. Denote by $A(P)$ the image under A of P, i.e.,

$$A(P)=\{Ax;\ x \in P\}.$$

Clearly $0 \in A(P)$ and $A(P)$ is convex by conditions (a) and (b).

A system of components $\{x_j\}$ of x is called to be *full* if

$$x=\sum_{j=1}^{n} x_j.$$

In this case we say that x has a full system of components.

First we have

THEOREM 1. *Let u and x be elements of P and set $x^*=\varepsilon u+(1-\varepsilon)x$ with $0<\varepsilon<1$. If $\{f_i\}$ is u-independent, then $\{f_i\}$ is x^*-independent.*

PROOF. Let $\{u_j\}$ be a system of components of u. Then $\varepsilon u_j \in P$ and $x^*-\varepsilon u_j=\varepsilon(u-u_j)+(1-\varepsilon)x \in P$ for each j. It is clear that $\{A(\varepsilon u_j)\}$ is linearly independent, so that $\{\varepsilon u_j\}$ is a system of components of x^*.

We shall prepare

LEMMA 1. *Let B be a convex set in R^n such that $0 \in B$. If B contains a set of n vectors in B which is linearly independent, then B° is nonempty.*

PROOF. Let $\{z_j\}$ be the set of n vectors in B which is linearly independent. Then the set V defined by

$$V=\{\sum_{j=1}^{n} r_j z_j;\ (r_1, \ldots, r_n) \in R_0^n \text{ and } \sum_{j=1}^{n} r_j \leqq 1\}$$

is contained in B. Since V° is nonempty, our assertion is clear.

LEMMA 2. *Let B be a convex set in R^n such that $0 \in B$. If $z_0 \in B^\circ$ and $z_0 \neq 0$, then there exist a set $\{z_j\}$ of n vectors in B° and a set $\{a_j\}$ of n strictly positive numbers such that $\{z_j\}$ is linearly independent,*

$$z_0=\sum_{j=1}^{n} a_j z_j \quad \textit{and} \quad \sum_{j=1}^{n} a_j<1.$$

PROOF. Let us put $z_0=(c_1, \ldots, c_n)$ and choose ε_j in such a way that $\varepsilon_j=$

1 if $c_j \geqq 0$ and $\varepsilon_j = -1$ if $c_j < 0$. Let $r > 0$ be a number such that $z_0 + r\varepsilon_j e_j \in B^\circ$ for each j, m be the number of vanishing c_j's and v be the vector in R^n whose i-th coordinate b_i is equal to 0 if $c_i \neq 0$ and $b < 0$ if $c_i = 0$. We choose $|b|$ so small that $z_0 + v + r\varepsilon_j e_j \in B^\circ$ for each j and $r + mb > 0$. Let us take $z_j = z_0 + v + r\varepsilon_j e_j$ for each j. Writing

$$c_0 = \sum_{j=1}^n |c_j|, \qquad a_0 = c_0/(c_0 + r + mb),$$

$$a_j = [|c_j|(1 - a_0) - b_j \varepsilon_j a_0]/r,$$

we have $a_j > 0$ for each j,

$$\sum_{j=1}^n a_j = a_0 < 1 \qquad \text{and} \quad z_0 = \sum_{j=1}^n a_j z_j.$$

We show that $\{z_j\}$ is linearly independent. Suppose that $\sum_{j=1}^n t_j z_j = 0$ for $\{t_j\} \subset R$. Setting $t_0 = \sum_{j=1}^n t_j$, we have

$$\begin{aligned} 0 &= t_0(z_0 + v) + \sum_{j=1}^n r\varepsilon_j t_j e_j \\ &= t_0 \sum_{j=1}^n (c_j + b_j) e_j + \sum_{j=1}^n r\varepsilon_j t_j e_j \\ &= \sum_{j=1}^n [t_0(c_j + b_j) + r\varepsilon_j t_j] e_j. \end{aligned}$$

Since $\{e_j\}$ is linearly independent, we have

$$t_0(c_j + b_j) + r\varepsilon_j t_j = 0 \tag{1}$$

for each j. Multiplying both sides by ε_j, we have

$$t_0(|c_j| + \varepsilon_j b_j) + r t_j = 0,$$

so that

$$0 = \sum_{j=1}^n [t_0(|c_j| + \varepsilon_j b_j) + r t_j] = t_0(c_0 + mb + r).$$

Since $c_0 + mb + r > 0$, we have $t_0 = 0$ and hence $t_j = 0$ by (1). Namely $\{z_j\}$ is linearly independent and $\{z_j\}$ and $\{a_j\}$ satisfy our requirements.

We have

THEOREM 2. *If $z_0 \in A(P)^\circ$, then there exists $x \in P$ such that $Ax = z_0$ and $\{f_i\}$ is x-independent.*

PROOF. First we consider the case where $z_0 \neq 0$. Applying Lemma 2 with $B=A(P)$, we can find a set $\{z_j\}$ of n vectors in $A(P)^\circ$ and a set $\{a_j\}$ of n strictly positive numbers such that $\{z_j\}$ is linearly independent,

$$z_0 = \sum_{j=1}^{n} a_j z_j \qquad \text{and} \qquad \sum_{j=1}^{n} a_j < 1.$$

There exists $x_j \in P$ such that $Ax_j = z_j$ for each j. Taking $x = \sum_{j=1}^{n} a_j x_j$, we see that $x \in P$ and $Ax = z_0$ by conditions (a) and (b). Since $\{Ax_j\}$ is linearly independent, we see that $\{a_j x_j\}$ is a full system of components of x. In case $z_0 = 0$, let us choose $r > 0$ so small that

$$r \sum_{j=1}^{n} e_j \in A(P)^\circ \qquad \text{and} \qquad -re_j \in A(P)^\circ$$

for each j. There exists a set $\{u_j\}$ of $n+1$ elements in P such that

$$Au_j = -re_j \quad (j=1, \dots, n),$$

$$Au_{n+1} = r \sum_{j=1}^{n} e_j.$$

Writing $x_j = u_j/(n+1)$ for each j and $x = \sum_{j=1}^{n+1} x_j$, we have $x \in P$ and $Ax = 0$. It is clear that $\{x_j;\ j=1, \dots, n\}$ is a system of components of x. This completes the proof.

COROLLARY. *If $z_0 \in A(P)^\circ$, then there exists $x \in P$ and $t > 0$ such that $Ax = z_0$, $(1+t)x \in P$ and $\{f_i\}$ is $(1+t)x$-independent.*

PROOF. Since $z_0 \in A(P)^\circ$, there exists $t > 0$ such that $(1+t)z_0 \in A(P)^\circ$. On account of Theorem 2, there exists $x^* \in P$ such that $Ax^* = (1+t)z_0$ and $\{f_i\}$ is x^*-independent. Taking $x = x^*/(1+t)$, we see that $x \in P$, $Ax = z_0$ and $(1+t)x = x^* \in P$.

We shall often use

LEMMA 3. *Let $\{v_j;\ j=1, \dots, n\}$ be linearly independent in R^n. If $((v_j, w)) = 0$ for each j, then $w = 0$.*

PROOF. Let $v_j = (a_{1j}, \dots, a_{nj})$ and $w = (r_1, \dots, r_n)$. Then we have

$$\sum_{i=1}^{n} r_i a_{ij} = 0.$$

Since $\det(a_{ij}) \neq 0$, we have $r_i = 0$ for each i, i.e., $w = 0$.

We have

THEOREM 3. *If $\{f_i\}$ is x-independent, then $tAx \in A(P)^\circ$ for every t, $0 < t < 1$.*

PROOF. Let us put $z_0=Ax$ and let $\{x_j\}$ be a system of components of x. Since $\{Ax_j\}\subset A(P)$, $A(P)^\circ$ is nonempty by Lemma 1. Suppose that there exists t_0 such that $0<t_0<1$ and $t_0z_0\notin A(P)^\circ$. By means of the separation theorem ([1], p. 71, Proposition 1), we can find a nonzero $w\in R^n$ and $\alpha\in R$ such that

$$((t_0z_0,\ w))=\alpha\leqq((z,\ w))$$

for all $z\in A(P)$. Since $tz_0\in A(P)$ for every t, $0\leqq t\leqq 1$, we have $((z_0,\ w))=0$, so that $\alpha=0$. Since $Ax_j\in A(P)$ and $A(x-x_j)\in A(P)$ for each j, we have

$$0\leqq((Ax_j,\ w))\leqq((Ax,\ w))=((z_0,\ w))=0,$$

and hence $w=0$ by Lemma 3. This is a contradiction. Therefore $tz_0\in A(P)^\circ$ for every t, $0<t<1$.

COROLLARY. *Assume that $\{f_i\}$ is x-independent. If there exists $t>0$ such that $(1+t)x\in P$, then $z_0=Ax\in A(P)^\circ$.*

PROOF. Let ε be a number such that $0<\varepsilon<1$ and $\varepsilon<t$. It is valid that $(1+\varepsilon)x\in P$, so that $(1+\varepsilon)z_0=A((1+\varepsilon)x)\in A(P)$. Since $A(P)$ is convex, $\varepsilon z_0\in A(P)^\circ$ by Theorem 3 and z_0 lies on the segment connecting εz_0 and $(1+\varepsilon)z_0$, z_0 belongs to $A(P)^\circ$ ([1], p. 51, Proposition 15). This completes the proof.

§3. A decomposition theorem

We shall be concerned with the existence of a full system of components of x in case $\{f_i\}$ is x-independent.

For $x\in P$, we define $C[x]$ by

$$C[x]=\{u\in P;\ x-u\in P\}.$$

It is clear that $C[x]$ is convex and contains 0 and x. Denote by $Q(x)$ the convex cone generated by $A(C[x])$, i.e., $z\in Q(x)$ if and only if there exist $t>0$ and $u\in C[x]$ such that $z=tAu$.

We have

LEMMA 4. *If $\{f_i\}$ is x-independent, then $Ax\in Q(x)^\circ$.*

PROOF. Let $\{x_j\}$ be a system of components of x. Since $\{Ax_j\}\subset Q(x)$, $Q(x)^\circ$ is nonempty by Lemma 1. Suppose that $Ax\notin Q(x)^\circ$. There exist a nonzero $w\in R^n$ and $\alpha\in R$ such that

$$((Ax,\ w))=\alpha\leqq((z,\ w))$$

for all $z\in Q(x)$ by the separation theorem ([1], p. 71, Proposition 1). Since $Q(x)$

is a cone, we have $\alpha=0$. From $x_j \in C[x]$ and $x-x_j \in C[x]$ for each j, it follows that

$$0 \leqq ((Ax_j, w)) \leqq ((Ax, w)) = 0,$$

so that $w=0$ by Lemma 3. This is a contradiction. Therefore $Ax \in Q(x)^\circ$.

Now we shall prove the following decomposition theorem.

THEOREM 4. *Let $x \in P$ and $Ax \neq 0$. If $\{f_i\}$ is x-independent, then x has a full system of components.*

PROOF. Our assertion is clear in case $n=1$, so we assume $n \geqq 2$. We first show that there exist a set $\{x_j\}$ of n elements in $C[x]$ and a set $\{s_j\}$ of n strictly positive numbers such that $\{Ax_j\}$ is linearly independent, $nx_j \in C[x]$ for each j and

$$z_0 = Ax = \sum_{j=1}^{n} s_j Ax_j.$$

We can apply Lemma 2 with $B=Q(x)$, since $Q(x)$ is convex and contains 0, $z_0 \neq 0$ and $z_0 \in Q(x)^\circ$ by Lemma 4. There exist a set $\{z_j\}$ of n vectors in $Q(x)^\circ$ and a set $\{a_j\}$ of n strictly positive numbers such that $\{z_j\}$ is linearly independent and

$$z_0 = \sum_{j=1}^{n} a_j z_j.$$

There exist $u_j \in C[x]$ and $t_j > 0$ such that $t_j A u_j = z_j$ for each j. Taking $x_j = u_j/n$ and $s_j = n a_j t_j$, we see easily that $\{x_j\}$ and $\{s_j\}$ satisfy our requirements. In case $s_0 = \sum_{j=1}^{n} s_j \leqq 1$, we have $\bar{x} = \sum_{j=1}^{n} s_j x_j \in C[x]$ and $A\bar{x} = z_0$. By choosing

$$x_j^* = s_j x_j + (x - \bar{x})/n$$

for each j, we see that $\{x_j^*\}$ is a full system of components of x. In case $s_0 > 1$, we have

$$x_0 = \sum_{j=1}^{n} (s_j/s_0) x_j \in C[x] \qquad \text{and} \qquad Ax_0 = z_0/s_0.$$

Let us define x_j^* by

$$x_j^* = (s_j/s_0) x_j + (x - x_0)/n$$

for each j. Since $x - x_0 \in C[x]$ and $nx_j \in C[x]$, it is valid that $x_j^* \in C[x]$ for each j and $x = \sum_{j=1}^{n} x_j^*$. In order to prove that $\{x_j^*\}$ is a full system of components of x, it is enough to show that $\{Ax_j^*\}$ is linearly independent. Suppose that

$$\sum_{j=1}^{n} b_j A x_j^* = 0$$

for $\{b_j\} \subset \mathrm{R}$. Then it follows that

$$\begin{aligned} 0 &= \sum_{j=1}^{n} b_j (s_j/s_0) A x_j + (1-1/s_0) z_0 \sum_{k=1}^{n} (b_k/n) \\ &= \sum_{j=1}^{n} b_j (s_j/s_0) A x_j + (1-1/s_0) \sum_{j=1}^{n} s_j A x_j \sum_{k=1}^{n} (b_k/n) \\ &= \sum_{j=1}^{n} [b_j (s_j/s_0) + (1-1/s_0) s_j \sum_{k=1}^{n} (b_k/n)] A x_j. \end{aligned}$$

Since $\{Ax_j\}$ is linearly independent, we have

$$b_j (s_j/s_0) + (1-1/s_0) s_j \sum_{k=1}^{n} (b_k/n) = 0$$

for each j, so that

$$(2) \qquad b_j + (s_0 - 1) b_0/n = 0 \qquad \text{with} \quad b_0 = \sum_{j=1}^{n} b_j.$$

Thus we have

$$0 = \sum_{j=1}^{n} [b_j + (s_0 - 1) b_0/n] = s_0 b_0,$$

and hence $b_0 = 0$. Therefore $b_j = 0$ by (2). Namely $\{Ax_j^*\}$ is linearly independent. This completes the proof.

This is a generalization of Lemma 4 in [3].

THEOREM 5. *Let $x \in P$ and $Ax = 0$. If $\{f_i\}$ is x-independent, then x does not have a full system of components.*

PROOF. Suppose that there exists a full system of components $\{x_j\}$ of x. Then we have

$$0 = f_i(x) = \sum_{j=1}^{n} f_i(x_j)$$

for each i, so that

$$\begin{aligned} \sum_{j=1}^{n} A x_j &= (\sum_{j=1}^{n} f_1(x_j), \ldots, \sum_{j=1}^{n} f_n(x_j)) \\ &= (0, \ldots, 0) = 0. \end{aligned}$$

Namely $\{Ax_j\}$ is linearly dependent. This is a contradiction. Therefore x does not have a full system of components.

§4. Semi-infinite programs

Let $g(x)$ be a real-valued function defined on P. Given $z_0 \in R^n$, let us consider the following semi-infinite program:

(I) Minimize $g(x)$

subject to $x \in P$ and $Ax = z_0$.

Problem (I) was studied in [3] in the case where P is a convex cone and $g(x)$ is positively homogeneous and convex. We investigated the problem how the value of problem (I) changes as g and f_i change by using a duality theorem. We shall be concerned with an analogous problem by using our decomposition theorem.

Assume that $g(x)$ satisfies conditions (a) and (b) in §2. Let $\{f_i^{(p)}\}$ and $\{g^{(p)}\}$ be sequences of real-valued functions on P which converge (pointwise) to f_i and g respectively and $\{z^{(p)}\}$ be a sequence of vectors in R^n which converges to z_0. Here we assume that $f_i^{(p)}$ and $g^{(p)}$ satisfy conditions (a) and (b). Let us put

$$A_p x = (f_1^{(p)}(x), \dots, f_n^{(p)}(x)),$$

$$S^{(p)} = \{x \in P;\ A_p x = z^{(p)}\}, \qquad S = \{x \in P;\ Ax = z_0\},$$

$$M_p = \inf\{g^{(p)}(x);\ x \in S^{(p)}\}, \qquad M = \inf\{g(x);\ x \in S\}.$$

Here we use the convention that the infimum of a real-valued function on the empty set is equal to ∞.

We shall prove

THEOREM 6. *If $z_0 \in A(P)^\circ$ and $z_0 \neq 0$, then it is valid that $\overline{\lim}_{p\to\infty} M_p \leqq M$.*

PROOF. We may assume that $M < \infty$. For any $\alpha > M$, there is $x \in S$ such that $g(x) < \alpha$. Since $z_0 \in A(P)^\circ$, there exist $\bar{x} \in P$ and $t > 0$ such that $(1+t)\bar{x} \in P$, $A\bar{x} = z_0$ and $\{f_i\}$ is $(1+t)\bar{x}$-independent by the corollary of Theorem 2. Writing $x^* = \varepsilon(1+t)\bar{x} + (1-\varepsilon)x$ with $0 < \varepsilon < 1$, we see that $x^* \in P$, $Ax^* = (1+\varepsilon t)z_0$ and $\{f_i\}$ is x^*-independent by Theorem 1. Since $Ax^* \neq 0$, there exists a full system of components $\{x_j^*\}$ of x^* by Theorem 4. First we show that there exists p_0 such that $\{A_p x_j^*\}$ is linearly independent for all $p \geqq p_0$. Supposing the contrary, we may assume that there exists $w^{(p)} = (s_1^{(p)}, \dots, s_n^{(p)}) \in R^n$ such that $\|w^{(p)}\| = 1$ and

$$\sum_{j=1}^{n} s_j^{(p)} A_p x_j^* = 0$$

for infinitely many p. By choosing a subsequence if necessary, we may assume that $\{w^{(p)}\}$ converges to $w = (s_1, \dots, s_n)$. Then we have

$$\|w\|=1 \qquad \text{and} \quad \sum_{j=1}^{n} s_j A x_j^* = 0,$$

since $\lim_{p\to\infty} A_p x_j^* = Ax_j^*$ for each j. This contradicts the fact that $\{Ax_j^*\}$ is linearly independent. Therefore there exists p_0 such that $\{A_p x_j^*\}$ is linearly independent for all $p \geqq p_0$. Let D_p^* be the convex set in R^n defined by

$$D_p^* = \{\sum_{j=1}^{n} r_j A_p x_j^*; (r_1, \dots, r_n) \in R_0^n \qquad \text{and} \quad \sum_{j=1}^{n} r_j x_j^* \in P\}.$$

We show that there exists p_1 such that $z^{(p)} \in D_p^*$ for all $p \geqq p_1$. Supposing the contrary, we have $z^{(p)} \notin D_p^*$ for infinitely many p. In case $z^{(p)} \notin D_p^*$ and $p \geqq p_0$, we can find $y^{(p)} \in R^n$ and $\alpha_p \in R$ such that $\|y^{(p)}\|=1$ and

$$((z^{(p)}, y^{(p)})) = \alpha_p \leqq ((z, y^{(p)}))$$

for all $z \in D_p^*$ by the separation theorem ([1], p. 71, Proposition 1), since $(D_p^*)^\circ$ is nonempty by Lemma 1. By choosing a subsequence of $\{y^{(p)}\}$ if necessary, we may assume that $\{y^{(p)}\}$ and $\{\alpha_p\}$ converge to y and α respectively. It follows that $\|y\|=1$, $\alpha \leqq 0$,

$$((z_0, y)) = \alpha \leqq ((Ax_j^*, y))$$

for each j and

$$\alpha \leqq ((\sum_{j=1}^{n} Ax_j^*, y)) = ((Ax^*, y)) = (1+\varepsilon t)\alpha.$$

Therefore $\alpha \geqq 0$ and hence $\alpha = 0$. Thus we have $((Ax_j^*, y)) = 0$ for each j, so that $y=0$ by Lemma 3. This is a contradiction. Therefore there exists p_1 such that $p_1 > p_0$ and $z^{(p)} \in D_p^*$ for all $p \geqq p_1$. For $z^{(p)} \in D_p^*$, there exists a unique $v^{(p)} = (r_1^{(p)}, \dots, r_n^{(p)}) \in R_0^n$ such that

$$z^{(p)} = \sum_{j=1}^{n} r_j^{(p)} A_p x_j^* \qquad \text{and} \quad x^{(p)} = \sum_{j=1}^{n} r_j^{(p)} x_j^* \in P. \tag{3}$$

It is valid that $x^{(p)} \in S^{(p)}$ and

$$M_p \leqq g^{(p)}(x^{(p)}) = \sum_{j=1}^{n} r_j^{(p)} g^{(p)}(x_j^*) \tag{4}$$

for all $p \geqq p_1$. Next we show that $\{\|v^{(p)}\|\}$ is bounded. Supposing the contrary, we may assume that $v^{(p)} \neq 0$ for all p and $\|v^{(p)}\| \to \infty$ as $p \to \infty$. Setting $u^{(p)} = v^{(p)}/\|v^{(p)}\| = (b_1^{(p)}, \dots, b_n^{(p)})$, we have $\|u^{(p)}\|=1$ and

$$\sum_{j=1}^{n} b_j^{(p)} A_p x_j^* = z^{(p)}/\|v^{(p)}\|.$$

By choosing a subsequence if necessary, we may assume that $\{u^{(p)}\}$ converges to $u=(b_1, \dots, b_n)$. It follows that $\|u\|=1$ and

$$\sum_{j=1}^{n} b_j A x_j^* = 0,$$

which contradicts that $\{Ax_j^*\}$ is linearly independent. Therefore $\{\|v^{(p)}\|\}$ is bounded. Let $v=(r_1, \dots, r_n)$ be an accumlation vector of $\{v^{(p)}\}$. Then we have by (3)

$$\sum_{j=1}^{n} r_j A x_j^* = z_0. \tag{5}$$

On the other hand, we have

$$\sum_{j=1}^{n} A x_j^* = A x^* = (1+\varepsilon t) z_0. \tag{6}$$

Since $\{Ax_j^*\}$ is linearly independent, we have by (5) and (6) that $r_j = 1/(1+\varepsilon t)$ for each j. Thus we have shown that

$$\lim_{p\to\infty} r_j^{(p)} = 1/(1+\varepsilon t) \tag{7}$$

for each j. On account of (4) and (7), we have

$$\begin{aligned}\overline{\lim_{p\to\infty}}\, M_p &\leqq \lim_{p\to\infty} \sum_{j=1}^{n} r_j^{(p)} g^{(p)}(x_j^*) = g(x^*)/(1+\varepsilon t) \\ &= [\varepsilon(1+t) g(\bar{x}) + (1-\varepsilon) g(x)]/(1+\varepsilon t).\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\overline{\lim_{p\to\infty}}\, M_p \leqq g(x) < \alpha.$$

By the arbitrariness of α, we obtain the desired inequality.

This is an improvement of Theorem 10 in [3].

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