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On the K-Ring of S^{4n+3}/H_m

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§ 1. Introduction

The purpose of this note is to study the K-ring $K(N^n(m))$ of complex vector bundles over the $(4n+3)$-dimensional quotient manifold

$$N^n(m)=S^{4n+3}/H_m, \qquad (m\geqq 2).$$

Here, H_m is the generalized quaternion group generated by the two elements x and y with the two relations

$$x^{2^{m-1}}=y^2 \quad \text{and} \quad xyx=y,$$

that is, H_m is the subgroup of the unit sphere S^3 in the quaternion field $\boldsymbol{H}$ generated by the two elements

$$x=\exp(\pi i/2^{m-1}) \quad \text{and} \quad y=j,$$

and the action of H_m on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space $\boldsymbol{H}^{n+1}$ is given by the diagonal action.

Recently, the problem of immersing or embedding this manifold $N^n(m)$ in euclidean spaces is studied in [8].

Let α' and β' be the complex line bundles over $N^n(m)$ whose first Chern classes are the generators of $H^2(N^n(m);\ Z)=Z_2\oplus Z_2$, and $\delta'=\pi^!\lambda$ be the complex plane bundle over $N^n(m)$ induced from the canonical complex plane bundle λ over the quaternion projective space HP^n by the natural projection

$$\pi\colon N^n(m)\longrightarrow HP^n.$$

Then we have the following

THEOREM 1.1. *The reduced K-ring* $\tilde{K}(N^n(m))$ $(m\geqq 2)$ *is generated multiplicatively by the three elements*

$$\alpha=\alpha'-1, \quad \beta=\beta'-1 \quad \textit{and} \quad \delta=\delta'-2.$$

This theorem shows that the natural ring homomorphism

$$\xi\colon \tilde{R}(H_m)\longrightarrow \tilde{K}(N^n(m))$$

is an epimorphism, where $\tilde{R}(H_m)$ is the reduced (unitary) representation ring.

For the case $m=2$, $H_2=\{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group and we have

THEOREM 1.2. *As an abelian group,*

$$\tilde{K}(N^n(2))=Z_{2^{n+1}}\oplus Z_{2^{n+1}}\oplus Z_{2^{2n+1}}\oplus Z_{2^{n-1}}$$

for $n\geqq 1$ and the direct summands are generated by the elements

$$\alpha,\ \beta,\ \delta \quad \text{and} \quad \delta^2+4\delta+2^{n+1}\delta,$$

respectively. Also $\tilde{K}(N^0(2))=Z_2\oplus Z_2$ is generated by the two elements α and β. The multiplicative structure of $\tilde{K}(N^n(2))$ $(n\geqq 0)$ is given by

$$\alpha^2=-2\alpha,\quad \beta^2=-2\beta,\quad \alpha\beta=-2\alpha-2\beta+4\delta+\delta^2,$$

$$\alpha\delta=-2\alpha,\quad \beta\delta=-2\beta,\quad \delta^{n+1}=0.$$

This note is constructed as follows. In §2, a CW-decomposition of $N^n(m)$ is given to have the cohomology groups of this manifold. Moreover, the order of $\tilde{K}(N^n(m))$ is determined by using the Atiyah-Hirzebruch spectral sequence. In §3, the unitary representation rings $R(H_m)$, $R(Z_{2^m})$ and $R(S^3)$ of the groups H_m, Z_{2^m} and S^3 are considered. Considering the inclusions $\rho\colon Z_{2^m}\longrightarrow H_m$, $\rho'\colon Z_4\longrightarrow H_m$ defined by $\rho(z)=x$, $\rho'(z)=y$ for the generator z of the cyclic groups, and the natural projections

$$\rho\colon L^{2n+1}(2^m)\longrightarrow N^n(m),\quad \rho'\colon L^{2n+1}(4)\longrightarrow N^n(m),$$

where $L^{2n+1}(k)=S^{4n+3}/Z_k$ is the standard lens space mod k, we determine the images of α, β and δ by the induced ring homomorphisms $\rho^!$ and $\rho'^!$ in §4. Then, Theorem 1.1 is proved in §5 by the induction on the skeletons of $N^n(m)$. Finally, Theorem 1.2 is proved in §6 by using the above results.

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§2. A CW-decomposition and the cohomology groups of $N^n(m)$

The generalized quaternion group H_m $(m\geqq 2)$ is the subgroup of the unit sphere S^3 in the quaternion field $\boldsymbol{H}$, generated by the two elements

$$x=\exp(\pi i/2^{m-1}) \quad \text{and} \quad y=j.$$

In this note, we consider the diagonal action of H_m on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space $\boldsymbol{H}^{n+1}$, given by

$$q(q_1,\dots, q_{n+1})=(qq_1,\dots, qq_{n+1})$$

for $q\in H_m$ and $(q_1,\dots, q_{n+1})\in S^{4n+3}$. Regarding $\boldsymbol{H}$ as the complex 2-space $\boldsymbol{C}^2$, by the replacement $q=z+jz'$, this H_m-action is given by

$$x(z_1, z_2,\dots, z_{2n+1}, z_{2n+2})=(xz_1, x^{-1}z_2,\dots, xz_{2n+1}, x^{-1}z_{2n+2}),$$

$$y(z_1, z_2,\dots, z_{2n+1}, z_{2n+2})=(-z_2, z_1,\dots, -z_{2n+2}, z_{2n+1})$$

for $(z_1, z_2,\dots, z_{2n+1}, z_{2n+2})\in S^{4n+3}$.

In this section, we give an H_m-equivariant CW-decomposition of S^{4n+3}, which induces a CW-decomposition of the manifold $N^n(m)=S^{4n+3}/H_m$, and we determine the cohomology groups of $N^n(m)$.

We consider the following cells in S^{4n+3}, for $0\leqq k\leqq n$, $0\leqq j<2^m$ and $\varepsilon=0, 1$:

$$e^{4k}_{j,\varepsilon}=\{(z_1,\dots, z_{2k+2}, 0,\dots, 0);\ z_{2k+1+\varepsilon}\neq 0,\ z_{2k+2-\varepsilon}=0,$$
$$\arg z_{2k+1+\varepsilon}=(-1)^{\varepsilon}j\pi/2^{m-1}\},$$

$$e^{4k+1}_{j,\varepsilon}=\{(z_1,\dots, z_{2k+2}, 0,\dots, 0);\ z_{2k+1+\varepsilon}\neq 0,\ z_{2k+2-\varepsilon}=0,$$
$$(-1)^{\varepsilon}j\pi/2^{m-1}<\arg z_{2k+1+\varepsilon}<(1+(-1)^{\varepsilon}j)\pi/2^{m-1}\},$$

$$e'^{4k+1}_{j,\varepsilon}=\{(z_1,\dots, z_{2k+2}, 0,\dots, 0);\ z_{2k+1}\neq 0,\ z_{2k+2}\neq 0,$$
$$\arg z_{2k+1}-\varepsilon\pi=-\arg z_{2k+2}=j\pi/2^{m-1}\},$$

$$e^{4k+2}_{j,\varepsilon}=\{(z_1,\dots, z_{2k+2}, 0,\dots, 0);\ z_{2k+1}\neq 0,\ z_{2k+2}\neq 0,$$
$$\arg z_{2k+2-\varepsilon}=\varepsilon\pi+(-1)^{\varepsilon+1}j\pi/2^{m-1},$$
$$(-1)^{\varepsilon}j\pi/2^{m-1}<\arg z_{2k+1+\varepsilon}<\pi+(-1)^{\varepsilon}j\pi/2^{m-1}\},$$

$$e'^{4k+2}_{j,\varepsilon}=\{(z_1,\dots, z_{2k+2}, 0,\dots, 0);\ z_{2k+1}\neq 0,\ z_{2k+2}\neq 0,$$
$$(j-\varepsilon)\pi/2^{m-1}<\arg z_{2k+1}-\varepsilon\pi=-\arg z_{2k+2}<(j+1-\varepsilon)\pi/2^{m-1}\},$$

$$e^{4k+3}_{j,\varepsilon}=\{(z_1,\dots, z_{2k+2}, 0,\dots, 0);\ z_{2k+1}\neq 0,\ z_{2k+2}\neq 0,$$
$$(j-\varepsilon)\pi/2^{m-1}<\arg z_{2k+1}-\varepsilon\pi+(\varepsilon-1)\theta=$$
$$\varepsilon\theta-\arg z_{2k+2}<(j+1-\varepsilon)\pi/2^{m-1},\ 0<\theta<\pi\}.$$

Set $e^{4k+s}=e^{4k+s}_{0,0}$ and $e'^{4k+t}=e'^{4k+t}_{0,0}$. Then, it is easy to show that

$$e^{4k+s}_{j,\varepsilon}=x^j y^{\varepsilon} e^{4k+s},\quad e'^{4k+t}_{j,\varepsilon}=x^j y^{\varepsilon} e'^{4k+t},$$

and $\{e^{4k+s}_{j,\varepsilon}, e'^{4k+t}_{j,\varepsilon};\ 0\leqq k\leqq n,\ 0\leqq j<2^m,\ 0\leqq s\leqq 3,\ t=1, 2,\ \varepsilon=0, 1\}$ gives an H_m-

equivariant CW-decomposition of S^{4n+3}, with the boundary formulas

$$\partial e^{4k} = \Sigma_{q\in H_m} q e^{4k-1},$$

$$\partial e^{4k+1} = (x-1)e^{4k}, \quad \partial e'^{4k+1} = (y-1)e^{4k},$$

$$\partial e^{4k+2} = (1+x+x^2+\cdots+x^{2^{m-1}-1})e^{4k+1} - (y+1)e'^{4k+1},$$

$$\partial e'^{4k+2} = (xy+1)e^{4k+1} + (x-1)e'^{4k+1},$$

$$\partial e^{4k+3} = (x-1)e^{4k+2} - (xy-1)e'^{4k+2}.$$

Let ξ: $S^{4n+3} \to S^{4n+3}/H_m = N^n(m)$ be the natural projection, and set

$$e^{4k+s} = \xi(e^{4k+s}) \qquad \text{for } s=0, 3,$$

$$e_1^{4k+t} = \xi(e^{4k+t}), \quad e_2^{4k+t} = \xi(e'^{4k+t}) \qquad \text{for } t=1, 2.$$

Then, we have obtained the following

LEMMA 2.1. *The set* $\{e^{4k+s}, e_1^{4k+t}, e_2^{4k+t};\ 0\leqq k\leqq n,\ s=0, 3,\ t=1, 2\}$ *is a CW-decomposition of the manifold* $N^n(m)$, *with the boundary formulas*:

$$\partial e^{4k} = 2^{m+1} e_1^{4k-1}, \quad \partial e_1^{4k+1} = \partial e_2^{4k+1} = 0,$$

$$\partial e_1^{4k+2} = 2^{m-1} e_1^{4k+1} - 2e_2^{4k+1}, \quad \partial e_2^{4k+2} = 2e_1^{4k+1}, \quad \partial e^{4k+3} = 0.$$

This implies

PROPOSITION 2.2. [3, Ch.XII, §7] *The integral cohomology groups of* $N^n(m)$ *are given by*

$$H^k(N^n(m); Z) = \begin{cases} Z & \text{for } k=0, 4n+3, \\ Z_{2^{m+1}} & \text{for } k\equiv 0(4),\ 0<k<4n+3, \\ Z_2\oplus Z_2 & \text{for } k\equiv 2(4),\ 0<k<4n+3, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $K(X)$ be the K-ring of complex vector bundles over a topological space X, and $\tilde{K}(X)$ be its reduced K-ring. Let $\{E_r^{p,q}\}$ be the Atiyah-Hirzebruch spectral sequence for $\tilde{K}(N^n(m))$ (cf. [2, §2]). Then, by the above proposition, we have

$$E_2^{p,q} = H^p(N^n(m); K^{-q}(P))$$

$$= \begin{cases} Z & \text{for } q \text{ even},\ p=0, 4n+3, \\ Z_{2^{m+1}} & \text{for } q \text{ even},,\ p\equiv 0(4),\ 0<p<4n+3, \end{cases}$$

$$\begin{cases} Z_2 \oplus Z_2 & \textit{for } q \textit{ even, } p \equiv 2(4),\ 0 < p < 4n+3, \\ 0 & \textit{otherwise,} \end{cases}$$

where P is a single point. Therefore, the differentials of this spectral sequence are trivial, and we have easily

PROPOSITION 2.3. *$\tilde{K}(N^n(m))$ consists of $2^{n(m+3)+2}$ elements, and $\tilde{K}^1(N^n(m)) = Z$.*

Since $H^2(N^n(m); Z) = Z_2 \oplus Z_2$, there are two complex line bundles α' and β' over $N^n(m)$, whose first Chern classes generate $H^2(N^n(m); Z)$. Then, by using the above spectral sequence and the Chern classes, we have easily the following

LEMMA 2.4. *For the 3-manifold $N^0(m)$,*

$$\tilde{K}(N^0(m)) = Z_2 \oplus Z_2,$$

generated by $\alpha = \alpha' - 1$ and $\beta = \beta' - 1$, and $\alpha^2 = \beta^2 = \alpha\beta = 0$.

§ 3. The representation ring $R(H_m)$

In this section, we consider the (unitary) representation ring $R(H_m)$ of H_m ($m \geqq 2$) (cf. [4, §47.15, Example 2]).

The conjugate classes of H_m are given by

$$C_0 = \{x^{2i}y;\ i = 0, 1, \dots, 2^{m-1} - 1\},$$

$$C_1 = \{x^{2i+1}y;\ i = 0, 1, \dots, 2^{m-1} - 1\},$$

$$C_{j+2} = \{x^j, x^{-j}\} \qquad \text{for } j = 0, 1, \dots, 2^{m-1}.$$

Also, H_m has four representations of degree 1:

F_0 = the unit representation,

$$\begin{cases} F_1(x) = 1 \\ F_1(y) = -1, \end{cases} \qquad \begin{cases} F_2(x) = -1 \\ F_2(y) = 1, \end{cases} \qquad \begin{cases} F_3(x) = -1 \\ F_3(y) = -1, \end{cases}$$

and $2^{m-1} - 1$ representations of degree 2:

$$F_{i+3}(x) = \begin{pmatrix} x^i & 0 \\ 0 & x^{-i} \end{pmatrix}, \qquad F_{i+3}(y) = \begin{pmatrix} 0 & (-1)^i \\ 1 & 0 \end{pmatrix}$$

for $i = 1, 2, \dots, 2^{m-1} - 1$.

Then, we see that these are all of the irreducible representations of H_m, by the following character table, where χ_j is the character of F_j for $j = 0, 1, \dots, 2^{m-1} + 2$.

	C_0	C_1	C_{j+2} $(j=0,\dots,2^{m-1})$
$\chi_0=1$	1	1	1
χ_1	-1	-1	1
χ_2	1	-1	$(-1)^j$
χ_3	-1	1	$(-1)^j$
χ_{i+3} $(i=1,\dots,2^{m-1}-1)$	0	0	$x^{ij}+x^{-ij}$

Furthermore, the multiplicative structure of $R(H_m)$, which is given by the tensor product of characters, can be determined by the routine calculations using the above table, and we have the following

PROPOSITION 3.1. (cf. [8, §1]) *The representation ring $R(H_m)$ is a free Z-module genrerated by χ_j, $j=0, 1,\dots, 2^{m-1}+2$, with relations*:

$$\chi_0=1,\quad \chi_i\chi_j=\chi_j\chi_i,\quad \chi_1^2=\chi_2^2=1,$$

$$\chi_3=\chi_1\chi_2,\quad \chi_1\chi_4=\chi_4,\quad \chi_2\chi_4=\chi_{2^{m-1}+2},$$

$$\chi_4^2=\begin{cases}1+\chi_1+\chi_2+\chi_3 & \text{for } m=2,\\ 1+\chi_1+\chi_5 & \text{for } m\geqq 3,\end{cases}$$

$$\chi_{i+1}=\chi_4\chi_i-\chi_{i-1}\qquad \text{for } i\geqq 5.$$

REMARK 3.2. *The following equality can be proved by the above relations.*

$$\chi_i=\sum_{j=0}^{[(i-4)/2]}(-1)^j\left\{\binom{i-4-j}{j}+\binom{i-4-j}{i-1}\right\}\chi_4^{i-2j-3}+\varepsilon(i)(-1)^{[(i+1)/2]}(\chi_1+1)$$

for $m\geqq 3$, $i\geqq 5$, where $\varepsilon(i)=0$ if i is even and $=1$ if i is odd.

For the reduced representation ring $\tilde{R}(H_m)$, which is the kernel of the augmentation homomorphism

$$\deg\colon R(H_m)\longrightarrow Z,$$

we have

PROPOSITION 3.3. *The commutative ring $\tilde{R}(H_m)$ is a free Z-module generated by*

$$\alpha=\chi_1-1,\quad \beta=\chi_2-1,\quad \gamma=\chi_1+\chi_2+\chi_3-3,$$

$$\delta_i = \chi_{i+3} - 2 \qquad \textit{for } 1 \leqq i < 2^{m-1},$$

with relations

$$\alpha^2 = -2\alpha, \quad \beta^2 = -2\beta, \quad \gamma = \alpha\beta + 2\alpha + 2\beta,$$

$$\alpha\delta_1 = -2\alpha, \quad \beta\delta_1 = -2\beta + \delta_{2^{m-1}-1} - \delta_1,$$

$$\delta_1^2 = \begin{cases} -4\delta_1 + \gamma & \textit{for } m = 2, \\ -4\delta_1 + \delta_2 + \alpha & \textit{for } m \geqq 3, \end{cases}$$

$$\delta_{i+1} = \delta_1\delta_i + 2\delta_1 + 2\delta_i - \delta_{i-1} \qquad \textit{for } m \geqq 3,\ i \geqq 2.$$

These show that $\tilde{R}(H_m)$ *is generated by* α, β *and* δ_1 *as a ring.*

Now, let

$$\pi\colon H_m \longrightarrow S^3 \tag{3.4}$$

be the inclusion, and let

$$\rho\colon Z_{2^m} \longrightarrow H_m, \qquad \rho'\colon Z_4 \longrightarrow H_m \tag{3.5}$$

be the inclusions such that $\rho(z) = x$, $\rho'(z) = y$ for the generator z of the cyclic group Z_k. The ring homomorphisms induced by these inclusions are denoted by the same letters:

$$\begin{aligned} &\pi\colon R(S^3) \longrightarrow R(H_m), \\ &\rho\colon R(H_m) \longrightarrow R(Z_{2^m}), \qquad \rho'\colon R(H_m) \longrightarrow R(Z_4). \end{aligned} \tag{3.6}$$

The following lemmas are well known:

Lemma 3.7. (cf. [5, Ch.13, Th. 3.1]) *$R(S^3)$ is the polynomial ring $Z[\zeta]$, where ζ is given by the representation*

$$z_1 + jz_2 \longrightarrow \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \qquad \textit{for } z_1 + jz_2 \in S^3.$$

Lemma 3.8. (cf. [1, §8]) *$R(Z_{2^m})$ is the truncated polynomial ring $Z[\chi]/ <\chi^{2^m} - 1>$, where χ is given by $z \to \exp(\pi i/2^{m-1})$ for the generator z of Z_{2^m}.*

By the definitions, we have easily the following equalities for the homomorphisms of (3.6):

$$\pi(\zeta) = \chi_4;$$

$$\rho(\chi_1)=1, \quad \rho(\chi_2)=\chi^{2^{m-1}}, \quad \rho(\chi_4)=\chi+\bar{\chi};$$

$$\rho'(\chi_1)=\chi^2, \quad \rho'(\chi_2)=1, \quad \rho'(\chi_4)=\chi+\bar{\chi};$$

where $\bar{\chi}$ is the conjugation of χ. These show the following

Proposition 3.9. *For the induced homomorphisms*

$$\pi: \tilde{R}(S^3) \longrightarrow \tilde{R}(H_m),$$

$$\rho: \tilde{R}(H_m) \longrightarrow \tilde{R}(Z_{2^m}), \quad \rho': \tilde{R}(H_m) \longrightarrow \tilde{R}(Z_4)$$

of (3.6), *we have the following equalities*:

$$\pi(\zeta-2)=\delta_1;$$

$$\rho(\alpha)=0, \quad \rho(\beta)=(\sigma+1)^{2^{m-1}}-1, \quad \rho(\delta_1)=\sigma^2/(1+\sigma);$$

$$\rho'(\alpha)=(\sigma+1)^2-1, \quad \rho'(\beta)=0, \quad \rho'(\delta_1)=\sigma^2/(1+\sigma),$$

where $\sigma=\chi-1$.

§ 4. Some elements of $\tilde{K}(N^n(m))$

Assume that a topological group G acts on a topological space X without fixed point. Then, the natural projection

$$p: X \longrightarrow X/G$$

defines the ring homomorphisms

$$p: R(G) \longrightarrow K(X/G), \quad p: \tilde{R}(G) \longrightarrow \tilde{K}(X/G)$$

as follows (cf. [5, Ch. 12, 5.4]): For an n-dimensional representation ω of G, $p(\omega)$ is the complex n-plane bundle induced from the principal G-bundle $p: X \to X/G$ by the group homomorphism $\omega: G \to GL(n, \boldsymbol{C})$. Furthermore, if H is a subgroup of G, then the inclusion $i: H \to G$ and the natural projections $p': X \to X/H$, $i: X/H \to X/G$ induce the following commutative diagram

$$\begin{array}{ccc} \tilde{R}(G) & \xrightarrow{p} & \tilde{K}(X/G) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i^!} \\ \tilde{R}(H) & \xrightarrow{p'} & \tilde{K}(X/H). \end{array}$$

Now, considering the projection

$$\xi: S^{4n+3} \longrightarrow N^n(m)=S^{4n+3}/H_m,$$

we define the elements

(4.1) $\alpha=\xi(\alpha),\quad \beta=\xi(\beta),\quad \delta=\xi(\delta_1)\qquad \text{in } \tilde{K}(N^n(m)),$

which are the images of α, β and δ_1 in Proposition 3.3 by the ring homomorphism $\xi\colon \tilde{R}(H_m)\to\tilde{K}(N^n(m))$. It is easy by the definitions to show that $\alpha'=\xi(\chi_1)$ and $\beta'=\xi(\chi_2)$ are the complex line bundles over $N^n(m)$ whose first Chern classes generate $H^2(N^n(m); Z)=Z_2\oplus Z_2$, where χ_1 and χ_2 are the representations in Proposition 3.1 (cf. [1, Appendix, (3)]). Therefore,

LEMMA 4.2. *Lemma* 2.4 *holds for the elements α and β of* (4.1).

The K-ring $K(HP^n)$ of the quaternion projective space $HP^n=S^{4n+3}/S^3$ is given by

(4.3) $K(HP^n)=Z[\nu]/<\nu^{n+1}>,$

where $\nu=\lambda-2$ and λ is the canonical complex plane bundle over HP^n (cf. [9, Th. 3.12]).

LEMMA 4.4. $\pi^!(\nu)=\delta,$
where $\pi^!\colon \tilde{K}(HP^n)\to\tilde{K}(N^n(m))$ is the induced homomorphism of the natural projection $\pi\colon N^n(m)\to HP^n$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{R}(S^3) & \xrightarrow{\xi'} & \tilde{K}(HP^n) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^!} \\ \tilde{R}(H_m) & \xrightarrow{\xi} & \tilde{K}(N^n(m)), \end{array}$$

where $\xi'\colon S^{4n+3}\to HP^n$ is the projection. Then we have easily $\xi'(\zeta-2)=\nu$ by definitions, where ζ is the representation of Lemma 3.7 (cf. [5, Ch. 13, Th. 3.1]). Since $\pi(\zeta-2)=\delta_1$ by Proposition 3.9, we have the desired result. q.e.d.

Let N^k be the k-skeleton of the CW-complex $N^n(m)$ of Lemma 2.1 and $i\colon N^k\to N^n(m)$ be the inclusion. For an element $a\in\tilde{K}(N^n(m))$, we denote its image $i^!a$ by the same letter a.

The following lemma is used in the next section.

LEMMA 4.5. *The element $\alpha^i\beta^j\delta^k$ is zero in $\tilde{K}(N^{2i+2j+4k-1})$, where α, β and δ are the elements of* (4.1).

PROOF. It is clear that α and β are zero in $\tilde{K}(N^1)=0$. The fact that δ is zero in $\tilde{K}(N^3)=\tilde{K}(N^0(m))$ follows immediately from Lemma 4.4. Therefore, we have the lemma by the obvious fact that ab is zero in $\tilde{K}(N^{p+q-1})$ if a is zero in $\tilde{K}(N^{p-1})$ and b is zero in $\tilde{K}(N^{q-1})$ (cf. [2, (5) in p. 20]). q.e.d.

The K-ring $K(L^n(k))$ of the standard lens space mod k $L^n(k)=S^{2n+1}/Z_k$ is given by

$$K(L^n(k))=Z[\sigma]/<\sigma^{n+1},\ (\sigma+1)^k-1>, \tag{4.6}$$

where $\sigma=\mu-1$ and μ is the canonical complex line bundle over $L^n(k)$ (cf. [7, Lemma 3.3]).

LEMMA 4.7. *For the natural projection ρ: $L^{2n+1}(2^m)\to N^n(m)$, induced by the first inclusion ρ of* (3.5), *we have*

$$\rho^!(\alpha)=0, \quad \rho^!(\beta)=(\sigma+1)^{2^{m-1}}-1, \quad \rho^!(\delta)=\sigma^2/(1+\sigma).$$

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{R}(H_m) & \xrightarrow{\xi} & \tilde{K}(N^n(m)) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho^!} \\ \tilde{R}(Z_{2^m}) & \xrightarrow{\xi''} & \tilde{K}(L^{2n+1}(2^m)), \end{array}$$

where ξ'': $S^{4n+3}\to L^{2n+1}(2^m)$ is the projection. Then the equality $\xi''(\chi-1)=\mu-1$ can be proved easily by the definitions, since the first Chern class of μ generates $H^2(L^{2n+1}(2^m);Z)=Z_{2^m}$ (cf. [1, §2 and Appendix, (3)]). Hence, we obtain the desired equalities by (4.1) and Proposition 3.9. q.e.d.

For the second inclusion ρ': $Z_4\to H_m$ of (3.5), and the natural projection ρ': $L^{2n+1}(4)\to N^n(m)$, we have the following lemma similarly to the above lemmas.

LEMMA 4.8. $\rho'^!(\alpha)=(\sigma+1)^2-1, \quad \rho'^!(\beta)=0,$
$\rho'^!(\delta)=\sigma^2/(1+\sigma).$

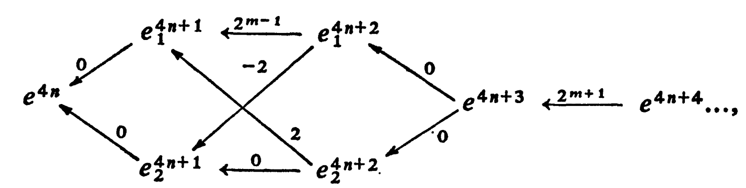
§ 5. Proof of Theorem 1.1

The CW-decompositions of $N^n(m)$ for $n\geqq 0$ of Lemma 2.1 define naturally a CW-decomposition of $N^\infty(m)=\cup_n N^n(m)$. Let N^k be the k-skeleton of the CW-complex $N^\infty(m)$. Then

$$N^{4n+3}=N^n(m)$$

and the cell structure of $N^\infty(m)$ is given by

(*)



where $e^{i+1} \xrightarrow{k} e^i$ means that the attaching map

$$S^i = \dot{e}^{i+1} \longrightarrow N^i / N^i - e^i = \bar{e}^i / \dot{e}^i = S^i$$

is the map of degree k.

We denote by $\# A$ the number of the elements of a finite set A.

LEMMA 5.1. $\tilde{K}^1(N^{4n+2}/N^{4n-2})=0, \quad \#\tilde{K}(N^{4n+2}/N^{4n-2})=2^{m+3}$.

PROOF. In the Puppe exact sequence of the pair $(N^{4n}/N^{4n-2}, N^{4n-1}/N^{4n-2})$

$$\tilde{K}^{-1}(S^{4n}) \longrightarrow \tilde{K}^{-1}(N^{4n}/N^{4n-2}) \longrightarrow \tilde{K}^{-1}(S^{4n-1})$$
$$\xrightarrow{\delta} \tilde{K}(S^{4n}) \longrightarrow \tilde{K}(N^4/N^{4n-2}) \longrightarrow \tilde{K}(S^{4n-1}),$$

we see by (*) that the coboundary δ is the multiplication by 2^{m+1}. Hence, we have $\tilde{K}^1(N^{4n}/N^{4n-2})=0$ and $\tilde{K}(N^{4n}/N^{4n-2})=Z_{2^{m+1}}$. Furthermore, by the Puppe sequence of $(N^{4n+1}/N^{4n-2}, N^{4n}/N^{4n-2})$

$$Z_{2^{m+1}} \longrightarrow \tilde{K}^{-1}(S^{4n+1} \vee S^{4n+1}) \longrightarrow \tilde{K}^{-1}(N^{4n+1}/N^{4n-2}) \longrightarrow 0 \longrightarrow 0$$
$$\longrightarrow \tilde{K}(N^{4n+1}/N^{4n-2}) \longrightarrow Z_{2^{m+1}} \longrightarrow \tilde{K}^1(S^{4n+1} \vee S^{4n+1}),$$

we have

$$\tilde{K}^1(N^{4n+1}/N^{4n-2}) = \tilde{K}^1(S^{4n+1} \vee S^{4n+1}) = Z \oplus Z,$$
$$\tilde{K}(N^{4n+1}/N^{4n-2}) = Z_{2^{m+1}}.$$

Consider the Puppe sequence of $(N^{4n+2}/N^{4n-2}, N^{4n+1}/N^{4n-2})$

$$0 \longrightarrow \tilde{K}^{-1}(N^{4n+2}/N^{4n-2}) \longrightarrow \tilde{K}^{-1}(N^{4n+1}/N^{4n-2}) \xrightarrow{\delta}$$
$$\tilde{K}(S^{4n+2} \vee S^{4n+2}) \longrightarrow \tilde{K}(N^{4n+2}/N^{4n-2}) \longrightarrow \tilde{K}(N^{4n+1}/N^{4n-2}) \longrightarrow 0.$$

Then we see from the attaching maps of (*) that the coboundary $\delta: Z \oplus Z \rightarrow Z \oplus Z$ is the multiplication by 2. Hence, we have the lemma. q.e.d.

LEMMA 5.2. $\tilde{K}^1(N^{4n+2})=0, \quad \#\tilde{K}(N^{4n+2})=2^{n(m+3)+2}$.

PROOF. Since $N^1=S^1 \vee S^1$, we have the lemma for $n=0$, by using the Puppe sequence of (N^2, N^1)

$$0 \longrightarrow \tilde{K}^{-1}(N^2) \longrightarrow \tilde{K}^{-1}(S^1 \vee S^1) \xrightarrow{\times 2} \tilde{K}(S^2 \vee S^2) \longrightarrow \tilde{K}(N^2) \longrightarrow 0.$$

The lemma for $n \geqq 1$ can be proved inductively by Lemma 5.1 and the Puppe sequence

$$0 \longrightarrow \tilde{K}^{-1}(N^{4n+2}) \longrightarrow \tilde{K}^{-1}(N^{4n-2}) \longrightarrow \tilde{K}(N^{4n+2}/N^{4n-2})$$

$$\longrightarrow \tilde{K}(N^{4n+2}) \longrightarrow \tilde{K}(N^{4n-2}) \longrightarrow 0. \qquad \text{q.e.d.}$$

LEMMA 5.3. *The induced homomorphism* $i^!$: $\tilde{K}(N^{4n+3}) \to \tilde{K}(N^{4n+2})$ *is an isomorphism, where* i: $N^{4n+2} \to N^{4n+3}$ *is the inclusion.*

PROOF. This follows immediately from Lemma 5.2 and the Puppe sequence

$$0 \longrightarrow \tilde{K}(N^{4n+3}) \xrightarrow{i^!} \tilde{K}(N^{4n+2}) \longrightarrow \tilde{K}^1(S^{4n+3}). \qquad \text{q.e.d.}$$

LEMMA 5.4. $\tilde{K}^1(N^{4n})=0, \quad \# \tilde{K}(N^{4n})=2^{n(m+3)}.$

PROOF. This follows from Proposition 2.3, Lemma 5.2 and the Puppe sequence of (N^{4n}, N^{4n-1})

$$0 \to \tilde{K}^{-1}(N^{4n}) \to \tilde{K}^{-1}(N^{4n-1}) \xrightarrow{\times 2^{m+1}} \tilde{K}(S^{4n}) \to \tilde{K}(N^{4n}) \to \tilde{K}(N^{4n-1}) \to 0.$$

q.e.d.

LEMMA 5.5. $\tilde{K}^1(N^{4n+1})=Z \oplus Z$ *and* $i^!$: $\tilde{K}(N^{4n+1}) \to \tilde{K}(N^{4n})$ *is an isomorphism, where* i: $N^{4n} \to N^{4n+1}$ *is the inclusion.*

PROOF. The lemma follows from Lemma 5.4 and the Puppe sequence

$$0 \longrightarrow \tilde{K}(N^{4n+1}) \xrightarrow{i^!} \tilde{K}(N^{4n}) \longrightarrow \tilde{K}^1(S^{4n+1} \vee S^{4n+1}) \longrightarrow \tilde{K}^1(N^{4n+1}) \longrightarrow 0.$$

q.e.d.

Now, we are ready to prove Theorem 1.1.

PROOF OF THEOERM 1.1. Consider the natural projection π: $N^n(m) \to HP^n$ of Lemma 4.4 and the commutative diagram of the Puppe sequence

$$\begin{array}{ccccc} \tilde{K}(N^{4n}/N^{4n-1}) & \longrightarrow & \tilde{K}(N^{4n}) & \xrightarrow{i^!} & \tilde{K}(N^{4n-1}) \\ \uparrow \pi^! & & \uparrow \pi^! & & \uparrow \pi^! \\ 0 \longrightarrow \tilde{K}(HP^n/HP^{n-1}) & \longrightarrow & \tilde{K}(HP^n) & \longrightarrow & \tilde{K}(HP^{n-1}). \end{array}$$

For the element $\nu \in \tilde{K}(HP^n)$ of (4.3), it is easy to show that the element $\nu^n \in \tilde{K}(HP^n)$ is the image of a generator of $\tilde{K}(HP^n/HP^{n-1})=\tilde{K}(S^{4n})=Z$, by using the Chern character (cf. [9, Proof of (3.12)]). Also, it is easy to show that the restriction

$$\pi: \ (N^{4n}, N^{4n-1}) \longrightarrow (HP^n, HP^{n-1})$$

is a relative homomorphism, by the definitions of the cells of Lemma 2.1, and so $\pi^!$ in the left is an isomorphism. These show that $\operatorname{Ker} i^!$ is generated by $\delta^n = \pi^!(\nu^n)$. Therefore, if the ring $\tilde{K}(N^{4n-1})$ is generated multiplicatively by α, β and

δ, then so is $\tilde{K}(N^{4n})$.

Now, we prove that the ring $\tilde{K}(N^{4n+2})$ is generated multiplicatively by α, β and δ if $\tilde{K}(N^{4n+1})$ is so. Then, the theorem is proved by the induction on n, using Lemmas 4.2, 5.3 and 5.5.

In the Puppe sequence

$$0 \longrightarrow \tilde{K}^{-1}(N^{4n+1}) \xrightarrow{\times 2} \tilde{K}(S^{4n+2} \vee S^{4n+2}) \longrightarrow \tilde{K}(N^{4n+2}) \xrightarrow{i^!} \tilde{K}(N^{4n+1}) \longrightarrow 0,$$

we have $\operatorname{Ker} i^! = Z_2 \oplus Z_2$ by Lemmas 5.2, 5.4 and 5.5. Since $\alpha\delta^n$ and $\beta\delta^n$ belong to $\operatorname{Ker} i^!$ by Lemma 4.5, it is sufficient to show that $\alpha\delta^n \neq 0$, $\beta\delta^n \neq 0$ and $\alpha\delta^n \neq \beta\delta^n$ in $\tilde{K}(N^{4n+2}) = \tilde{K}(N^{4n+3}) = \tilde{K}(N^n(m))$.

Considering the induced homomorphism

$$\rho^!: \tilde{K}(N^n(m)) \longrightarrow \tilde{K}(L^{2n+1}(2^m)),$$

we have

$$\rho^!(\beta\delta^n) = ((\sigma+1)^{2^{m-1}} - 1)(\sigma^2/(1+\sigma))^n = 2^{m-1}\sigma^{2n+1}$$

by Lemma 4.7 and (4.6). Since σ^{2n+1} in $\tilde{K}(L^{2n+1}(2^m))$ is of order 2^m (cf. [6, Prop. 2.6]), we have $\beta\delta^n \neq 0$. Also, $\rho^!(\alpha\delta^n) = 0$ by Lemma 4.7, and so we have $\alpha\delta^n \neq \beta\delta^n$ in $\tilde{K}(N^n(m))$. Considering the induced homomorphism

$$\rho'^!: \tilde{K}(N^n(m)) \longrightarrow \tilde{K}(L^{2n+1}(4)),$$

we have

$$\rho'^!(\alpha\delta^n) = ((\sigma+1)^2 - 1)(\sigma^2/(1+\sigma))^n = 2\sigma^{2n+1}$$

by Lemma 4.8 and (4.6). Since σ^{2n+1} in $\tilde{K}(L^{2n+1}(4))$ is of order 4 (cf. [6, Prop. 2.6]), we have $\alpha\delta^n \neq 0$ in $\tilde{K}(N^n(m))$. q.e.d.

§6. Proof of Theorem 1.2

In this section, we deal with the special case $N^n(2) = S^{4n+3}/H_2$, where $H_2 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group.

The elements α, β and δ in $\tilde{K}(N^n(2))$ $(n \geqq 1)$ of (4.1) have the following relations by Proposition 3.3:

$$\alpha^2 = -2\alpha, \quad \beta^2 = -2\beta, \tag{6.1}$$

$$\alpha\beta = -2\alpha - 2\beta + 4\delta + \delta^2, \tag{6.2}$$

$$\alpha\delta = -2\alpha, \quad \beta\delta = -2\beta. \tag{6.3}$$

By these relations, we have

$$(6.4)\qquad \delta^3+6\delta^2+8\delta=0.$$

Moreover, Lemma 4.5 shows that

$$(6.5)\qquad \delta^{n+1}=0.$$

LEMMA 6.6. (i) $2^{n+1}\alpha=0,\quad 2^{n+1}\beta=0.$

(ii) $2^{2i+3}\delta^{n-i}=0$ *for* $0\leqq i\leqq n-1.$

(iii) $2^{2i+2}\delta^{n-i}=\pm 2^{2i+4}\delta^{n-i-1}$ *for* $0\leqq i\leqq n-2.$

PROOF. (i) By (6.3) and (6.5), we have $2^{n+1}\alpha=-2^n\alpha\delta=\cdots=(-1)^{n+1}\alpha\delta^{n+1}=0$ and $2^{n+1}\beta=0$.

(ii) The equality $(6.4)\times\delta^{n-1}$ and (6.5) show that $8\delta^n=0$. The desired equality is obtained by the induction on i, by using $(6.4)\times 2^{2i}\delta^{n-i-1}$.

(iii) This is obtained inductively by using $(6.4)\times 2^{2i-1}\delta^{n-i-1}$ and (ii). q.e.d.

LEMMA 6.7. $2^{n-1}(\delta^2+4\delta+2^{n+1}\delta)=0.$

PROOF. We consider the element $\delta(1)=\delta^2+4\delta$. Then $\delta(1)\delta=-2\delta(1)$ and $\delta(1)^2=-4\delta(1)$ by (6.4). Therefore,

$$(6.8)\qquad \delta(1)\delta^i=(-1)^i 2^i\delta(1),\quad \delta(1)^{i+1}=(-1)^i 2^{2i}\delta(1).$$

For the case $n=2m\geqq 2$, we have

$$\begin{aligned}
-2^{n-1}\delta(1) &= (-1)^{m-1}\delta(1)^m\delta && \text{by (6.8)}\\
&= (-1)^{m-1}\textstyle\sum_{i=0}^{m-1}\binom{m-1}{i}2^{2i}\delta^{n-i-1}\delta(1) && \text{by } \delta(1)=\delta^2+4\delta\\
&= (-1)^{m-1}\delta^{n-1}\delta(1) && \text{by (iii) of Lemma 6.6}\\
&= \pm 4\delta^n && \text{by (6.5)}\\
&= \pm 2^{2n}\delta && \text{by (iii) of Lemma 6.6.}
\end{aligned}$$

For the case $n=2m+1\geqq 3$, we have in the same way

$$2^{n-1}\delta(1)=(-1)^{m-1}\delta(1)^m\delta^2=\pm 2^{2n}\delta.$$

For the case $n=1$, we have $\delta(1)=\pm 4\delta$ by (6.5) and (ii) of Lemma 6.6. These show the lemma. q.e.d.

PROOF OF THEOREM 1.2. The theorem for $n=0$ has been proved in Lemmas 4.2 and 4.5. Let $n\geqq 1$. By Theorem 1.1 and the relations (6.1)–(6.4), we see that every element of $\tilde{K}(N^n(2))$ is a linear combination of α, β, δ and $\delta^2+4\delta+2^{n+1}\delta$. The order of these elements are not greater than 2^{n+1}, 2^{n+1}, 2^{2n+1} and 2^{n-1}, respectively, by the above lemmas. Since $2^{n+1}\times 2^{n+1}\times 2^{2n+1}\times 2^{n-1}=2^{5n+2}$, we have the theorem by Proposition 2.3. q.e.d.

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