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# Existence of Solutions of Heavily Nonlinear Volterra Integral Equations

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## 1. Introduction

The objective of this paper is to show the existence of solutions (in a BANACH function space) of VOLTERRA integral equations of the form

$$x(t)=f(t)+\int_0^t K(t, s, x(s))ds, \tag{1.1}$$

where $x$, $f$, $K$ are $n$-dimensional vectors. To achieve this, we assume that "admissibility" conditions hold for a linear equation associated with (1.1). By "admissibility" we mean here the concept introduced by MILLER [12].

Our results are particularly useful in the case of equations of the form

$$x(t)=f(t)+\int_0^t K(t, s, x(s))x(s)ds, \tag{1.2}$$

(where $K$ is now an $n\times n$ matrix), provided that we know un upper bound for the norm of the linear operator $I-R_u$, where $I$ is the identity operator, and $R_u$ is the resolvent kernel associated with the linear equation

$$x(t)=f(t)+\int_0^t K(t, s, u(s))x(s)ds. \tag{1.2$_u$}$$

The function $u(t)$ above lies in a suitable closed ball of a Banach function space. We also show that the same method can be applied to nonlinear perturbations of linear systems.

## 2. Preliminaries

In what follows, $J=[0, \infty)$, $E=\{(t, s)\in J^2;\ t\geq s\}$, and $R=(-\infty, \infty)$. For a vector $x\in R^n$ we put $\|x\|=\sum_i|x_i|$, and for a real $n\times n$ matrix $A=[a_{ij}]$, $\|A\|=\sup_k\sum_i|a_{ik}|$. We denote by $C_c$ the space of all continuous functions $f$: $J\rightarrow R^n$, associated with the topology of uniform convergence on compact subintervals of $J$. The letter $B$ will always denote a BANACH space contained in $C_c$, stronger than $C_c$, and with norm $\|\cdot\|_B$. $C$ will stand for the space of all bounded $f \in C_c$ under the sup-norm $\|\cdot\|_C$. For a finite interval $J' \subset J$ and a continuous $R^n$-valued function $x(t)$ on $J'$ we put $\|x\|_{J'} = \sup_{t \in J'} \|x(t)\|$. It is known that if $f \in C_c$ and $K(t, s)$ is an $n \times n$ real matrix defined and continuous on $E$, then equation

$$x(t) = f(t) + \int_0^t K(t, s)x(s)ds \tag{2.1}$$

has a unique solution $x \in C_c$, given by the formula

$$x(t) = [(I - R)f](t), \tag{2.2}$$

where $R$ is the resolvent kernel operator associated with the kernel $K(t, s)$, i.e.,

$$(Rf)(t) = \int_0^t r(t, s)f(s)ds \tag{2.3}$$

for every $f \in C_c$, where $r(t, s)$ is the solution of

$$r(t, s) = -K(t, s) + \int_s^t K(t, u)r(u, s)du. \tag{2.4}$$

The pair $(B, B)$ is said to be "$K$-admissible", if for every $f \in B$, the solution $x(t)$ of (2.1) belongs to $B$. Thus $K$-admissibility is equivalent to the admissibility of the operator $R$, i.e., $RB \subset B$.

## 3. Main Results

We first give a result connecting $\|x\|_B$ to $\|f\|_B$ in (2.1), under the assumption of $K$-admissibility.

THEOREM 3.1. *For the equation* (2.1) *assume the following*:

(i) *$K(t, s)$ is an $n \times n$ real matrix defined and continuous on the set $E$;*

(ii) *the pair $(B, B)$ is $K$-admissible;*

(iii) *let $Y$ be the linear manifold consisting of all $x \in B$ such that $(I - T)x \in B$, where $T$ is the linear operator in* (2.1) *defined on $C_c$.*

*Then there exists a positive constant $M_0$ such that for each $f \in B$ the solution $x(t)$, $t \in J$ of* (2.1) *satisfies $\|x\|_Y \leq M_0 \|f\|_B$, where*

$$\|x\|_Y = \|x\|_B + \|(I - T)x\|_B.$$

*Proof.* We first show that $Y$ becomes a Banach space under the above norm. In fact, let $\{x_n\}$, $n = 1, 2, \ldots$ be a Cauchy sequence in $Y$. Then for every $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that

$$\|x_n - x_m\|_Y = \|x_n - x_m\|_B + \|f_n - f_m\|_B < \varepsilon$$

for every $m, n > N(\varepsilon)$, where $f_n = (I-T)x_n$. This implies that there exist $x \in B$ and $f \in B$ such that $\|x_n - x\|_B \to 0$ and $\|f_n - f\|_B \to 0$ as $n \to \infty$. Since $B$ is stronger than $C_c$, $x_n \to x$, $f_n \to f$ as $n \to \infty$, uniformly on every finite subinterval of $J$. Now let $J' = [0, b]$, for some $b > 0$, and $(I-T)x = y$. Then for $t \in J'$ we have

$$\|f_n(t) - y(t)\| \leq \|x_n(t) - x(t)\| + \int_0^t \|K(t, s)(x_n(s) - x(s))\| ds$$

$$\leq \|x_n - x\|_{J'} + \lambda_{J'} \|x_n - x\|_{J'}$$

$$= (1 + \lambda_{J'}) \|x_n - x\|_{J'},$$

where $\lambda_{J'} = \sup_{t \in J'} \int_0^t \|K(t, s)\| ds$. It follows that $f_n \to y$ uniformly on every finite subinterval of $J$, and this implies that $(I-T)x = y = f \in B$. Consequently, $x \in Y$, and this implies that $Y$ is complete. Now consider the restriction $Q$ of $I-T$ on $Y$. Then $Q$ maps $Y$ onto $B$, and is linear and bounded with norm $\|Q\| \leq 1$. From the fact that solutions of linear VOLTERRA integral equations are unique, it follows, that $Q^{-1}$ (the inverse of $Q$ on $B$) exists, and is a bounded linear operator on $B$. Letting $M = \|Q^{-1}\| - 1$ ($\|Q^{-1}\|$ denotes the norm of $Q^{-1}$), we obtain

$$\|x\|_Y = \|Q^{-1}Qx\|_Y \leq (M+1)\|f\|_B = M_0 \|f\|_B,$$

where $f = Qx$.

It should be noted that $Q^{-1}$ in he above proof is the operator $I-R$ defined on $B$ and with values onto $Y$. Thus, $M+1 = \|I-R\| \leq 1 + \|R\|$, because $R: B \to B$ is also bounded (cf. MILLER [12, Lemma 2]).

The following theorem is the main result of this paper. The subsequent results are important applications of it.

THEOREM 3.2. *Assume that the hypotheses of* Th. 3.1 *are satisfied, and for the kernel $K$ in the equation* (1.1) *assume that*

(i) *$K: E \times R^n \to R^n$, continuous, $K(t, s, 0) = 0$ for every $(t, s) \in E$* and

$$\|T(x_1 - x_2) - [T_0 x_1 - T_0 x_2]\|_B \leq \delta \|x_1 - x_2\|_B, \tag{3.1}$$

*for every $x_1, x_2 \in S_\gamma = \{x \in B;\ \|x\|_B \leq \gamma\}$, where $\delta$ is a positive constant with* $0 < \delta M_0 < 1$ *($M_0 = M+1$ is the constant of* Th. 3.1), *and $T$, $T_0$ are the operators*

$$(Tu)(t) = \int_0^t K(t,s)\, u(s) ds, \quad (T_0 u)(t) = \int_0^t K(t, s, u(s)) ds, \tag{3.2}$$

*defined on $C_c$.*

*Then if $f \in B$ is such that*

$$\|f\|_B \leq \gamma(1 - \delta M_0)/M_0 = \gamma/M_1,$$

*there exists at least one solution* $x\in B$ *of* (1.1) *such that* $\|x\|_B\leq M_1\|f\|_B\leq\gamma$.

*Proof.* Consider the operator $Q=I-T$ of the proof of Th. 3.1 defined on the Banach space $Y\subset B$, and the operator $Q_0=I-T_0$ defined also on $Y$. Now let $x_1, x_2\in S_\gamma$. Then we have

$$\begin{aligned}(3.3)\qquad &\|Q(x_1-x_2)-[Q_0x_1-Q_0x_2]\|_B\\ &=\|T(x_1-x_2)-[T_0x_1-T_0x_2]\|_B\leq\delta\|x_1-x_2\|_B\\ &\leq\delta\|x_1-x_2\|_Y,\end{aligned}$$

and

$$\begin{aligned}(3.4)\qquad \|Q_0x_1-Q_0x_2\|_B&\leq\|Q(x_1-x_2)\|_B+\delta\|x_1-x_2\|_B\\ &\leq(\|Q\|+\delta)\|x_1-x_2\|_Y.\end{aligned}$$

Thus, $Q_0$ is continuous on $S_\gamma$. Since we also have $Q_0 0=0$, it follows from Th. 1 of GRAVES [8] that there exists at least one solution $x\in S_\gamma$ of (1.1) such that $\|x\|_B\leq\|x\|_Y\leq M_1\|f\|_B\leq\gamma$. This completes the proof.

There is a large class of kernels for which the above theorem can be applied. This is the content of the following.

THEOREM 3.3. *Assume that the hypotheses of* Th. 3.1 *are satisfied for* $B=C$, *and for the kernel* $K(t, s, x)$ *in* (1.1) *assume that* $K(t, s, x)=K_1(t, s, x)x$, *where* $K_1$ *is a real* $n\times n$ *matrix defined and continuous on* $E\times R^n$. *Moreover, assume that for each* $u\in S_\gamma=\{u\in C;\ \|u\|_C\leq\gamma\}$,

$$\sup_{t\in J}\int_0^t\|K(t, s)-K_1(t, s, u(s))\|ds\leq\delta,$$

*where* $\delta$ *is as in* Th. 3.2 *and independent of* $u(t)$. *Furthermore, for each finite interval* $J'\subseteq J$ *and each* $t_0,\ t\in J'$,

$$\lim_{t\to t_0}\sup_{u\in S_\gamma}\int_{J'}\|K_1(t, s, u(s))-K_1(t_0, s, u(s))\|ds=0.$$

*Then if* $\|f\|_C\leq\gamma(1-\delta M_0)/M_0$, *there exists at least one solution* $x\in S_\gamma$ *of the equation* (1.1).

*Proof.* Let $J_m=[0, m]$, $m=1, 2,\ldots$, $S_m=\{u\in C[J_m, R^n];\ \|u\|_{J_m}\leq\gamma\}$, and $U_m$ be the operator which maps each function $u\in S_m$ into the unique solution $x_m\in S_m$ of the equation

$$x(t)=f(t)+\int_0^t K_1(t, s, u(s))\,x(s)\,ds.$$

The function $x_m(t)$ is the restriction on $J_m$ of the solution on $J$ guaranteed by Th. 3.2. Now fix $m$ and let $S_m^0$ be the set consisting of all $u \in S_m$ such that for each $t_0$, $t \in J_m$,

$$\begin{aligned}\|u(t)-u(t_0)\| &\leq \|f(t)-f(t_0)\|+\gamma P_{J_m}|t-t_0| \\ &+\gamma \sup_{u\in S_\gamma}\int_0^m \|K_1(t, s, u(s))-K_1(t_0, s, u(s))\|ds \\ &\equiv \lambda(t, t_0),\end{aligned}$$

where $P_{J_m}=\sup \|K_1(t, s, u)\|$, $(t, s)\in J_m\times J_m\cap E$, $\|u\|\leq\gamma$. Then it is easy to show (cf. KARTSATOS [9]) that the set $S_m^0$ is closed. It follows that $S_m^0$ is compact because it consists of equicontinuous functions. Moreover, the operator $U_m$ maps $S_m^0$ into $S_m^0$. To show that $U_m$ is continuous, let $u_n \in S_m^0$, $n=1, 2,\ldots$ be such that $\|u_n-u\|_{J_m}\to 0$ as $n\to\infty$. Let $U_m u_n=y_n$, $n=1, 2,\ldots$ and $U_m u=z$. Since $U_m S_m^0$ is a set of equicontinuous and uniformly bounded functions, there is a subsequence $\{y_{k_n}\}$, $n=1, 2,\ldots$ of $\{y_n\}$ and $y\in S_m^0$ such that $\|y_{k_n}-y\|_{J_m}\to 0$ as $n\to\infty$. Consequently, we have

$$\begin{aligned}\|y_{k_n}(t)-z(t)\| &= \left\|\int_0^t K_1(t, s, u_{k_n}(s))y_{k_n}(s)-K_1(t, s, u(s))z(s)ds\right\| \\ &\leq \int_0^t \|K_1(t, s, u_{k_n}(s))y_{k_n}(s)-K_1(t, s, u(s))z(s)\|ds \\ &\leq \int_0^m \|K_1(t, s, u_{k_n}(s))y_{k_n}(s)-K_1(t, s, u(s))z(s)\|ds.\end{aligned} \tag{3.5}$$

Since the integrand in the last member of (3.5) tends to zero uniformly on $J_m\times J_m\cap E$, it follows that $\|y_{k_n}-z\|_{J_m}\to 0$ as $n\to\infty$. Since we could have started with any subsequence of $\{y_n\}$ instead of $\{y_n\}$ itself, it follows that every subsequence of $\{y_n\}$, contains a subsequence converging to $z(t)$ uniformly on $J_m$. It follows that $\|y_n-z\|_{J_m}\to 0$ as $n\to\infty$.

Thus, $\|U_m u_n-Uu\|_{J_m}\to 0$ as $n\to\infty$, and $U_m$ is continuous. From Schauder's fixed point theorem, it follows that $U_m$ has a fixed point $x_m=U_m x_m\in S_m^0$. Since the sequence $\{x_m\}$ $m=1, 2,\ldots$ so obtained is uniformly bounded by $\gamma$, it follows from Lemma 2.1 of KARTSATOS [9] that there exists at least one solution $x(t)$, $t\in J$ of the equation (1.1), and this completes the proof.

The above theorem can be now easily extended to equations of the form

$$x(t)=f(t)+\int_0^t K_1(t, s, x(s))\,[x(s)+g(s, x(s))]ds, \tag{3.6}$$

by using Theorem 3.1.

If the kernel in (1.1) is continuously differentiable with respect to $x$ and

$K(t, s, 0)\equiv 0$, then there exists an $n\times n$ matrix $K_1(t, s, x)$ as above. For a proof, the reader is referred to LAKSHMIKANTHAM and LEELA [11]. There would be no essential difficulty in extending these results to equations under CARATHEODORY conditions, by replacing the space $C_c$ by the space $L$ of all locally LEBESGUE integrable $R^n$-valued functions defined on $J$. One could also extend the above results to integro-differential equations of the form

$$x'(t)=f(t)+A(t)x(t)+\int_0^t K(t, s, x(s))ds,$$

where $A(t)$ is an $n\times n$ matrix. However, a more complete study in this case would require taking int consideration the subspace $R_{0B}$ of $R^n$ consisting of initial values of $B$-solutions of the homogeneous equation ($f\equiv 0$). For further results concerning admissibility of VOLTERRA integral equations, the reader is referred to AVRAMESCU [1]-[4], BOWNDS and CUSHING [5], CORDUNEANU [7], MILLER [12], [13] and KARTSATOS [9]. For results concerning the contents of this paper, but for differential equations, the reader is referred to KARTSATOS [10].

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