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Harmonic Functions and the Borel-Weil Theorem

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§ 1. Introduction

In the previous paper [3], we proved that, for non-zero eigenvalues, arbitrary eigenfunctions of the laplacian can be given by the "Poisson integral" of elements of a certain space $\tilde{\mathscr{B}}(S^{n-1})$ which contains the space of hyperfunctions on the $(n-1)$ dimensional unit sphere as a proper subspace.

In case the eigenvalue is zero, however, the Poisson integral gives only constant functions.

In this paper, we shall give the modification of the Poisson integral so that, using the Borel-Weil theorem, the modified "Poisson integral" gives the canonical isomorphism between the space of all homogeneous harmonic polynomials on $\boldsymbol{R}^n$ of degree m and the space of all holomorphic sections of a certain $SO(n, \boldsymbol{C})$ -homogeneous holomorphic line bundle L_m over the Grassmann manifold $SO(n)/SO(2)\times SO(n-2)$. In the last section, we shall consider a certain space $\oplus\sum_{m\geq 0}\Gamma(L_m)$ and show that every harmonic function on $\boldsymbol{R}^n$ can be represented by an analogue of the "Poisson integral" of the unique element of $\oplus\sum_{m\geq 0}\Gamma(L_m)$.

§ 2. Homogeneous harmonic polynomials

In this section we shall refer to general properties about harmonic polynomials which we need in the following sections. In this paper, we denote by G the rotation group of degree n, where n is a positive integer. For each non-negative integer m, let $\mathscr{H}^{n,m}$ denote the space of all homogeneous harmonic polynomials on $\boldsymbol{R}^n$ of degree m. By left translations, one obtains an irreducible (unitary) representation τ_m of G on $\mathscr{H}^{n,m}$. The representation τ_m is of class one with respect to the subgroup H' of G consisting of all elements of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \quad : \quad h \in SO(n-1, \boldsymbol{R}),$$

and every irreducible representation of G of class one with respect to H' is equivalent to τ_m for some non-negative integer m.

Let P^n be the ring of polynomial function on $\boldsymbol{R}^n$ with coefficients in the complex field $\boldsymbol{C}$, and $P^{n,m}$ be the subspace of P^n consisting of all m-homogeneous elements. We define the harmonic projection H_p of $P^{n,m}$ into $\mathscr{H}^{n,m}$ by

$$H_p f(x)=\sum_{k=0}^{[m/2]}\frac{(-1)^k r^{2k}(\Delta^k f)(x)}{2^k k!(n+2m-4)\cdots(n+2m-2k-2)}$$

for $x=(x_1, \ldots, x_n)\in \boldsymbol{R}^n$ where r is the norm of x with respect to the usual Euclidean metric and Δ is the Laplace-Beltrami operator. Then the following sequence is exact (see Vilenkin [8]):

$$0\longrightarrow r^2 p^{n,m-2}\longrightarrow p^{n,m}\xrightarrow{H_p}\mathscr{H}^{n,m}\longrightarrow 0 \qquad (1)$$

The group G acts on P^n by left-translations, and this projection H_p is a G-homomorphism of $P^{n,m}$ onto $\mathscr{H}^{n,m}$ for each m. In this paper, we write $[f]$ instead of $H_p(f)$ for every $f\in p^{n,m}$.

For each non-negative integer m, there exists a set J_m of multi-indices $(i_1, \ldots, i_n)$ of non-negative integers such that 1) $i_1+\ldots+i_n=m$ and 2) $\{[f_{i_1\ldots i_n}]: (i_1, \ldots, i_n)\in J_m\}$ is a basis of $\mathscr{H}^{n,m}$, where $f_{i_1\ldots i_n}$ is a polynomial function on $\boldsymbol{R}^n$ defined by $f_{i_1\ldots i_n}(x)=x_1^{i_1}\cdots x_n^{i_n}$ for $x=(x_1, \ldots, x_n)\in \boldsymbol{R}^n$.

§ 3. The Borel-Weil theorem for $SO(n, \boldsymbol{R})$

In this section we shall construct a G-irreducible subspace of $C^\infty(SO(n, \boldsymbol{R})/SO(n-2, \boldsymbol{R}))$ equivalent to τ_m.

Define subgroups H and K of G by

$$H=\left\{\begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ & 0 & h\end{pmatrix} : h\in SO(n-2, \boldsymbol{R})\right\},$$

$$K=\left\{\begin{pmatrix} k_1 & 0 \\ 0 & k_2\end{pmatrix}: K_1\in SO(2, \boldsymbol{R}),\ k_2\in SO(n-2, \boldsymbol{R})\right\}.$$

The group $SO(2, \boldsymbol{R})$ acts on the Stiefel manifold G/H as right-translations:

$$(gH)\cdot u_\theta=g\begin{pmatrix} u_\theta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1\end{pmatrix}H$$

where $g\in G$ and $u_\theta=\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\in SO(2, \boldsymbol{R})$.

The space G/H is a fibre bundle over G/K with fibre $SO(2, \boldsymbol{R})$. For each non-negative integer m, let ξ_m be the unitary character of $SO(2, \boldsymbol{R})$ defined by

$$\tilde{\xi}_m(u_\theta) = e^{im\theta} \quad \text{for} \quad u_\theta \in SO(2, \boldsymbol{R})$$

then we have an associated line bundle $\tilde{L}_m$ on G/K. The space $C^\infty(\tilde{L}_m)$ of all C^∞-sections on $\tilde{L}_m$ becomes a G-module by left-translations and it is isomorphic to the G-module;

$$\{f \in C^\infty(G/H);\ f(pu_\theta) = \tilde{\xi}_m(u_\theta)^{-1} f(p);\ p \in G/H,\ u_\theta \in SO(2, \boldsymbol{R})\}$$

Thus, we regard $C^\infty(\tilde{L}_m)$ as a subspace of $C^\infty(G/H)$.

Now the space G/K has a G-invariant complex structure holomorphically isomorphic to $G^C/K^C P_+$, where G^C and K^C are complexifications of G and K respectively and P_+ is the subgroup of G^C, consisting of all elements of the form;

$$\left\{\begin{pmatrix} 1-\frac{1}{2}(z_3^2+\cdots+z_n^2). & \frac{i}{2}(z_3^2+\cdots+z_n^2), & -z_3, \dots, -z_n \\ \frac{i}{2}(z_3^2+\cdots+z_n^2), & 1+\frac{1}{2}(z_3^2+\cdots+z_n^2), & iz_3, \dots, iz_n \\ z_3, & -iz_3, & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \vdots & \vdots & \\ z_n, & -iz_n, & \end{pmatrix} : z_3, \dots, z_n \in \boldsymbol{C}\right\}$$

For each non-negative integer m, we define the holomorphic character ξ_m of $K^C P_+$ by

$$\xi_m(uz) = e^{im\theta} \text{ for every } u = \begin{pmatrix} u_\theta & 0 \\ 0 & u' \end{pmatrix} \in K^C \quad \text{and} \quad z \in P_+$$

Where $$u_\theta = \begin{pmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{pmatrix} \in SO(2, \boldsymbol{C}) \text{ and } u' \in SO(n-2, \boldsymbol{C}).$$

Then we obtain a G^C-homogeneous holomorphic line bundle L_m over $G^C/K^C P_+$, which is C^∞-isomorphic to $\tilde{L}_m$. The space $\Gamma(L_m)$ of all holomorphic sections of L_m, may be identified with the space.

$$\{f \in \mathrm{Hol}(SO(n, \boldsymbol{C}):\ f(\omega\gamma) = \xi_m^{-1}(\gamma) f(\omega), \omega \in SO(n, \boldsymbol{C}), \gamma \in K^C P_+\},$$

and the group G acts on them by left-translations. Thus, we obtain the following relations:

$$\Gamma(L_m) \hookrightarrow C^\infty(L_m) \cong C^\infty(L_m) \hookrightarrow C^\infty(G/H),$$

where $\hookrightarrow$ or $\cong$ implies a G-module inclusion or a G-module isomorphism respectively. By the well-known Borel-Weil theorem, the representation π_m of G on $\Gamma(L_m)$ is irreducible and equivalent to τ_m.

For a multi-index $(i_1 \ldots i_n)$ of non-negative integers, we define a holomorphic function $\varphi_{i_1 \ldots i_n}$ on $SO(n, \boldsymbol{C})$ by

$$\varphi_{i_1 \ldots i_n}(g) = (x_1 - iy_1)^{i_1} \cdots (x_n - iy_n)^{i_n} \quad \text{for each } g = \begin{pmatrix} x_1 & y_1 & \\ \vdots & \vdots & * \\ x_n & y_n & \end{pmatrix} \in SO(n, \boldsymbol{C})$$

It is easily seen that $\varphi_{i_1 \ldots i_n}$ satisfies $\varphi_{i_1 \ldots i_n}(\omega\gamma) = \xi_m(\gamma)^{-1} \varphi_{i_1 \ldots i_n}(\omega)$ for every ω in $SO(n, \boldsymbol{C})$ and γ in $K^{\boldsymbol{C}}P_+$ and so $\varphi_{i_1 \ldots i_n}$ is included in $\Gamma(L_m)$.

Moreover $\{\varphi_{i_1 \ldots i_n} : (i_1 \ldots i_n) \in J_m\}$ forms a basis of $\Gamma(L_m)$ since the space $\Gamma(L_m)$ can be identified with the space $\boldsymbol{C}[z_1 \ldots z_n]/(z_1^2 + \cdots + z_n^2)$; where $\boldsymbol{C}[z_1, \ldots, z_n]$ denotes the polynomial ring of n-variables $z_1, \ldots, z_n$ and $(z_1^2 + \cdots + z_n^2)$ is the ideal in $\boldsymbol{C}[z_1, \ldots, z_n]$ generated by $z_1^2 + \cdots + z_n^2$. This identification is given by the assignment of $z_1^{i_1} \cdots z_n^{i_n}$ to $\varphi_{i_1 \ldots i_n}$.

§ 4. Poisson integral

In view of § 2 and § 3, the representation of G on $\Gamma(L_m)$ is equivalent to $(\tau_m, \mathscr{H}^{n,m})$. In this section, we shall show that the Poisson integral gives an intertwining operater between them. We fix $\omega_0 = {}^t(1, i, 0, \ldots, 0)$, once for all.

PROPOSITION 4.1. *For each holomorphic section φ in $\Gamma(L_m)$, we define a function f on $\boldsymbol{R}^n$ by the following integral:*

$$f(x) = \int_{G/H} e^{i<x,\omega>} \varphi(\omega) d\omega \quad \text{for each} \quad x \in \boldsymbol{R}^n$$

where $d\omega$ is the G-invariant measure on G/H normalized by $\int_{G/H} d\omega = 1$ and $<x, \omega>$ denotes the complex-bilinear inner product $<x, g\,\omega_0>$ for $\omega = gH$. Then f is in $\mathscr{H}^{n,m}$.

PROOF. For each x in $\boldsymbol{R}^n$ we can regard $e^{i<x,\omega>}\varphi(\omega)$ as a function on G,

$$f(x) = \int_G e^{i<x,g\omega_0>} \varphi(g) dg$$

where dg is the Haar measure on G normalized by $\int_G dg = 1$. Since dg is a Haar measure on G, we have

$$\begin{aligned} f(x) &= \int_G \left[\frac{1}{2\pi} \int_0^{2\pi} e^{i<x,g\omega_0>e^{i\theta}} e^{-im\theta} d\theta \right] \varphi(g) dg \\ &= \frac{i^m}{m!} \int_G (<x, g\omega_0>)^m \varphi(g) dg \\ &= \frac{i^m}{m!} \int_{G/H} (<x, \omega>)^m \varphi(\omega) d\omega, \end{aligned}$$

and so f is a homogeneous polynomial of the degree m on $\boldsymbol{R}^n$. The fact that f is in $\mathscr{H}^{n,m}$ is an immediate consequence from $\Delta(<x, \omega>)^m=0$ where Δ is the Laplacian with respect to the variable x.

By Proposition 4.1, the correspondence $\varphi \to f$ defines a linear transformation $\mathscr{P}$ of $\Gamma(L_m)$ to $\mathscr{H}^{n,m}$.

THEOREM 1. *The map $\mathscr{L}$ is a G-isomorphism of $\Gamma(L_m)$ onto $\mathscr{H}^{n,m}$.*

PROOF. Both $\Gamma(L_m)$ and $\mathscr{H}^{n,m}$ are irreducible G-module and, as one can see easily from its definition, $\mathscr{P}$ commutes with the action of G, and so it is sufficient for the proof of this theorem to show that there exists φ in $\Gamma(L_m)$ such that $\mathscr{P}(\varphi)\neq 0$. Indeed, for $\varphi=\varphi_{m,0\ldots 0}$, we have

$$\mathscr{P}(\varphi)x_0=\frac{i^m}{m!}\int_G (x_1^2+y_1^2)^m dg \neq 0$$

where $x_0={}^t(1, 0 \ldots 0)\in \boldsymbol{R}^n$ and $g=\begin{pmatrix} x_1 & y_1 & \\ \vdots & \vdots & * \\ x_n & y_n & \end{pmatrix}\in G$.

This completes the proof of the theorem.

We set

$$C_m=\begin{cases} \dfrac{2^{p-1}\Gamma(p)\Gamma(2p-2)i^m(2m)!2^m\Gamma(m+p-1)}{\Gamma(p-1)m!(2m+2p-2)!\Gamma(m+2p-2)} & \text{(if } n \text{ is an even integer } 2p) \\[2ex] \dfrac{\sqrt{\pi}2^{p-1}\Gamma\left(p+\frac{1}{2}\right)\Gamma(2p-1)i^m(2m)!2^m\Gamma\left(m+p-\frac{1}{2}\right)}{\Gamma\left(p-\frac{1}{2}\right)m!(2m+2p-2)!\Gamma(m+2p-1)}\dfrac{2m+2p-2}{2m+2p-1}\cdots\dfrac{3}{4}\dfrac{1}{2} & \\ & \text{(if } n \text{ is an odd integer } 2p+1) \end{cases}$$

Then we have

COROLLARY 4.2

$$\mathscr{P}(\varphi_{i_1\ldots i_n})=C_m[f_{i_1\ldots i_n}], \text{ for every } (i_1, \ldots, i_n)\in J_m.$$

PROOF. It is not difficult to see that $\{\varphi_{i_1\ldots i_n}: (i_1 \ldots i_n)\in J_m\}$ and $\{[f_{i_1\ldots i_n}]: (i_1 \ldots i_n)\in J_m\}$ are bases of $\Gamma(L_m)$ and $\mathscr{H}^{n,m}$ equivalent under the action of G. And so there exists a non-zero constant C'_m, which depends only n, m, such that $\mathscr{P}(\varphi_{i_1\ldots i_n})=C'_m[f_{i_1\ldots i_n}]$ for every $(i_1 \ldots i_n)\in J_m$. In order to know this constant, we shall calculate the value of $\mathscr{P}(\varphi_{m,0\ldots 0})$ at the point $(1, 0, \ldots, 0)$ in $\boldsymbol{R}^n$. It is shown in Vilenkin [8] that

$$[f_{m,0\ldots 0}](1, 0 \ldots 0)=\frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma(m+n-2)}{2^m\Gamma\left(m+\frac{n-2}{2}\right)\Gamma(n-2)}$$

On the other hand, we have

$$\mathscr{P}(\varphi_{m,0\dots 0})(1,0\dots 0)=\frac{i^m}{m!}\int_{S^{n-1}}(\omega_1^2+\omega_2^2)^m d\omega$$

$$=\begin{cases} 2^{p-1}\Gamma(p)\dfrac{i^m(2m)!}{m!(2m+2p-2)!} & (\text{if } m=0(\text{mod}2)) \\ \sqrt{\pi}2^{p-1}\Gamma\left(p+\dfrac{1}{2}\right)\dfrac{i^m(2m)!}{m!(2m+2p-2)!}\dfrac{2m+2p-2}{2m+2p-1}\cdots\dfrac{3}{4}\dfrac{1}{2} & (\text{if } m=1\ (\text{mod } 2)). \end{cases}$$

Thus we have $C'_m=C_m$.

§ 5. Harmonic functions and Poisson transform

Let us consider the differential equation

$$\Delta f=0 \qquad f\in C^\infty(\boldsymbol{R}^n)$$

where
$$\Delta=-\sum_{i=1}\frac{\partial}{\partial x_i^2}.$$

We denote by $C^\infty(\boldsymbol{R}^n)_\Delta$ the space of all C^∞-differentiable functions f which satisfy $\Delta f=0$, and by $\oplus\sum_{m\geq 0}\mathscr{H}^{n,m}$ the space of the series $\sum_{m\geq 0}f_m(f_m\in\mathscr{H}_{n,m})$ which converges absolutely and uniformly on every compact subset in $\boldsymbol{R}^n$. Then we have the following

PROPOSITION 5.1. $C^\infty(\boldsymbol{R}^n)_\Delta=\oplus\sum_{m\geq 0}\mathscr{H}^{n,m}$

PROOF. By definition it is easy to see that $C^\infty(\boldsymbol{R}^n)_\Delta$ contains $\oplus\sum_{m\geq 0}\mathscr{H}^{n,m}$. So we have only to prove that $\oplus\sum_{m\geq 0}\mathscr{H}^{n,m}$ contains $C^\infty(\boldsymbol{R}^n)_\Delta$. Since the laplacian Δ is an elliptic defferential operator, each element in $C^\infty(\boldsymbol{R}^n)_\Delta$ is a real analytic function on $\boldsymbol{R}^n$. It is well-known that a harmonic function f has an expansion $f=\sum_{m\geq 0}f_m$ $(f_m\in P^{n,m})$ which converges absolutely and uniformly on every compact subsets in $\boldsymbol{R}^n$. From $\Delta f=0$, we have $\Delta f_m=0$ each m. Therefore f_m is in $\mathscr{H}^{n,m}$, and as f is in $\oplus\sum_{m\geq 0}\mathscr{H}^{n,m}$. This completes the proof of the lemma.

Let $\{\varphi_{i_1\dots i_n}: (i_1\dots i_n)\in J_m\}$ be the basis of $\Gamma(L_m)$, which is defined in § 3. We denote by $\oplus\sum_{m\geq 0}\Gamma(L_m)$ the space of all formal series $\sum_{m\geq 0}\sum_{(i_1\dots i_n)\in J_n}a_{i_1\dots i_n}\varphi_{i_1\dots i_n}$ with complex coefficients satisfying $\sum_{m\geq 0}\frac{\|a_m\|}{m!}s^m<+\infty$ for all $s>0$ where $\|a_m\|=\max_{(i_1\dots i_n)\in J_m}|a_{i_1\dots i_n}|$. We remark here that every element $\sum_{m\geq 0}\sum_{(i_1\dots i_n)\in J_m}a_{i_1\dots i_n}\varphi_{i_1\dots i_n}$

in $\oplus \sum_{m\geqq 0} \Gamma(L_m)$ satisfies $\sum_{m\geqq 0} \frac{|p(m)|\|a_m\|}{m!} s^m < +\infty$ for any polynomial P in m and for all $s>0$.

The following proposition assures that the Poisson integral $\mathscr{P}$ may be extended to a linear transformation of $\oplus \sum_{m\geqq 0} \Gamma(L_m)$ into $C^\infty(\boldsymbol{R}^n)_\Delta$

PROPOSITION 5.2. *For every* $\sum_{m\geqq 0} \sum_{(i_1\dots i_n)\in J_m} a_{i_1\dots i_n}\varphi_{i_1\dots i_n} \in \oplus \sum_{m\geqq 0} \Gamma(L_m)$, *the series*

$$f(x)=\sum_{m\geqq 0} \sum_{(i_1\dots i_n)\in J_m} a_{i_1\dots i_n}\int_{G/H} e^{i<x,\omega>}\varphi_{i_1\dots i_n}(\omega)d\omega$$

converges absolutely and uniformly on compact subsets in $\boldsymbol{R}^n$, *and* f *is an element of* $C^\infty(\boldsymbol{R}^n)_\Delta$.

PROOF. For non-negative integers k and m and for a multi-index $(i_1, \dots, i_n)$ in J_m, we have

$$|(\Delta^k f_{i_1\dots i_n})(x)| \leqq n^2 m^{2k} r^{m-2k}$$

where $x_1^2+x_2^2+\cdots+x_n^2=r^2$. It follows from the above inequality and the definition of the harmonic projection that $|[f_{i_1\dots i_n}](x)| \leqq n^2 e^{m/2} r^m$ for every $(i_1 \dots i_n) \in J_m$. We fix $r_0>0$. For any x in $\boldsymbol{R}^n$ such that $\|x\|<r_0$, we have

$$\begin{aligned}
&\sum_{m\geqq 0} \sum_{(i_1\dots i_n)\in J_m} |a_{i_1\dots i_n}\int_{G/H} e^{i<x,\omega>}\varphi_{i_1\dots i_n}(\omega)d\omega| \\
&= \sum_{m\geqq 0} \sum_{(i_1\dots i_n)\in J_m} |C_m a_{i_1\dots i_n}[f_{i_1\dots i_n}](x)| \\
&\leqq \sum_{m\geqq 0} \sum_{(i_1\dots i_n)\in J_m} C_m |a_{i_1\dots i_n}| |[f_{i_1\dots i_n}](x)| \\
&\leqq \sum_{m\geqq 0} b_n d(m) \frac{\|a_m\|}{m!} (2\sqrt{e}r)^m \qquad \text{(where } d(m)=\dim \mathscr{H}^{n,m}) \\
&< \sum_{m\geqq 0} b_n d(m) \frac{\|a_m\|}{m!} (2\sqrt{e}r_0)^m
\end{aligned}$$

where

$$b_n=\begin{cases} a_n=\dfrac{2^{p+1}p^2\Gamma(p)\Gamma(2p-2)}{\Gamma(p-1)} & (\text{if } n=2p) \\ \dfrac{\sqrt{\pi}2^{p-1}(2p+1)^2\Gamma\left(p+\frac{1}{2}\right)\Gamma(2p-1)}{\Gamma\left(p-\frac{1}{2}\right)} & (\text{if } n=2p+1) \end{cases}$$

which is convergent since d is a polynomial function is m, thus the series in the proposition converges absolutely and uniformly on every compact subset in $\boldsymbol{R}^n$. Moreover, f belongs to $C^\infty(\boldsymbol{R}^n)_\Delta$, since each term in the expansion of f is a harmonic function on $\boldsymbol{R}^n$. This completes the proof of Proposition 5.2.

Now we can define the Poisson transform $\mathscr{P}$ of $\oplus \sum_{m\geqq 0} \Gamma(L_m)$ into $C^\infty(\boldsymbol{R}^n)_\Delta$:

$$(\mathscr{P}\varphi)(x)=\sum_{m\geqq 0}\sum_{(i_1\dots i_n)\in J_m} a_{i_1\dots i_n}\int_{G/H} e^{i<x,\omega>}\varphi_{i_1\dots i_n}(\omega)d\omega$$

for every $\varphi=\sum_{m\geqq 0}\sum_{(i_1\dots i_n)\in J_m} a_{i_1\dots i_n}\varphi_{i_1\dots i_n}$ in $\oplus\sum_{m\geqq 0}\Gamma(L_m)$. Then the following theorem says that every solution of the differential equation $\Delta f=0$ can be given by the "Poisson transform" of an element in $\oplus\sum_{m\geqq 0}\Gamma(L_m)$.

THEOREM 2. *The map* $\mathscr{P}$ *is a linear isomorphism of* $\oplus\sum_{m\geqq 0}\Gamma(L_m)$ *onto* $C^\infty(\boldsymbol{R}^n)_\Delta$.

PROOF. From Corollary 4.2. and Proposition 5.2, $\mathscr{P}$ is injective, and so it suffices to show that $\mathscr{P}$ is surjective.

Let f be an arbitrary element of $C^\infty(\boldsymbol{R}^n)_\Delta$. By Proposition 5.1, f has an absolutely convergent expansion:

$$f=\sum_{m\geqq 0}\sum_{(i_1\dots i_n)\in J_m} a_{i_1\dots i_n}[f_{i_1\dots i_n}]$$

where $a_{i_1\dots i_n}\in\boldsymbol{C}$.

Since each term $a_{i_1\dots i_n}[f_{i_1\dots i_n}]$ $((i_1\dots i_n)\in J_m)$ is a polynomial of degree m, the series $\sum_{m\geqq 0}\sum_{(i_1\dots i_n)\in J_m} a_{i_1\dots i_n}[f_{i_1\dots i_n}]$ converges absolutely not only on $\boldsymbol{R}^n$ but also on $\boldsymbol{C}^n$. Especially the above series converges absolutely at the point $(t, \omega t, \dots, \omega^{n-1}t)$ in $\boldsymbol{C}^n$, where t is a positive real number and $\omega=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$. Thus we have

$$\sum_{m\geqq 0}\sum_{(i_1\dots i_n)\in J_m}|a_{i_1\dots i_n}||[f_{i_1\dots i_n}](t, \omega t, \dots, \omega^{n-1}t)|<+\infty$$

By the exactness of the sequence (1) in §2, we have

$$[f_{i_1\dots i_n}]-f_{i_1\dots i_n}\in r^2p^{n,m-2},$$

so we have

$$|[f_{i_1\dots i_n}](t, \omega t, \dots, \omega^{n-1}t)|=|f_{i_1\dots i_n}(t,\dots, \omega^{n-1}t)|=t^m$$

Therefore

$$\sum_{m\geqq 0}\sum_{(i_1\dots i_n)\in J_m}|a_{i_1\dots i_n}|t^m<+\infty \qquad \text{(for any } t>0)$$

Hence $\sum_{m\geqq 0} \|a_m\| t^m < +\infty$ for any $t>0$ where $\|a_m\| = \max\{|a_{i_1\dots i_n}| : (i_1 \dots i_n) \in J_m\}$

From Cauchy-Hadamard's test, we have

$$\overline{\lim_{m\to\infty}} (\|a_m\|)^{1/m} = 0$$

and so,

$$\overline{\lim_{m\to\infty}} \left(\frac{\|a_m\|}{|C_m| m!} \right)^{1/m} = 0$$

This implies that

$$\sum_{m\geqq 0} \frac{1}{m!} \left\| \frac{a_m}{c_m} \right\| s^m < +\infty \qquad \text{for any } s>0.$$

Now we put
$$\varphi = \sum_{m\geqq 0} \sum_{(i_1\dots i_n)\in J_m} C_m^{-1} a_{i_1\dots i_n} \varphi_{i_1\dots i_n}.$$

Then φ lies in $\oplus \sum_{m\geqq 0} \Gamma(L_m)$ and satisfies $\mathscr{P}\varphi = f$.

This completes the proof of the theorem.

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