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Locally Coalescent Classes of Lie Algebras

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Introduction

In the recent study of infinite-dimensional Lie algebras, an important role has been played by the notion of coalescent classes of Lie algebras introduced in [5, 6]. A more general notion, locally coalescent classes, has been introduced by B. Hartley [5] and investigated by R. K. Amayo [2, 3, 4]. In this paper we shall develop a number of characterizations of locally coalescent classes of Lie algebras and investigate the radicals of Lie algebras defined by such classes.

In Section 2, we shall show several lemmas on locally coalescent and persistent classes for the subsequent sections. In Section 3, we shall show characterizations of locally coalescent classes by making use of the closure operations M and N. Namely, we show that a class $\mathfrak{X}$ is locally coalescent if and only if any class $\mathfrak{Y}$ such that $\mathfrak{X} \leq \mathfrak{Y} \leq \mathrm{M}\mathfrak{X}$ is locally coalescent and that, when the basic field is of characteristic 0 and $\mathfrak{X}$ is I-closed, $\mathfrak{X}$ is locally coalescent if and only if $\mathrm{M}\mathfrak{X} = \mathrm{N}\mathfrak{X}$, and if and only if $\mathrm{M}\mathfrak{X}$ is persistent (Theorems 3.2, 3.3 and 3.5). We also show that, if the basic field is of characteristic 0, $\mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$ is locally coalescent and persistent (Proposition 3.7).

In [3] it is stated that, when the basic field is of characteristic 0, B. Hartley has introduced the radical $\beta^*(L)$ of a Lie algebra L as the subalgebra generated by all the $\mathrm{L}\mathfrak{N}$ subideals of L, where $\mathrm{L}\mathfrak{N}$ is locally coalescent. More generally, we define the radical $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ of L for any locally coalescent class $\mathfrak{X}$ in a similar way and investigate its properties in Sections 4 and 5. We show that if $\mathfrak{X}$ is complete (resp. strongly complete), then $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ is invariant under every locally finite derivation (resp. every derivation) of L (Theorems 5.1 and 5.5). We also treat of the special case where $\mathfrak{X}$ is $\mathfrak{S} \cap \mathfrak{F}$, $\mathfrak{N}$ or $\mathfrak{S}$ (Corollaries 5.3 and 5.6).

§1. Preliminaries

We shall be concerned with Lie algebras over a field Φ which are not necessarily finite-dimensional. Throughout this paper, L will be an arbitrary Lie algebra over a field Φ and $\mathfrak{X}$ will be an arbitrary class of Lie algebras over Φ, that is, an arbitrary collection of Lie algebras over Φ such that $(0) \in \mathfrak{X}$ and if $H \in \mathfrak{X}$ and $H \simeq K$ then $K \in \mathfrak{X}$, unless otherwise specified.

We mainly employ the terminology and notations which were used in [8, 9].

$\mathrm{L}\mathfrak{X}$ is the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subalgebra of L. $\mathrm{M}\mathfrak{X}$ is the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subideal of L [3, 9]. $\mathrm{N}\mathfrak{X}$ is the class of Lie algebras which are generated by their $\mathfrak{X}$ subideals. L, M and N are closure operations in the sense of [7].

$\mathfrak{X}$ is I-closed provided every subideal of an $\mathfrak{X}$ algebra is always an $\mathfrak{X}$ algebra. $\mathfrak{X}$ is N_0-closed provided the sum of any two $\mathfrak{X}$ ideals of L is an $\mathfrak{X}$ ideal of L.

$\mathfrak{X}$ is locally coalescent [3, 5] provided for any $\mathfrak{X}$ subideals H, K of L and for any finite subset F of $<H, K>$ there exists an $\mathfrak{X}$ subideal X of L such that $F \subseteq X \leq <H, K>$. $\mathfrak{X}$ is persistent [4] provided for any $\mathfrak{X}$ subideals H, K of L $<H, K>$ is an $\mathfrak{X}$ subalgebra of L. Hence every coalescent class of Lie algebras is locally coalescent and persistent.

$\mathfrak{F}$, $\mathfrak{N}$ and $\mathfrak{S}$ are the classes of all finite-dimensional, all nilpotent and all solvable Lie algebras respectively. The universal class $\mathfrak{D}$ is the class of all Lie algebras. When the basic field is of characteristic 0, it is known that $\mathfrak{N}$, $\mathfrak{S}$, $\mathrm{L}\mathfrak{N}$, $\mathrm{L}\mathfrak{F}$ and $\mathfrak{D}$ are locally coalescent [3, 4] and $\mathfrak{S}$ is persistent [1].

§ 2. Lemmas

We begin with the following

Lemma 2.1. *Let* A *and* B *be closure operations. If* $\mathrm{A} \leq \mathrm{B}$, *then*

$$\mathrm{AB} = \mathrm{BA} = \mathrm{B}.$$

Proof. If $\mathrm{A} \leq \mathrm{B}$, for any class $\mathfrak{X}$ of Lie algebras we have

$$\mathrm{B}\mathfrak{X} \leq \begin{Bmatrix} \mathrm{AB}\mathfrak{X} \\ \mathrm{BA}\mathfrak{X} \end{Bmatrix} \leq \mathrm{BB}\mathfrak{X} = \mathrm{B}\mathfrak{X}$$

and therefore

$$\mathrm{AB}\mathfrak{X} = \mathrm{BA}\mathfrak{X} = \mathrm{B}\mathfrak{X}.$$

We now show the following lemma, the first part of which is stated in [3].

Lemma 2.2. (1) *If $\mathfrak{X}$ is locally coalescent, then* $\mathrm{M}\mathfrak{X} = \mathrm{N}\mathfrak{X} \leq \mathrm{L}\mathfrak{X}$.
(2) *If* $\mathrm{L}\mathfrak{X}$ *is locally coalescent, then* $\mathrm{ML}\mathfrak{X} = \mathrm{NL}\mathfrak{X} = \mathrm{L}\mathfrak{X}$.

Proof. (1) Assume that $L \in \mathrm{N}\mathfrak{X}$. L is then generated by $\mathfrak{X}$ subideals L_α of L. Let F be any finite subset of L. Then there exist $L_{\alpha_1}, \dots, L_{\alpha_n}$ among L_α's such that

$$F \subseteq < L_{\alpha_1}, \ldots, L_{\alpha_n} > .$$

Since $\mathfrak{X}$ is locally coalescent, there exists an $\mathfrak{X}$ subideal H of L such that

$$F \subseteq H \leq < L_{\alpha_1}, \ldots, L_{\alpha_n} > .$$

Therefore $L \in \mathrm{M}\mathfrak{X}$. Thus we have $\mathrm{N}\mathfrak{X} \leq \mathrm{M}\mathfrak{X}$. The converse inclusion is evident.

(2) By the first statement,

$$\mathrm{L}\mathfrak{X} \leq \mathrm{ML}\mathfrak{X} = \mathrm{NL}\mathfrak{X} \leq \mathrm{LL}\mathfrak{X} = \mathrm{L}\mathfrak{X} .$$

Hence $\mathrm{ML}\mathfrak{X} = \mathrm{NL}\mathfrak{X} = \mathrm{L}\mathfrak{X}$.

This completes the proof.

LEMMA 2.3. *If $\mathfrak{X}$ is locally coalescent, then $\mathrm{M}\mathfrak{X}$ is N-closed and especially N_0-colsed.*

PROOF. By Lemmas 2.1 and 2.2,

$$\mathrm{N}(\mathrm{M}\mathfrak{X}) = \mathrm{N}\mathfrak{X} = \mathrm{M}\mathfrak{X} .$$

LEMMA 2.4. *If $\mathfrak{X}$ is I-closed, then $\mathrm{L}\mathfrak{X}$ and $\mathrm{M}\mathfrak{X}$ are I-closed.*

PROOF. Assume that $L \in \mathrm{M}\mathfrak{X}$ (resp. $\mathrm{L}\mathfrak{X}$) and H si L. For any finite subset F of H, there exists a $K \in \mathfrak{X}$ such that

$$F \subseteq K \text{ si } L \quad (\text{resp. } \leq L)$$

Then $H \cap K$ si K. Since $\mathfrak{X}$ is I-closed, $H \cap K \in \mathfrak{X}$. Furthermore

$$F \subseteq H \cap K \text{ si } H \quad (\text{resp. } \leq H),$$

whence $H \in \mathrm{M}\mathfrak{X}$ (resp. $\mathrm{L}\mathfrak{X}$). Thus $\mathrm{M}\mathfrak{X}$ (resp. $\mathrm{L}\mathfrak{X}$) is I-closed.

LEMMA 2.5. *$\mathrm{N}\mathfrak{X}$ is persistent.*

PROOF. Assume that H, K are $\mathrm{N}\mathfrak{X}$ subideals of an arbitrary Lie algebra L. Then H (resp. K) is generated by $\mathfrak{X}$ subideals H_α of H (resp. K_β of K). It follows that H_α and K_β are $\mathfrak{X}$ subideals of L and therefore of $<H, K>$ and generate $<H, K>$. Hence $<H, K> \in \mathrm{N}\mathfrak{X}$. Thus $\mathrm{N}\mathfrak{X}$ is persistent.

§ 3. Characterizations of locally coalescent classes

In this section we shall study local coalescence of classes of Lie algebras and present several conditions for a class to be locally coalescent.

THEOREM 3.1. *Let $\mathfrak{Y}$ be any class such that $\mathfrak{X}\leq\mathfrak{Y}\leq \mathrm{N}\mathfrak{X}$. Then $\mathfrak{Y}$ is locally coalescent if and only if, for any finite family $\{H_1,\dots, H_n\}$ of $\mathfrak{X}$ subideals of an arbitrary Lie algebra L and for any finite subset F of $<H_1,\dots, H_n>$, there exists a $\mathfrak{Y}$ subideal X of L such that*

$$F\subseteq X\leq <H_1,\dots, H_n>.$$

PROOF. If $\mathfrak{Y}$ is locally coalescent, then the condition is obviously satisfied since $\mathfrak{X}\leq\mathfrak{Y}$. Conversely assume that the condition is satisfied. Let H and K be $\mathfrak{Y}$ subideals of L and let F be any finite subset of $<H, K>$. Since $\mathfrak{Y}\leq \mathrm{N}\mathfrak{X}$, H (resp. K) is generated by $\mathfrak{X}$ subideals H_α of H (resp. K_β of K). Hence there exists a finite subset $\{H_{\alpha_1},\dots, H_{\alpha_m}, K_{\beta_1},\dots, K_{\beta_l}\}$ among H_α's and K_β's such that

$$F\subseteq <H_{\alpha_1},\dots, H_{\alpha_m}, K_{\beta_1},\dots, K_{\beta_l}>.$$

It follows that H_{α_i} si L and K_{β_j} si L. Therefore by our assumption there exists a $\mathfrak{Y}$ subideal X of L such that

$$F\subseteq X\leq <H_{\alpha_1},\dots, H_{\alpha_m}, K_{\beta_1},\dots, K_{\beta_l}>.$$

Hence

$$F\subseteq X\leq <H, K>.$$

Therefore $\mathfrak{Y}$ is locally coalescent.

The proof is complete.

THEOREM 3.2. *Let $\mathfrak{Y}$ be any class of Lie algebras such that $\mathfrak{X}\leq\mathfrak{Y}\leq \mathrm{M}\mathfrak{X}$. Then $\mathfrak{X}$ is locally coalescent if and only if $\mathfrak{Y}$ is locally coalescent.*

PROOF. Let $\mathfrak{Y}$ be locally coalescent. Assume that H and K are $\mathfrak{X}$ subideals of an arbitrary Lie algebra L and put $J=<H, K>$. Then H and K are $\mathfrak{Y}$ subideals of L. Therefore for any finite subset F of J there exists a $\mathfrak{Y}$ subideal X of L such that

$$F\subseteq X\leq J.$$

Since $X\in\mathfrak{Y}\leq \mathrm{M}\mathfrak{X}$, there exists an $\mathfrak{X}$ subideal Y of X containing F. It follows that Y is an $\mathfrak{X}$ subideal of L such that

$$F\subseteq Y\leq J.$$

Thus $\mathfrak{X}$ is locally coalescent.

Conversely if $\mathfrak{X}$ is locally coalescent, the condition in Theorem 3.1 is obviously satisfied. Therefore by Theorem 3.1 $\mathfrak{Y}$ is locally coalescent.

This completes the proof.

THEOREM 3.3. *Let the basic field be of characteristic* 0. *If* $\mathfrak{X}$ *is* I-*closed, then the following statements are equivalent*:

(1) $\mathfrak{X}$ *is locally coalescent.*

(2) $\mathrm{M}\mathfrak{X}=\mathrm{N}\mathfrak{X}$.

(3) *For any* $\mathfrak{X}$ *subideals* H, K *of an arbitrary Lie algebra* L, $<H, K> \in \mathrm{M}\mathfrak{X}$.

PROOF. (1)$\Rightarrow$(2). This follows from Lemma 2.2.

(2)$\Rightarrow$(3). This is immediate since $<H, K> \in \mathrm{N}\mathfrak{X}$.

(3)$\Rightarrow$(1). Assume the statement (3). Suppose that H and K are $\mathfrak{X}$ subideals of L and put $J=<H, K>$. Then $J \in \mathrm{M}\mathfrak{X}$. Therefore for any finite subset F of J, there exists an $\mathfrak{X}$ subideal X of J containing F. It is known that the universal class $\mathfrak{D}$ is locally coalescent. Therefore, regarding H, K as members of $\mathfrak{D}$, we see that there exists a subideal Y of L such that

$$F \subseteq Y \leq J.$$

It follows that

$$X \cap Y \text{ si } Y \text{ si } L.$$

Furthermore $X \cap Y$ si X and $X \in \mathfrak{X}$. Since $\mathfrak{X}$ is I-closed, $X \cap Y \in \mathfrak{X}$. Thus

$$F \subseteq X \cap Y \text{ si } L, \in \mathfrak{X}.$$

Therefore $\mathfrak{X}$ is locally coalescent.

This completes the proof.

As a consequence of this theorem we have the following result which is due to R. K. Amayo [4].

COROLLARY 3.4. *Let the basic field be of characteristic* 0. *If* $\mathfrak{X}$ *is* I-*closed and persistent, then* $\mathfrak{X}$ *is locally coalescent.*

PROOF. If $\mathfrak{X}$ is persistent, then we have the statement (3) of Theorem 3.3, since $\mathfrak{X} \leq \mathrm{M}\mathfrak{X}$. Hence the statement is immediate from Theorem 3.3.

We now show the following two theorems by using Corollary 3.4.

THEOREM 3.5. *If* $\mathfrak{X}$ *is locally coalescent, then* $\mathrm{M}\mathfrak{X}$ *is persistent. When the basic field is of characteristic* 0 *and* $\mathfrak{X}$ *is* I-*closed*, $\mathfrak{X}$ *is locally coalescent if and only if* $\mathrm{M}\mathfrak{X}$ *is persistent.*

PROOF. If $\mathfrak{X}$ is locally coalescent, by Lemma 2.2

$$\mathrm{M}\mathfrak{X} = \mathrm{N}\mathfrak{X}.$$

Therefore $\mathrm{M}\mathfrak{X}$ is persistent by Lemma 2.5.

Now assume that $\mathfrak{X}$ is I-closed and $\mathrm{M}\mathfrak{X}$ is persistent. Then by Lemma 2.4 $\mathrm{M}\mathfrak{X}$ is I-closed. We now use Corollary 3.4 to see that $\mathrm{M}\mathfrak{X}$ is locally coalescent. Theorem 3.2 tells us that $\mathfrak{X}$ is then locally coalescent.

This completes the proof.

Corollary 3.6. *If $\mathrm{L}\mathfrak{X}$ is locally coalescent, then it is persistent. When the basic field is of characteristic* 0 *and $\mathrm{L}\mathfrak{X}$ is* I-*closed, $\mathrm{L}\mathfrak{X}$ is locally coalescent if and only if it is persistent.*

Proof. If $\mathrm{L}\mathfrak{X}$ is locally coalescent, by Lemma 2.2

$$\mathrm{L}\mathfrak{X} = \mathrm{ML}\mathfrak{X}.$$

Therefore the statement follows from Corollary 3.4 and Theorem 3.5.

We here recall the definition of completeness of a class of Lie algebras.

Let L be a Lie algebra over a field Φ of characteristic 0. Let Φ_0 be the field of formal power series

$$a = \sum_{i=m}^{\infty} a_i t^i, \quad a_i \in \Phi, \quad m \in Z$$

and $L^{\#}$ be the Lie algebra over Φ_0 consisting of all formal power series

$$x = \sum_{i=m}^{\infty} x_i t^i, \quad x_i \in L, \quad m \in Z$$

[5]. Put

$$L^* = L \otimes_{\Phi} \Phi_0$$

[7]. L^* is a Lie algebra over Φ_0 and is naturally embedded as a subalgebra of $L^{\#}$. Then the elements of L^* are of the form

$$x = \sum_{i=m}^{\infty} x_i t^i$$

where all x_i lie in some finite-dimensional subspace of L. For $M \leq L$ M^* (resp. $M^{\#}$) is the set of all elements $x \in L^*$ (resp. $L^{\#}$) with $x_i \in M$ for all i. Then $M^* \leq L^*$ (resp. $M^{\#} \leq L^{\#}$). For $K \leq L^{\#}$ $K^{\flat}$ is the set of all leading coefficients of elements of K, together with 0. Then $K^{\flat} \leq L$. We have several properties of M^* and $K^{\flat}$ which are quite similar to those of $M^{\#}$ and $K^{\flat}$ in [5, 8].

A class $\mathfrak{X}$ of Lie algebras is said to be complete [3] provided the following conditions are satisfied:

(i) If $L \in \mathfrak{X}$, then $L^* \in \mathfrak{X}$ as a Lie algebra over Φ_0.

(ii) If $H \leq L^*$ and $H \in \mathfrak{X}$ as a Lie algebra over Φ_0, then $H^\flat \in \mathfrak{X}$.

The classes $\mathfrak{F}$, $\mathfrak{N}$, $\mathfrak{S}$, $\mathrm{L}\mathfrak{N}$ and $\mathrm{L}\mathfrak{F}$ are known to be complete [3]. Now we show the following

Proposition 3.7. *If the basic field Φ is of characteristic* 0, $\mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$ *is complete, locally coalescent and persistent.*

Proof. Assume that $L \in \mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$. For any finite subset $\{x_1, \ldots, x_n\}$ of L^*,

$$x_i = \sum_{j=1}^{m_i} x_{ij} \otimes \lambda_{ij}, \quad x_{ij} \in L, \quad \lambda_{ij} \in \Phi_0 .$$

There exists an $\mathfrak{S} \cap \mathfrak{F}$ subalgebra H of L such that

$$< x_{11}, \ldots, x_{1m_1}, \ldots, x_{n1}, \ldots, x_{nm_n} > \leq H \leq L .$$

Then

$$\{x_1, \ldots, x_n\} \subseteq H^* \leq L^*$$

and since $\mathfrak{S} \cap \mathfrak{F}$ is complete,

$$H^* \in \mathfrak{S} \cap \mathfrak{F} .$$

Therefore $L^* \in \mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$.

Conversely, assume that $H \leq L^*$ and $H \in \mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$. For any finite subset $\{y_1, \ldots, y_m\}$ of $H^\flat$, there exists $y_i' \in H$ with y_i as its leading coefficient. Since $H \in \mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$, there exists a $K \in \mathfrak{S} \cap \mathfrak{F}$ such that

$$\{y_1', \ldots, y_m'\} \subseteq K \leq H .$$

Then

$$\{y_1, \ldots, y_m\} \subseteq K^\flat \leq H^\flat$$

and since $\mathfrak{S} \cap \mathfrak{F}$ is complete,

$$K^\flat \in \mathfrak{S} \cap \mathfrak{F} .$$

Therefore $H^\flat \in \mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$. Thus we see that $\mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$ is complete.

$\mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$ is obviously I-closed and also N_0-closed. In [3] it has been shown that if $\mathfrak{X}$ is complete and $\{\mathrm{I}, \mathrm{N}_0\}$-closed and if $\mathfrak{X} \leq \mathrm{L}\mathfrak{F}$ then $\mathfrak{X}$ is locally coalescent. Therefore $\mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$ is locally coalescent. By Corollary 3.6 $\mathrm{L}(\mathfrak{S} \cap \mathfrak{F})$ is also persistent.

Thus the proof is complete.

§4. $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ over fields of arbitrary characteristic

In this section we shall first define $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ for a locally coalescent class $\mathfrak{X}$ in the same way as we did it in [8, 10] for a coalescent class, and we shall show its properties which are analogous to the properties of the corresponding radical defined for a coalescent class in Section 3 in [9].

Definition 4.1. *If $\mathfrak{X}$ is locally coalescent, we denote by* $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ *the subalgebra generated by all the $\mathfrak{X}$ subideals of L.*

Proposition 4.2. *If $\mathfrak{X}$ is locally coalescent, then* $\mathrm{Rad}_{\mathrm{M}\mathfrak{X}}(L)$ *is the unique maximal* $\mathrm{M}\mathfrak{X}$ *ideal of L.*

Proof. $\mathrm{M}\mathfrak{X}$ is N_0-closed by Lemma 2.3. Hence $\mathrm{Rad}_{\mathrm{M}\mathfrak{X}}(L)$ is defined as the sum of all the $\mathrm{M}\mathfrak{X}$ ideals of L [8]. Since $\mathrm{Rad}_{\mathrm{M}\mathfrak{X}}(L)$ belongs to $\mathrm{N}(\mathrm{M}\mathfrak{X})$, it belongs to $\mathrm{M}\mathfrak{X}$ by Lemma 2.3. Therefore it is the unique maximal $\mathrm{M}\mathfrak{X}$ ideal.

Proposition 4.3. *If $\mathfrak{X}$ is locally coalescent, then* $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ *is the union of all the $\mathfrak{X}$ subideals of L and belongs to* $\mathrm{M}\mathfrak{X}$.

Proof. By Lemma 2.2

$$\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)\in \mathrm{N}\mathfrak{X} = \mathrm{M}\mathfrak{X}.$$

Let x be any element of $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$. Then there exist $\mathfrak{X}$ subideals $H_1,\dots, H_n$ of L such that

$$x\in \langle H_1,\dots, H_n\rangle .$$

Since $\mathfrak{X}$ is locally coalescent, there exists an $\mathfrak{X}$ subideal K of L such that

$$x\in K \leq \langle H_1,\dots, H_n \rangle .$$

Thus $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ is contained in the union of $\mathfrak{X}$ subideals of L. The converse inclusion is evident and the proof is complete.

Proposition 4.4. *Let $\mathfrak{X}$ be locally coalescent. If* $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ *is a subideal of L, then it is the unique maximal* $\mathrm{M}\mathfrak{X}$ *subideal of L. If* $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ *is an ideal of L, then it is the unique maximal* $\mathrm{M}\mathfrak{X}$ *ideal of L.*

Proof. By Proposition 4.3 $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ is an $\mathrm{M}\mathfrak{X}$ subalgebra of L. Lemma 3.3 in [9] says that every $\mathrm{M}\mathfrak{X}$ subideal of L is a union of $\mathfrak{X}$ subideals of L. Therefore if $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ is a subideal (resp. an ideal) of L, then it is the unique maximal

$\mathrm{M}\mathfrak{X}$ subideal (resp. ideal) of L.

§ 5. $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ over fields of characteristic 0

Through this section, we assume that the basic field Φ is of characteristic 0 and study further properties of the radical $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$.

THEOREM 5.1. *If $\mathfrak{X}$ is complete and locally coalescent, then* $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ *is invariant under any locally finite derivation of L.*

PROOF. Let H be any $\mathfrak{X}$ subideal of L and D be any locally finite derivation of L. Put $\alpha = \exp tD$. Then α is an automorphism of H^*. Since $\mathfrak{X}$ is complete, H^* is an $\mathfrak{X}$ subideal of L^*. It follows that $H^{*\alpha}$ is an $\mathfrak{X}$ subideal of L^*. Hence

$$<H^*, H^{*\alpha}> \leq \mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L^*).$$

Let x be any element of H. Then xD is either 0 or the leading coefficient of $x^\alpha - x$. By Proposition 4.3, there exists a $K \in \mathfrak{X}$ such that

$$x^\alpha - x \in K \text{ si } L^*.$$

Hence

$$xD \in K^\flat \text{ si } L.$$

Since $\mathfrak{X}$ is complete, $K^\flat \in \mathfrak{X}$. Therefore

$$xD \in \mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L).$$

Thus $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ is invariant under D. The proof is complete.

COROLLARY 5.2. *If $\mathfrak{X}$ is complete and locally coalescent and if $L \in \mathrm{L}\mathfrak{F}$,*

$$\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L) \lhd L.$$

If $\mathfrak{X}$ is furthermore N*-closed,*

$$\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L) = \mathrm{Rad}_{\mathfrak{X}}(L).$$

PROOF. If $L \in \mathrm{L}\mathfrak{F}$, any inner derivation of L is locally finite. Hence $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L) \lhd L$ by Theorem 5.1. If $\mathfrak{X}$ is furthermore N-closed,

$$\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L) \in \mathrm{N}\mathfrak{X} = \mathfrak{X}$$

and $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ is an $\mathfrak{X}$ ideal of L. Hence $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L) \subseteq \mathrm{Rad}_{\mathfrak{X}}(L)$. The converse inclusion is evident and the proof is complete.

COROLLARY 5.3. *$\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})-\mathrm{si}}(L)$ and $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}-\mathrm{si}}(L)$ are invariant under any locally finite derivation of L. Especially if $L\in \mathrm{L}\mathfrak{F}$,*

$$\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})-\mathrm{si}}(L) = \mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L) \lhd L\,,$$

$$\mathrm{Rad}_{\mathrm{L}\mathfrak{N}-\mathrm{si}}(L) = \mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L) \lhd L.$$

PROOF. By Proposition 3.7, $\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$ is complete and locally coalescent. It is also known that $\mathrm{L}\mathfrak{N}$ is complete and locally coalescent. Hence by Lemma 2.2, they are N-closed. Therefore the statement follows from Theorem 5.1 and Corollary 5.2.

REMARK. $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}-\mathrm{si}}(L)$ is $\beta^*(L)$ and the result on $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}-\mathrm{si}}(L)$ in Corollary 5.3 is in [3].

For convenience sake we now introduce the new concept of strong completeness which is stronger than completeness, in the following

DEFINITION 5.4. *We say a class $\mathfrak{X}$ of Lie algebras to be strongly complete provided the following conditions are satisfied:*

(i) *If $L\in\mathfrak{X}$, then $L^\sharp\in\mathfrak{X}$ as a Lie algebra over Φ_0.*

(ii) *If $H\leq L^\sharp$ and $H\in\mathfrak{X}$ as a Lie algebra over Φ_0, then $H^\flat\in\mathfrak{X}$.*

Owing to Lemmas 6.4 and 6.5 in [8] we see that $\mathfrak{N}$ and $\mathfrak{S}$ are strongly complete. By making use of strong completeness we prove the following

THEOREM 5.5. *Let $\mathfrak{X}$ be strongly complete and locally coalescent. Then*

(1) *$\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ is a characteristic ideal of L.*

(2) *$\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ is the unique maximal $\mathrm{M}\mathfrak{X}$ subideal and the unique maximal $\mathrm{M}\mathfrak{X}$ ideal of L.*

(3) *It holds that*

$$\mathrm{Rad}_{\mathrm{M}\mathfrak{X}-\mathrm{si}}(L) = \mathrm{Rad}_{\mathrm{M}\mathfrak{X}}(L) = \mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L).$$

PROOF. (1) Let H be an $\mathfrak{X}$ subideal of L and D be a derivation of L. Put $\alpha=\exp tD$. Then α is an automorphism of $L^\sharp$. Since $\mathfrak{X}$ is strongly complete, $H^\sharp$ is an $\mathfrak{X}$ subideal of $L^\sharp$. It follows that $H^{\sharp\alpha}$ is an $\mathfrak{X}$ subideal of $L^\sharp$. Let x be any element of H. Then

$$\{x, x^\alpha\} \subseteq \langle H^\sharp, H^{\sharp\alpha}\rangle\,.$$

Since $\mathfrak{X}$ is locally coalescent, there exists an $\mathfrak{X}$ subideal K of $L^\sharp$ such that

$$\{x, x^\alpha\} \subseteq K \leq \langle H^\sharp, H^{\sharp\alpha}\rangle\,.$$

xD is either 0 or the leading coefficient of $x^\alpha - x$. Hence

$$xD \in K^{\flat} \text{ si } L.$$

Furthermore $K^{\flat} \in \mathfrak{X}$ since $\mathfrak{X}$ is strongly complete. Therefore

$$xD \in \mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L).$$

Thus $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ is invariant under D.

(2) The statement is immediate from (1) and Proposition 4.4.

(3) $\mathrm{M}\mathfrak{X}$ is locally coalescent by Theorem 3.2 and N_0-closed by Lemma 2.3. Therefore (3) follows from (2) and Proposition 4.2.

This completes the proof.

Corollary 5.6. *$\mathrm{Rad}_{\mathfrak{S}-\mathrm{si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{N}-\mathrm{si}}(L)$) is a characteristic ideal, the unique maximal $\mathrm{M}\mathfrak{S}$ (resp. $\mathrm{M}\mathfrak{N}$) subideal and the unique maximal $\mathrm{M}\mathfrak{S}$ (resp. $\mathrm{M}\mathfrak{N}$) ideal of L. And it holds that*

$$\mathrm{Rad}_{\mathrm{M}\mathfrak{S}-\mathrm{si}}(L) = \mathrm{Rad}_{\mathrm{M}\mathfrak{S}}(L) = \mathrm{Rad}_{\mathfrak{S}-\mathrm{si}}(L),$$

$$\mathrm{Rad}_{\mathrm{M}\mathfrak{N}-\mathrm{si}}(L) = \mathrm{Rad}_{\mathrm{M}\mathfrak{N}}(L) = \mathrm{Rad}_{\mathfrak{N}-\mathrm{si}}(L).$$

Proof. $\mathfrak{S}$ and $\mathfrak{N}$ are strongly complete and locally coalescent. Therefore the statement is immediate from Theorem 5.5.

Remark. $\mathrm{Rad}_{\mathfrak{N}-\mathrm{si}}(L)$ is the Baer radical $\beta(L)$ of L and the statement on unique maximality for $\mathrm{Rad}_{\mathfrak{N}-\mathrm{si}}(L)$ in Corollary 5.6 is Theorem 4.1 in [9].

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