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A Comparison Theorem on Generalized Capacity

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1. Introduction.

For any real function $\phi(r)$ which is continuous and monotone decreasing for $r>0$ with $\lim_{r\to 0+}\phi(r)=+\infty$, Frostman [2] defined capacity C^{ϕ} with respect to ϕ. Let ϕ_0 be a fixed function of the same type and let us consider the following two properties.

i) *If* $C^{\phi_0}(K)=0$ *for some compact set* $K\subset R^d$, *then* $C^{\phi}(K)=0$ *and the converse implication is also valid.*

ii) $M_1\phi_0(r)\geqq\phi(r)\geqq M_2\phi_0(r)$ *for each* $0<r<\delta_0$, *where* M_i, $i=1, 2$ *are positive constants.*

It is evident by the definition of capacity that ii) implies i). If ϕ_0 is such that $r^d\phi_0(r)$ is monotone increasing with $\lim_{r\to 0+} r^d\phi_0(r)=0$ and $r^{-d}\int_0^r \phi_0(s)s^{d-1}\,ds\leqq M_3\phi_0(r)$ for $0<r<\delta$, we see that i) implies $\phi(r)\leqq M_4\phi_0(r)$ by Theorem 4 and Remark in S. J. Taylor [6]. Our object in the present note is to show that i) implies $M_5\phi_0(r)\leqq\phi(r)$ for $0<r<\delta$ in case $r^p\phi_0(r)$ is monotone increasing for some $0<p<d$, which is a stronger assumption on ϕ_0 than S. J. Taylor's. Our result is as follows.

THEOREM. *Let* $\phi_0(r)$ *and* $\phi(r)$ *be such that they are monotone decreasing, right continuous with* $\lim_{r\to 0+}\phi_0(r)=\lim_{r\to 0+}\phi(r)=+\infty$ *and* $r^p\phi_0(r)$ *is monotone increasing for some* $d>p>0$. *Then* i) *implies* ii).

2. Definitions and known results.

We set

$\Phi=\{\phi;\ \phi(r)$ *is positive, monotone decreasing and right continuous with* $\lim_{r\to 0+}\phi(r)=+\infty\}$,

and

$$\phi_p=\{\phi\in\Phi;\ r^p\phi(r) \text{ is monotone increasing for } 0<r<\delta\}$$

For a compact set K in Euclidean d-space R^d we set

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$$M_K=\{\mu;\mu \text{ is a measure defined on } K \text{ such that } \mu(K)=1\}.$$

and for $\mu\in M_K$

$$V_\mu^\phi(K) = \sup_{x\in R^d}\int_K \phi(|x-y|)\mu(dy)$$

$$V^\phi(K) = \inf_{\mu\in M_K} V_\mu^\phi(K).$$

Then we define the ϕ-capacity of K, denoted $C^\phi(K)$ by

a) *if* $V^\phi(K)=+\infty$, *then* $C^\phi(K)=0$

b) *if* $V^\phi(K)<+\infty$, *then* $\phi(C^\phi(K))=V^\phi(K)$.

The following is known for $\phi\in\Phi$.

(2.1) *If* $C^\phi(K)>0$, *then there exists* $\mu\in M_K$ *such that* $\int_K\phi(|x-y|)\mu(dy)$ $<M$ *everywhere for some constant* M.

Let us put

$$\Phi^c=\{\phi\in\Phi;\ \varphi \text{ is continuous on } (0,+\infty)\},\ \Phi_p^c=\Phi_p\cap\Phi^c$$

For h such that $1/h\in\Phi^c$, we define the Hausdorff measure Λ_h by

$$\Lambda_h(K) = \lim_{\delta\to 0}\Big[\inf_{\substack{\cup C_i\supset K\\ d(C_i)<\delta}}\sum_{i=1}^{\infty} h[d(C_i)]\Big],$$

where $d(C_i)$ denotes the diameter of C_i and the infimum is taken over all coverings of K by sequences $\{C_i\}$ of spheres with diameter less than δ. Then Frostman [2] shows

(2.2) $$\Lambda_{1/\phi}(K) = 0 \Longrightarrow C^\phi(K) = 0.^{1)}$$

The following result obtained by S. J. Taylor [6] plays an essential role in our proof.

If $\phi_i(t)\in\Phi^c$, $i=1, 2$, *are such that* $\phi_2\in\Phi_d$ *with* $\lim_{r\to 0} r^d\phi_2(r)=0$ *and* $r^{-d}\int_0^r \phi_2(s)d^{d-1}ds\leqq M\phi_2(r)$ *for* $0<r<\delta$, *and*

(2.3) $$\liminf_{r\to 0+}\frac{\phi_2(r)}{\phi_1(r)} = 0,$$

1) It is known by S. Kametani [3] that $\Lambda_{1/\phi}(K)<\infty$ implies $C^\phi(K)=0$. But we do not need this sharper result in this paper.

then there exists a compact set $K \subset R^d$ *such that*

$$\Lambda_{1/\phi_1}(K) = 0, \qquad C^{\phi_2}(K) > 0.$$

3. Proof of Theorem.

Throughout this section we always assume that a subset K of R^d is compact and denote the closed sphere with radius r by Q_r. If $\phi \in \Phi$ is such that

$$\int_0^1 \phi(s) s^{d-\alpha} ds < +\infty \qquad \text{for some } 0 < \alpha \leqq d \text{ we define}$$

$$[\phi]_\alpha(r) = \frac{1}{r^{d-\alpha+1}} \int_0^r \phi(s) s^{d-\alpha} ds.$$

In case $\alpha = 1$, we omit the suffix α. Then we have

$$[\phi]_\alpha \in \Phi^c_{d-\alpha+1} \tag{3.1}$$

Indeed by the monotone property of ϕ it holds that

$$[\phi]_\alpha(r) > \frac{1}{d-\alpha+1} \phi(r) \tag{3.2}$$

and $[\phi]'_\alpha(r) = r^{-1}\{-(d-\alpha+1)[\phi]_\alpha(r) + \phi(r)\}$ almost everywhere. If $\phi \in \Phi$ is such that $\int_0^1 \phi(s) s^{d-1} ds < +\infty$, then

$$\Lambda_{1/[\phi]}(K) = 0 \Longrightarrow C^\phi(K) = 0. \tag{3.3}$$

This is proved as follows[2]. If μ is a measure on Q_r such that $\int_{Q_r} \phi(|x-y|)\mu(dy) \leqq M$ on Q_r, then we have

$$\mu(Q_r) \leqq \frac{2^d M}{[\phi](r)}. \tag{3.4}$$

Indeed it holds that $|Q_r|^{-1} \int_{Q_r} dx \int_{Q_r} \phi(|x-y|)\mu(dy) \leqq M$, where $|Q_r|$ denotes the volume of Q_r and

$$\inf_{y \in Q_r} |Q_r|^{-1} \int_{Q_r} \phi(|x-y|) dx \geqq \frac{1}{2^d} [\phi](r).$$

For a given $\varepsilon > 0$ we choose a countable number of spheres $\{Q_r\}$ with radii r_k such

2) In case $\phi \in \Phi^c$ L. Carleson [1] proved the sharper result than (3.3); that is $\Lambda_{1/[\phi]}(K) < +\infty \Rightarrow C^\phi(K) = 0$. Since $\phi \in \Phi$ now, we give the proof here for completeness though the method is same.

that $\cup Q_k \subset K$ and

$$\sum_{k=1}^{\infty} \frac{1}{[\phi](r_k)} < \varepsilon . \tag{3.5}$$

If $C^\phi(K)>0$, there exists $\mu \in M_K$ such that $\int_K \phi(|x-y|)\mu(dy) \leqq M$ on R^d by (2.1). Let μ_k be a measure which is the restriction of μ on Q_k. Then it follows from (3.4) that $\mu_k(Q_k) \leqq 2^d M[\phi](r_k)^{-1}$. Hence we have by (3.5)

$$1 = \mu(K) \leqq \Sigma \mu_k(Q_k) \leqq 2^d M \Sigma \frac{1}{[\phi](r_k)} \leqq 2^d M \varepsilon .$$

As ε is arbitrary, we can conclude that $C^\phi(K)=0$.

Now we prove our Theorem. Choose $\alpha_0 = d-p+1$ and α such that $\alpha_0 > \alpha > 1$. Then we may assume that

$$\phi_0 \in \Phi^c_{d-\alpha+1} ,$$

because it is easily checked that $M_0[\phi_0]_\alpha(r) \leqq \phi_0(r) \leqq M'_0[\phi_0]_\alpha(r)$, $0<r<\delta$. In the following we fix α and always assume that (*i*) of Theorem holds. Since it holds that

$$[\phi_0](r) = \frac{1}{r^d}\int_0^r \phi_0(s) s^{d-\alpha+1} s^{\alpha-2} ds \leqq \frac{1}{\alpha-1}\, \phi_0(r) ,$$

ϕ_0 satisfies S. J. Taylor's condition (2.3). Next we show

$$[\phi](r) \leqq M_1 \phi_0(r) \tag{3.6}$$

for $0<r<\delta_1$. If (3.6) did not hold for any M_1 and δ_1, then $\liminf_{r\to 0+} \phi_0(r)([\phi](r))^{-1} = 0$. Hence there exists a compact set $K \subset R^d$ such that $C^{\phi_0}(K)>0$ and $\Lambda_{1/[\phi]}(K) = 0$ by (2.3). Using (3.3) we see that $C^{\phi_0}(K)>0$ and $C^\phi(K)=0$, which contradicts to (*i*). Combining (3.2) with (3.6) we have

$$\phi(r) \leqq dM_1 \phi_0(r) \tag{3.7}$$

for $0<r<\delta_1$. Next consider $[\phi]_\beta$ for $\alpha>\beta>1$. Then

$$\begin{aligned} &[\phi]_\beta \in \Phi^c_{d-\beta+1} \\ &[[\phi]_\beta](r) \leqq M_2[\phi]_\beta(r), \qquad 0<r<\delta_2 . \end{aligned} \tag{3.8}$$

Indeed the first assertion follows from (3.1), because $\int_0^1 \phi(s) s^{d-\beta} ds < +\infty$ by (3.7). Now we have

$$[[\phi]_\beta](r) = \frac{1}{r^d}\int_0^r [\phi]_\beta(s)s^{d-\beta+1}s^{\beta-2}ds \leqq \frac{1}{\beta-1}[\phi]_\beta(r),$$

which is the second assertion. If $\liminf_{r\to 0+}[\phi]_\beta(r)\phi_0(r)^{-1}=0$, then by (2.3), there exists a compact set K such that $C^{[\phi]_\beta}(K)>0$ and $\Lambda_{1/\varphi_0}(K)=0$, which implies $C^{[\phi]_\beta}(K)>0$ and $C^{\phi_0}(K)=0$ by (2.2). Hence $C^\phi(K)>0$ and $C^{\phi_0}(K)=0$ by (3.2), which contradicts to (i). Therefore

$$(3.9) \qquad [\phi]_\beta(r) \geqq M_3\phi_0(r)$$

for $0<r<\delta$. On the other hand it holds by (3.7) that

$$(3.10) \qquad [\phi]_\beta(r) \leqq \frac{dM_1}{r^{d-\beta+1}}\int_0^r \phi_0(s)s^{d-\alpha+1}s^{\alpha-\beta-1}ds \leqq \frac{dM_1}{\alpha-\beta}\phi_0(r).$$

Combining (3.9) with (3.10), we have

$$(3.11) \qquad M_4\phi_0(r) \geqq [\phi]_\beta(r) \geqq M_3\phi_0(r)$$

for $0<r<\delta_4$. Note that (3.11) holds for arbitrary β such that $\alpha>\beta>1$, although M_3, M_4, δ_4 depend on the choice of β. Choose $\alpha>\beta>\beta'>1$ and fix them. Then

$$M_5[\phi]_{\beta'}(r) \geqq [\phi]_\beta(r) \geqq M_6[\phi]_{\beta'}(r)$$

for $0<r<\delta_5$. Hence

$$0 \leqq M_5[\phi]_{\beta'}(r)-[\phi]_\beta(r) = \int_0^1 \phi(rt)t^{d-\beta}[M_5t^{\beta-\beta'}-1]dt \equiv \int_0^1 \cdot$$

Choosing $c=(2M_5)^{-(\beta-\beta')^{-1}}$, we have

$$0 \leqq \int_0^1 \cdot = \int_0^c \cdot + \int_c^1 \cdot \leqq -\frac{1}{2}\int_0^c \phi(rt)t^{d-\beta}dt + M_5\int_c^1 \phi(rt)t^{d-\beta}dt,$$

from which we get

$$\frac{M_5}{d-\beta+1}\phi(cr) \geqq \frac{1}{2}c^{d-\beta+1}[\phi]_\beta(cr)$$

for $0<r<\delta_5$. Since $(d-\beta+1)[\phi]_\beta(cr)\geqq\phi(cr)$, it holds that

$$(3.12) \qquad M_7[\phi]_\beta(r) \geqq \phi(r) \geqq M_8[\phi]_\beta(r)$$

for $0<r<\delta_6$. Combining (3.11) with (3.12), we can finish the proof of (ii).

4. Remarks.

a) Let $X=(x_t, \zeta, M_t, P_x)$ be a Markov process and assume that X is a Hunt process and it has Green function $G(x, y)$ with respect to Lebesque measure. Suppose that, for each compact set K, there exists a measure $\mu_K(dy)$ on Ksuch that $P_x(\sigma_K < +\infty) = \int G(x, y)\mu_K(dy)$, where $\sigma_K = \inf(t>0, x_t \in K)$. Then we can define the capacity $C(K)$ of K relative to X as usual setting $C(K) = \mu_K(K)$. If $M_1\phi(|x-y|) \leqq G(x, y) \leqq M_2\phi(|x-y|)$ $(M_1 \geqq M_2 > 0)$ holds on a neighborhood of the diagonal set and $\phi(r)$ is a monotone decreasing function on $(0, +\infty)$ with $\lim_{r\to 0+}\phi(r) = +\infty$, then it is easy to check that $C(K)=0$ if and only if $C^{\phi}(K)=0$. Hence we can apply our theorem to Markov processes of the above mentioned type. For example Theorem 3 in [5] is a corollary of our theorem.

b) We can apply our Theorem to calculate the singularity of Green functions. Consider a Markov process X on R^d $(d \geqq 3)$ which is a process subordinate to Brownian motion by a subordinator whose exponent is $\Psi(s)$ on $[0, +\infty)$. It is known that X has Green function $G(x, y) = \phi(|x-y|)$, where $\phi(r)$ is continuous and monotone decreasing on $(0, +\infty)$ with $\lim_{r\to 0+}\phi(r) = +\infty$ in case $\sup\{\beta \geqq 0;\ s^{-\beta}\Psi(s) \to +\infty \text{ as } s \to +\infty\}$ is positive. (See the proof of Corollary [4].) Let X^b be such that $\Psi(s) = \int_{\alpha'}^{\alpha} b(\beta)s^{\beta}d\beta$, $1 \geqq \alpha > \alpha' \geqq 0$, where $b(\beta)$ is positive continuous on $[\alpha', \alpha]$. Then we have, for each sufficiently small r,

$$M_1 r^{2\alpha-d}\log 1/r \leqq \phi(r) \leqq M_2 r^{2\alpha-d}\log 1/r, \qquad M_2 \geqq M_1 > 0. \tag{4.1}$$

Indeed we proved (4.1) by a direct calculation in case $b(\beta) \equiv 1$ in § 6 [5] and it is easy to check that $C_b(K)=0$ if and only if $C_1(K)=0$ for each compact set K, where $C_b(K)$ (resp. $C_1(K)$) denotes the capacity of K relative to X^b (resp. X^1). Therefore (4.1) holds by our Theorem.

c) For a certain class of isotropic Lévy processes we can show that Green function $G(x, y) = \phi(|x-y|)$ exists, but it is difficult to check whether $\phi(r)$ is monotone decreasing or not. It is desirable to extend our theorem in some sense to the above processes for which $\phi(r)$ is not known to be monotonic (in this case Frostmann's capacity C^{ϕ} is not always defined, and so we denote by C^{ϕ} in i) the capacity defined in a)), although there exists an isotropic Lévy process for which $\phi(r)$ is not monotonic and i) does not imply ii) for $\phi_0(r) = r^{\alpha-d}$ for some fixed α, $0<\alpha<1/2$.

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