HIROSHIMA MATH. J.
4 (1974), 491–499

On the Oscillation of Second Order Nonlinear Ordinary Differential Equations

Takaŝi KUSANO, Hiroshi ONOSE and Hiroyuki TOBE

(Received January 31, 1974)

Introduction

In this paper we are concerned with the oscillatory behavior of solutions of the second order nonlinear differential equation

$$(r(t)x')' + a(t)f(x) = 0, \tag{A}$$

where the following assumptions are assumed to hold:

(a) $a \in C[0, +\infty)$;

(b) $r \in C^1[0, +\infty)$, and $r(t)>0$ for $t \geqq 0$;

(c) $f \in C(-\infty, +\infty) \cap C^1(-\infty, 0) \cap C^1(0, +\infty)$, $\operatorname{sgn} f(x) = \operatorname{sgn} x$, and $f'(x) \geqq 0$ for $x \neq 0$.

We restrict our attention to solutions of (A) which exist on some half-line $[t_0, +\infty)$, where t_0 may depend on the particular solution. Such a solution is called oscillatory if it has arbitrarily large zeros; otherwise, a solution is called nonoscillatory.

The problem of determining if all solutions of equation (A) are oscillatory has been the subject of intensive investigations since the pioneering work of Atkinson [1], and during the last decade an extensive amount of study has been devoted to obtain sufficient conditions for oscillation of all solutions of (A) when the coefficient $a(t)$ is allowed to assume negative values for arbitrarily large values of t. For results on the subject we cite those given in the papers [2–5, 7–13, 15–20] as being representative. In particular, we refer the reader to a general oscillation theorem of Kamenev [8] which yields as particular cases oscillation criteria of Kiguradze [12], Kamenev [7] and others.

The purpose of this paper is to derive from the above mentioned theorem of Kamenev and its variant several criteria for oscillation of all solutions (or all bounded solutions) of the damped differential equation

$$x'' + q(t)x' + p(t)f(x) = 0. \tag{B}$$

Our results include an improvement and an extension of some of the recent results of Erbe [6] and Naito [14] for equation (B). Our approach seems natural as equation (B) can be transformed into an equation of the form (A), and actually it enables us to determine the effect of the separate behavior of $p(t)$ and $q(t)$ on the oscillatory character of (B).

1. We begin with Kamenev's oscillation theorem [8] in which the following conditions are needed:

$$\text{(F)} \qquad \int_{+\varepsilon}^{+\infty}\frac{dx}{f(x)}<+\infty, \int_{-\varepsilon}^{-\infty}\frac{dx}{f(x)}<+\infty \qquad \text{for some } \varepsilon>0;$$

$$(\Phi) \qquad \int_{+0}^{+\varepsilon}\frac{dx}{f(x)}<+\infty, \int_{-0}^{-\varepsilon}\frac{dx}{f(x)}<+\infty \qquad \text{for some } \varepsilon>0.$$

THEOREM 1. (Kamenev) *Suppose there exists a function* $\rho\in C^2[t_0, +\infty)$, $\rho(t)>0$, *such that*

$$\int^{+\infty}\rho(t)a(t)dt = +\infty,$$

$$\int^{+\infty}\frac{dt}{\rho(t)r(t)} = +\infty.$$

Then the following statements are true:

(i) *If* $\rho'(t)\geqq 0$ *and* $R'(t)\leqq 0$, *where* $R(t)=r(t)\rho'(t)$, *then all bounded solutions of* (A) *are oscillatory. If in addition* (F) *holds, then all solutions of* (A) *are oscillatory.*

(ii) *If* $\rho'(t)\leqq 0$ *and* $R'(t)\geqq 0$, *then all bounded solutions of* (A) *are oscillatory or tend monotonically to zero as* $t\to+\infty$. *If in addition* (Φ) *holds, then all solutions of* (A) *are oscillatory.*

(iii) *If* (Φ) *holds and*

$$\int^{+\infty}|R'(t)|dt < +\infty,$$

then all bounded solutions of (A) *are oscillatory. If in addition* (F) *holds, then all solutions of* (A) *are oscillatory.*

We note that the original theorem of Kamenev does not contain the criteria for oscillation of bounded solutions of (A). However, a close look at his proof ensures the validity of the assertions of Theorem 1 as stated above.

We now apply Theorem 1 to produce some oscillation criteria for the damped differential equation (B) where $p, q\in C[0, +\infty)$ and f satisfies condition (c).

COROLLARY 1.1. *Assume that* $q(t)\leqq 0$, $q'(t)\geqq 0$ *for all large* t *and let* $\int^{+\infty}p(t)dt=+\infty$. *Then all bounded solutions of* (B) *are oscillatory. If in addition* (F) *holds, then all solutions of* (B) *are oscillatory.*

COROLLARY 1.2. *Assume that* $q(t)\geqq 0$, $q'(t)\leqq 0$ *for all large* t *and let* $\int^{+\infty} p(t)dt=+\infty$. *Then all bounded solutions of* (B) *are oscillatory or tend monotonically to zero as* $t\to\infty$. *If in addition* (Φ) *holds, then all solutions of* (B) *are oscillatory.*

COROLLARY 1.3. *Assume that either* $q(t)>0$, $q'(t)\geqq 0$ *for all large* t *and* $\lim_{t\to+\infty} q(t)<+\infty$, *or* $q(t)<0$, $q'(t)\leqq 0$ *for all large* t *and* $\lim_{t\to+\infty} q(t)>-\infty$. *Let* $\int^{+\infty} p(t)dt=+\infty$. *If* ($\Phi$) *holds, then all bounded solutions of* (A) *are oscillatory. If* (Φ) *and* (F) *hold, then all solutions of* (A) *are oscillatory.*

PROOF OF COROLLARIES 1.1, 1.2 and 1.3. Transform equation (B) into an equation of the form (A) where $r(t)=\exp\left(\int_T^t q(s)ds\right)$ and $a(t)=r(t)p(t)$. If we put $\rho(t)=1/r(t)$, then we see that

$$\rho'(t) = -q(t)/r(t), \qquad R'(t) = -q'(t),$$

$$\int^{+\infty}\rho(t)a(t)dt = \int^{+\infty} p(t)dt = +\infty,$$

$$\int^{+\infty}\frac{dt}{\rho(t)r(t)} = \int^{+\infty} dt = +\infty.$$

Now Corollaries 1.1, 1.2 and 1.3 follow from (i), (ii) and (iii) of Theorem 1, respectively.

REMARK 1.1. Corollaries 1.1 and 1.2 improve part of situations covered by Theorems 2.3 and 2.4 of Erbe [6].

COROLLARY 1.4. *Assume that* $tq(t)\leqq 1$ *and* $(tq(t))'\geqq 0$ *for all large* t *and let* $\int^{+\infty} tp(t)dt=+\infty$. *Then all bounded solutions of* (B) *are oscillatory. If in addition* (F) *holds, then all solutions of* (B) *are oscillatory.*

COROLLARY 1.5. *Assume that* $tq(t)\geqq 1$ *and* $(tq(t))'\leqq 0$ *for all large* t *and let* $\int^{+\infty} tp(t)dt=+\infty$. *Then all bounded solutions of* (B) *are oscillatory or tend monotonically to zero as* $t\to\infty$. *If in addition* (Φ) *holds, then all solutions of* (B) *are oscillatory.*

COROLLARY 1.6. *Assume that either* $tq(t)>1$, $(tq(t))'\geqq 0$ *for all large* t *and* $\lim_{t\to+\infty} tq(t)<+\infty$, *or* $tq(t)<1$, $(tq(t))'\leqq 0$ *for all large* t *and* $\lim_{t\to+\infty} tq(t)>-\infty$. *Let* $\int^{+\infty} tp(t)dt=+\infty$. *If* ($\Phi$) *holds, then all bounded solutions of* (A) *are oscillatory. If* (Φ) *and* (F) *hold, then all solutions of* (A) *are oscillatory.*

PROOF OF COROLLARIES 1.4, 1.5 and 1.6. Transform (B) into (A) where $r(t) = \exp\left(\int_T^t q(s)ds\right)$ and $a(t)=r(t)p(t)$. If we put $\rho(t)=t/r(t)$, then we have

$$\rho'(t) = (1-tq(t))/r(t), \qquad R'(t) = -(tq(t))',$$

$$\int^{+\infty} \rho(t)a(t)dt = \int^{+\infty} tp(t)dt = +\infty,$$

$$\int^{+\infty} \frac{dt}{\rho(t)r(t)} = \int^{+\infty} \frac{dt}{t} = +\infty.$$

Now Corollaries 1.4, 1.5 and 1.6 follow from (i), (ii) and (iii) of Theorem 1, respectively.

REMARK 1.2. Corollary 1.4 improves recent results of Erbe [6] (Theorem 2.8, Corollary 2.9).

Consider the generalized Emden-Fowler equation

$$\text{(C)} \qquad x'' + p(t)|x|^\alpha \operatorname{sgn} x = 0, \qquad \alpha > 0.$$

Kamenev [8] has shown that Kiguradze's criterion for (C) with $\alpha>1$ [12] can be derived from his main theorem. We now show that Belohorec's criterion for (C) with $0<\alpha<1$ [2] also follows from Kamenev's theorem.

COROLLARY 1.7. (Belohorec) *Consider equation* (C) *with* $0<\alpha<1$. *If*

$$\int^{+\infty} t^\beta p(t)dt = +\infty \qquad \text{for some} \quad \beta \leqq \alpha,$$

then all solutions of (C) *are oscillatory.*

PROOF. We put $x(t)=ty(t)$. If $x(t)$ satisfies (C), then $y(t)$ satisfies the differential equation

$$\text{(C}'\text{)} \qquad (t^2y')' + t^{1+\alpha} p(t)|y|^\alpha \operatorname{sgn} y = 0,$$

and $x(t)$ is oscillatory if and only if $y(t)$ is. It is easy to verify that the function $\rho(t)=t^{-\gamma}$, $\gamma\geqq 1$, satisfies

$$\rho'(t) = -\gamma t^{-\gamma-1} \leqq 0, \qquad R'(t) = (-\gamma)(1-\gamma)t^{-\gamma} \geqq 0,$$

$$\int^{+\infty} \rho(t)a(t)dt = \int^{+\infty} t^\beta p(t)dt, \qquad \beta = 1+\alpha-\gamma,$$

$$\int^{+\infty} \frac{dt}{\rho(t)r(t)} = \int^{+\infty} \frac{dt}{t^{2-\gamma}} = +\infty.$$

Therefore we can apply Theorem 1 (ii) to equation (C′) to conclude that all solutions of (C′), and hence all solutions of (C), are oscillatory. (The proof presented here was suggested by M. Naito.)

2. Let us again consider the equation

$$(r(t)x')' + a(t)f(x) = 0 \tag{A}$$

for which, in addition to (a) and (b), the following assumption is satisfied:

(c′) $f \in C(-\infty, +\infty) \cap C^1(-\infty, 0) \cap C^1(0. +\infty)$, $\operatorname{sgn} f(x) = \operatorname{sgn} x$, and there is a positive constant k such that $f'(x) \geqq k$ for $x \neq 0$.

The purpose of this section is to prove a variant of Kamenev's theorem which is an extension of a recent result of Naito [14].

THEOREM 2. *Assume there exists a function* $\rho \in C^2[t_0, +\infty)$, $\rho(t) > 0$, $\rho'(t) \geqq 0$, *such that*

$$\int^{+\infty} \rho(t)a(t)dt = +\infty, \tag{1}$$

$$\int^{+\infty} \frac{dt}{\rho(t)r(t)} = +\infty, \tag{2}$$

$$\int^{+\infty} \frac{r(t)(\rho'(t))^2}{\rho(t)} dt < +\infty. \tag{3}$$

Then all solutions of (A) *are oscillatory.*

PROOF. Suppose that $x(t)$ is a nonoscillatory solution of (A). Without loss of generality we may assume that $x(t) > 0$ for $t \geqq t_1 \geqq t_0$, since a parallel argument holds when $x(t) < 0$. Multiplying (A) by $\rho(t)/f(x(t))$ and integrating from t_1 to t, we obtain

$$\begin{aligned} &\frac{\rho(t)r(t)x'(t)}{f(x(t))} + \int_{t_1}^{t} \frac{\rho(s)r(s)f'(x(s))(x'(s))^2}{(f(x(s)))^2} ds \\ &\quad - \int_{t_1}^{t} \frac{\rho'(s)r(s)x'(s)}{f(x(s))} ds + \int_{t_1}^{t} \rho(s)a(s)ds = C, \end{aligned} \tag{4}$$

where C is a constant. By Schwarz's inequality, we note that

$$\begin{aligned} &\left| \int_{t_1}^{t} \frac{\rho'(s)r(s)x'(s)}{f(x(s))} ds \right| \\ &\quad \leqq \left\{ \int_{t_1}^{t} \frac{r(s)(\rho'(s))^2}{\rho(s)} ds \right\}^{1/2} \left\{ \int_{t_1}^{t} \frac{r(s)\rho(s)(x'(s))^2}{(f(x(s)))^2} ds \right\}^{1/2} \end{aligned} \tag{5}$$

$$\leqq K\left\{\int_{t_1}^{t}\frac{r(s)\rho(s)(x'(s))^2}{(f(x(s)))^2}ds\right\},$$

where we have set $K=\left\{\int_{t_1}^{+\infty}\frac{r(t)(\rho'(t))^2}{\rho(t)}dt\right\}^{1/2}$ which is finite because of (3). Moreover we have by (c')

$$\int_{t_1}^{t}\frac{\rho(s)r(s)f'(x(s))(x'(s))^2}{(f(x(s)))^2}ds \geqq k\int_{t_1}^{t}\frac{\rho(s)r(s)(x'(s))^2}{(f(x(s)))^2}ds. \tag{6}$$

From (4), (5) and (6) we obtain

$$\frac{\rho(t)r(t)x'(t)}{f(x(t))}+k\int_{t_1}^{t}\frac{\rho(s)r(s)(x'(s))^2}{(f(x(s)))^2}ds$$
$$-K\left\{\int_{t_1}^{t}\frac{\rho(s)r(s)(x'(s))^2}{(f(x(s)))^2}ds\right\}^{1/2}+\int_{t_1}^{t}\rho(s)a(s)ds \leqq C,$$

from which, using (1), we conclude that

$$\lim_{t\to+\infty}\frac{\rho(t)r(t)x'(t)}{f(x(t))}=-\infty,$$

that is, there is $t_2 \geqq t_1$ such that

$$x'(t)<0 \qquad \text{for} \quad t \geqq t_2. \tag{7}$$

The rest of the proof proceeds as in the proof of Theorem 1 of Kamenev [8]. From (4) and (7) it follows that there is $t_3 \geqq t_2$ such that

$$1+\int_{t_3}^{t}\frac{\rho(s)r(s)f'(x(s))(x'(s))^2}{(f(x(s)))^2}ds \leqq \frac{\rho(t)r(t)[-x'(t)]}{f(x(t))} \tag{8}$$

for $t \geqq t_3$. Multiplying (8) by

$$\frac{f'(x(t))[-x'(t)]}{f(x(t))}\left\{1+\int_{t_3}^{t}\frac{\rho(s)r(s)f'(x(s))(x'(s))^2}{(f(x(s)))^2}ds\right\}^{-1} \geqq 0$$

and integrating from t_3 to t, we have

$$\log\frac{f(x(t_3))}{f(x(t))} \leqq \log\left\{1+\int_{t_3}^{t}\frac{\rho(s)r(s)f'(x(s))(x'(s))^2}{(f(x(s)))^2}ds\right\},$$

which gives in view of (8)

$$f(x(t_3)) \leqq \rho(t)r(t)[-x'(t)], \qquad t \geqq t_3. \tag{9}$$

Dividing (9) through by $\rho(t)r(t)$ and integrating from t_3 to t, we have

$$x(t) \leqq x(t_3) - f(x(t_3)) \int_{t_3}^{t} \frac{ds}{\rho(s)r(s)}, \qquad t \geqq t_3,$$

which yields a contradiction in the limit as $t \to +\infty$.

We now consider the equation

$$\text{(B)} \qquad x'' + q(t)x' + p(t)f(x) = 0$$

where $p, q \in C[0, +\infty)$ and f satisfies condition (c′).

COROLLARY 2.1. *Assume that* $q(t) \leqq 0$ *for all large* t, $\int^{+\infty} (q(t))^2 dt < +\infty$ *and* $\int^{+\infty} p(t)dt = +\infty$. *Then all solutions of* (B) *are oscillatory.*

PROOF. We transform (B) into (A) where $r(t) = \exp\left(\int_T^t q(s)ds\right)$ and $a(t) = r(t)p(t)$. Put $\rho(t) = 1/r(t)$. Then conditions (1) and (2) are obviously satisfied, and moreover

$$\int^{+\infty} \frac{r(t)(\rho'(t))^2}{\rho(t)} dt = \int^{+\infty} (q(t))^2 dt < +\infty .$$

The assertion then follows from Theorem 2.

COROLLARY 2.2. *Assume that* $tq(t) \leqq 1$ *for all large* t *and*

$$\int^{+\infty} \frac{(tq(t)-1)^2}{t} dt < +\infty .$$

If $\int^{+\infty} tp(t)dt = +\infty$, *then all solutions of* (B) *are oscillatory.*

PROOF. Transform (B) into (A) where $r(t) = \exp\left(\int_T^t q(s)ds\right)$ and $a(t) = r(t)p(t)$. Put $\rho(t) = t/r(t)$. Then conditions (1) and (2) are satisfied, and moreover

$$\int^{+\infty} \frac{r(t)(\rho'(t))^2}{\rho(t)} dt = \int^{+\infty} \frac{(tq(t)-1)^2}{t} dt < +\infty ,$$

which is condition (3). Thus the required conclusion follows from Theorem 2.

REMARK 2.1. Corollary 2.2 was obtained in a recent paper by Naito [14] (Theorem 2).

References

[1] F. V. Atkinson, On second-order non-linear oscillation, Pacific J. Math. **5** (1955), 643–647.

[2] S. Belohorec, Two remarks on the properties of solutions of a nonlinear differential equation, Acta Fac. Rerum Natur. Univ. Comenian. Math. **12** (1969), 19–26.

[3] N. P. Bhatia, Some oscillation theorems for second order differential equations, J. Math. Anal. Appl. **15** (1966), 442–446.

[4] L. E. Bobisud, Oscillation of nonlinear second order equations, Proc. Amer. Math. Soc. **23** (1969), 501–505.

[5] W. J. Coles, Oscillation criteria for non-linear second order equations, Ann. Mat. Pura Appl. **82** (1969), 123–133.

[6] L. Erbe, Oscillation criteria for second order nonlinear differential equations, Ann. Mat. Pura Appl. **94** (1972), 257–268.

[7] I. V. Kamenev, On the oscillation of solutions of a nonlinear equation of second order, Moskov. Inst. Elektron. Mašinostroenija-Trudy MIÈM **5** (1969), 125–136. (Russian)

[8] I. V. Kamenev, On the oscillation of solutions of a nonlinear second order equation with coefficient of variable sign, Differenc′ialnye Uravnenija **6** (1970), 1718–1721. (Russian)

[9] I. V. Kamenev, Certain specifically nonlinear oscillation theorems, Mat. Zametki **10** (1971), 129–134. (Russian)

[10] A. G. Kartsatos, Properties of bounded solutions of nonlinear equations of second order, Proc. Amer. Math. Soc. **19** (1968), 1057–1059.

[11] A. G. Kartsatos and H. Onose, Remarks on oscillation of second order differential equations, Bull. Fac. Sci. Ibaraki Univ. Math. **5** (1973), 23–31.

[12] I. T. Kiguradze, A note on the oscillation of solutions of the equation $u''+a(t)|u|^n \operatorname{sgn} u=0$, Časopis Pěst. Mat.**92** (1967), 343–350. (Russian)

[13] G. G. Legatos and A. G. Kartsatos, Further results on oscillations of second order equations, Math. Japonicae **14** (1969), 1057–1059.

[14] M. Naito, Oscillation criteria for a second order differential equation with a damping term, Hiroshima Math. J. **4** (1974). (to appear)

[15] H. Onose, Osillation theorems for nonlinear second order differential equations, Proc. Amer. Math. Soc. **26** (1970), 461–464.

[16] H. Onose, On oscillation of nonlinear second order equations, J. Math. Anal. Appl. **39** (1972), 122–124.

[17] V. A. Staikos and Y. G. Sficas, Oscillations for forced second order nonlinear differential equations, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (to appear)

[18] P. Waltman, An oscillation criterion for a nonlinear second order equation, J. Math. Anal. Appl. **10** (1965), 439–441.

[19] J. S. W. Wong, On two theorems of Waltman, SIAM J. Appl. Math. **14** (1966), 724–728.

[20] J. S. W. Wong, A second order nonlinear oscillation theorem, Proc. Amer. Math. Soc. **40** (1973), 487–491.

Department of Mathematics,
Faculty of Science,
Hiroshima University,

Department of Mathematics,
College of General Education,
Ibaraki University,
and
Department of Mathematics,
Ibaraki Technical College