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Non-triviality of an Element in the Stable Homotopy Groups of Spheres

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Statement of results

In the stable homotopy groups G_* of spheres, two non-trivial families of p-primary elements, called α- and β-series, are known [6] (cf. [12]). These are constructed from the attaching classes α and β of the spectra $V(1)$ and $V(2)$ [12], whose cohomology groups are certain exterior algebras over the Steenrod algebra $\bmod p$ [11]. In a similar way, the existence of the spectrum $V\left(2\frac{1}{2}\right)$ assures to define an element called γ_1[12; § 5], which is the first element of the third family.

The purpose of this paper is to prove the following result.

MAIN THEOREM. *For every prime* $p \geqq 5$, *the element* $\gamma_1 \in G_{(p^2-1)q-3}$, $q=2(p-1)$, *is non-trivial.*

The result is an answer to a problem proposed by one of the authors [12; p. 237], and P. E. Thomas and R. Zahler [7] [13] also have obtained the same result in a quite different method. Our result states that γ_1 is a non-zero multiple of the element $\alpha_1\beta_{p-1}$ [12; (5.12)]. Also, one of the authors recently has proved more strict relation $\gamma_1=\alpha_1\beta_{p-1}$.

Originally, this paper was intended to prove $\gamma_1=0$ (cf. [4; II, Remark in p. 147], [7; § 0]), but the publication has been postponed by a contradiction to the result of P. E. Thomas and R. Zahler. We have re-examined our original proof, and after crucial investigations we have concluded the opposite result.

COROLLARY 1 ($p \geqq 5$). *The following relations hold in* G_*:

$$\alpha_1\beta_{p-1}\beta_s = 0 \qquad \text{for} \quad s \geqq 3,$$

and hence

$$\alpha_1\beta_1\beta_k = \alpha_1\beta_2\beta_{k-1} = 0 \qquad \text{for} \quad k \not\equiv -2 \bmod p \quad \text{and} \quad k \geqq p+1,$$

$$\alpha_1\beta_1^2\beta_k = \alpha_1\beta_1\beta_2\beta_{k-1} = 0 \qquad \text{for} \quad k \geqq p+1.$$

This is an easy restatement of Proposition 5.9 of [12]. Also, by Corollary 5.7, Theorem 5.1 and (5.4) of [12], we obtain the parallel relations in the algebra

$\mathscr{A}_*(M)$ [5] [12] of the Moore space mod p.

COROLLARY 2 ($p \geqq 5$). *The following relations hold in* $\mathscr{A}_*(M)$:

$$\alpha\delta\beta_{(p-1)}\delta\beta_{(s)} = 0 \qquad \text{for} \quad s \geqq 3,$$

and hence

$$\alpha\delta\beta_{(1)}\delta\beta_{(k)} = \alpha\delta\beta_{(2)}\delta\beta_{(k-1)} = 0 \qquad \text{for} \quad k \not\equiv -2 \bmod p \ \text{and} \ k \geqq p+1,$$

$$\alpha(\delta\beta_{(1)})^2\delta\beta_{(k)} = \alpha\delta\beta_{(1)}\delta\beta_{(2)}\delta\beta_{(k-1)} = 0 \quad \text{for} \quad k \geqq p+1.$$

§1. A secondary composition

Throughout this paper, p denotes a prime integer with $p \geqq 5$, and set $q = 2(p-1)$. n denotes a sufficiently large integer so that all spaces and maps considered are in the stable range.

For finite CW-complexes (spectra) X and Y, $[X, Y]$ denotes the set of homotopy classes of maps: $X \to Y$, and $\pi_k^S(X; Y)$ the limit group $\lim_n [\Sigma^{n+k}X, \Sigma^n Y]$ of stable classes of maps, where Σ^t denotes the t-fold suspension. Also denote by $\mathscr{A}_k(X)$ the group $\pi_k^S(X; X)$. The direct sum $\mathscr{A}_*(X) = \sum_k \mathscr{A}_k(X)$ forms naturally a graded ring, and in particular $G_* = \mathscr{A}_*(S^0)$ is the stable homotopy ring of spheres. A map and its stable class are written by the same letter.

There exist the following sequences of cofiberings of the spectra $V(0)$, $V(1)$ and $V(2)$ [12; p. 217]:

$$S^n \xrightarrow{p} S^n \xrightarrow{i} M^{n+1} \xrightarrow{\pi} S^{n+1},$$

$$M^{n+q} \xrightarrow{\alpha} M^n \xrightarrow{i_1} V(1)_{n-1} \xrightarrow{\pi_1} M^{n+q+1},$$

$$\Sigma^{(p+1)q} V(1)_n \xrightarrow{\beta} V(1)_n \longrightarrow V(2)_n \longrightarrow \Sigma^{(p+1)q+1} V(1)_n.$$

Here the Moore space

$$M^n = S^{n-1} \cup_p e^n$$

is the $(n-1)$-th component of the spectrum $V(0)$, and $V(k)_n$ denotes the n-th component of the spectrum $V(k)$.

In this paper, the notations and the results of the rings G_*, $\mathscr{A}_*(V(0))$ ($= \mathscr{A}_*(M^n)$) and $\mathscr{A}_*(V(1))$ are referred to [12] (cf. [4], [5], [8]). In particular, the families $\{\alpha_r\}$ and $\{\beta_r\}$ in G_* and $\{\beta_{(r)}\}$ in $\mathscr{A}_*(V(0))$ are defined from the elements α and β by

$$\alpha_r = \pi\alpha^r i, \qquad \beta_{(r)} = \pi_1\beta^r i_1, \qquad \beta_r = \pi\beta_{(r)} i.$$

Also our element γ_1 is defined by

$$\gamma_1 = \pi\gamma_{(1)}i, \qquad \gamma_{(1)} = \pi_1\gamma_{[1]}i_1,$$

where $\gamma_{[1]} \in \mathscr{A}_{p^2q-1}(V(1))$ is the element defined from the attaching class of $V\left(2\frac{1}{2}\right)$, and the following formula is Theorem 5.5 of [12].

(1.1) $\quad \gamma_{(1)} = x((\beta_{(1)}\delta)^p + (\delta\beta_{(1)})^p) + y\beta_{(p-1)}\delta\alpha$

for some integers $x \not\equiv 0 \bmod p$ *and* y.

Here we put

$$\delta = i\pi \in \mathscr{A}_{-1}(V(0)).$$

Now we consider the secondary composition

$$C = \{\pi\beta_{(1)}, \alpha i, \beta_1^p\} \subset G_{(p^2+p)q-3}.$$

From the results on G_* and $\mathscr{A}_*(V(0))$, we see that C is well defined and consists of a single element. Hence we have

$$C = \{\pi\beta_{(1)}, \alpha, i\beta_1^p\}$$

by the formula in [9; Prop. 1.2].

PROPOSITION 1.2.*) *The element* γ_1 *is non-trivial if and only if*

$$\{\pi\beta_{(1)}, \alpha i, \beta_1^p\} \neq 0 \ (\bmod\ zero).$$

PROOF. For x in (1.1), choose an integer x' such that $xx' \equiv 1 \bmod p$, and put

$$\lambda = \pi_0\beta \qquad \text{and} \qquad \mu = x'(\gamma_{[1]} + y\beta^{p-1}\alpha')i_0,$$

where $\pi_0 = \pi\pi_1$, $i_0 = i_1 i$ and $\alpha' = \alpha_1 \wedge 1_{V(1)}$ [12; pp. 218–219]. Then, $\lambda i_1 = \pi\beta_{(1)}$ and $\pi_1\mu = x'(\gamma_{(1)} - y\beta_{(p-1)}\delta\alpha)i = (\delta\beta_{(1)})^p i = i\beta_1^p$, since $\alpha' i_1 = -i_1\delta\alpha$ [12; (3.11)]. Therefore $C = \lambda\mu$ by the definition of C [9; p. 9]. By (5.11) of [12], the element $\gamma_{[1]}$ satisfies $\beta\gamma_{[1]} = 0$, and so

$$C = \lambda\mu = x'y\pi_0\beta^p\alpha' i_0 = x'y\beta_p\alpha_1.$$

The element $\beta_p\alpha_1$ is non-trivial by Theorem A of [4; II] and by the fact $\beta_p \neq 0$ of L. Smith [6]. Hence, $C \neq 0$ if and only if $y \not\equiv 0 \bmod p$, which is equivalent to $\gamma_1 \neq 0$ by (5.12) of [12]. Q.E.D.

§2. Extended powers of complexes

For a space X and a map f, we denote by $X^{(t)}$ and $f^{(t)}$ the t-times smash

*) The foot-note on p. 147 of [4; II] is incomplete. The tertiary composition $\{\beta_1, p\iota, \alpha_1, \beta_1^p\}$ has full indeterminancy, so this should be replaced by $\{\pi\beta_{(1)}, \alpha i, \beta_1^p\}$ above.

products $X \wedge \cdots \wedge X$ and $f \wedge \cdots \wedge f$. Let $\varphi_M: M^{m+n} \to M^m \wedge M^n$ be the map such that $(\pi \wedge 1_M)\varphi_M = (1_M \wedge \pi)\varphi_M = 1_M$ [12; Lemma 1.3]. We define

$$\varphi_M^t: M^{(t+1)n} \longrightarrow (M^n)^{(t+1)}$$

by $\varphi_M^1 = \varphi_M$ and $\varphi_M^t = (\varphi_M^{t-1} \wedge 1_M)\varphi_M$. Consider the operation $\theta: \mathscr{A}_k(M^n) \to \mathscr{A}_{k+1}(M^n)$ of [12].

PROPOSITION 2.1. *For any element* $\xi \in \mathscr{A}_*(M^n)$ *satisfying* $\theta(\xi) = 0$, *the relations*

$$(\pi\xi)^{(t)}\varphi_M^{t-1} = \pi\xi^t$$

hold.

PROOF. If $\theta(\xi) = 0$, then $(1_M \wedge \pi\xi)\varphi_M = \xi$ by [12; Th. 2.2, Lemma 1.3]. So we have inductively

$$(\pi\xi)^{(t+1)}\varphi_M^t = (\pi\xi)^{(t)}(1_{M^{(t)}} \wedge \pi\xi)(\varphi_M^{t-1} \wedge 1_M)\varphi_M$$
$$= (\pi\xi)^{(t)}\varphi_M^{t-1}(1_M \wedge \pi\xi)\varphi_M = \pi\xi^{t+1}.$$

Q.E.D.

We consider the extended p-th power functor $ep^r(\ )$ in [10]. In particular, ep^0 is the p-times smash product. Since the element $\alpha \in \mathscr{A}_q(M^n)$ lies in $\operatorname{Ker}\theta$, we have

$$ep^0(\pi\alpha)\varphi^{p-1}_M = \pi\alpha^p. \tag{2.2}$$

The (mod p) cell decomposition for $ep^r(S^n)$ is studied in [10; Lemmas 1–2]. For $r = q-1$, $q+1$, we have

LEMMA 2.3. $ep^{q-1}(S^n)$ *has a* mod p *summand* $S^{np} \vee S^{np+q-1}$. *If* $n \equiv 0$ mod p, *so is* $ep^{q+1}(S^n)$.

Here we say that X *has a* mod p *summand* Y if X is p-equivalent to a wedge $Y \vee Z$ for some Z.

Next we consider the complex $ep^r(M^n)$. Let $a \in H_{n-1}(M^n; Z_p)$ and $b \in H_n(M^n; Z_p)$ be the generators corresponding to the cells of M^n. Then a Z_p-basis for $\tilde{H}_*(ep^r(M^n); Z_p)$ is given by the following cycles [2; pp. 45–47] [10]:

(2.4) (i) $e_i \otimes_\pi a^p$, $e_i \otimes_\pi b^p$ *for* $0 \leqq i \leqq r$,

(ii) $e_0 \otimes_\pi (x_1 \otimes \cdots \otimes x_p)$,

(iii) $\partial(e_{r+1} \otimes_\pi (x_1 \otimes \cdots \otimes x_p))$, where $\pi=Z_p$, $x^p=x\otimes\cdots\otimes x$ (p-times), $x_j=a$ or b, $x_j\neq x_k$ for some j, k, and in (ii) and (iii) for odd r (resp. (iii) for even r), $(x_1,\dots, x_p)$ runs representatives of the classes obtained by the cyclic permutations; one representative (resp. $p-1$ repre-presentatives) being chosen from each class.

We consider the operation $P^1_*: H_i\to H_{i-q}$, the dual to the reduced power P^1, on $H_*(ep^r(M^n); Z_p)$. By using [10; Th. 1] (cf. [3]), we can calculate P^1_* on (2.4)(i). For example, we have

$$P^1_*\{e_i\otimes_\pi a^p\} = \begin{cases} 0 & \textit{for} \quad i<q, \\ -(n-1)/2\{e_0\otimes_\pi a^p\} & \textit{for} \quad i=q, \end{cases}$$

$$P^1_*\{e_i\otimes_\pi b^p\} = \begin{cases} 0 & \textit{for even } i<q, \\ \mu\{e_{i-p+2}\otimes_\pi a^p\},\ \mu\not\equiv 0 \bmod p, & \textit{for odd } i<q, \\ -n/2\{e_0\otimes_\pi b^p\} & \textit{for } i=q. \end{cases}$$

By dimensional reason, P^1_* on (2.4)(ii) is trivial. Since the elements (2.4)(iii) vanish in $H_*(ep^{r+1}(M^n); Z_p)$, it follows from the naturality of P^1_* that P^1_* on (2.4)(iii) is also trivial.

For the homology Bockstein operation Δ, the following relations are verified, up to sign ([10], [1; §5]):

$$\Delta\{e_i\otimes_\pi a^p\} = \begin{cases} 0 & \textit{for odd } i \textit{ and for } i=0, \\ \{e_{i-1}\otimes_\pi a^p\} & \textit{for even } i>0, \end{cases}$$

$$\Delta\{e_i\otimes_\pi b^p\} = \begin{cases} 0 & \textit{for odd } i<r \textit{ and for } i=0, \\ \{e_{i-1}\otimes_\pi b^p\} & \textit{for even positive } i<r, \end{cases}$$

$$\Delta\{e_r\otimes_\pi b^p\} = \{\partial(e_{r+1}\otimes_\pi(ab^{p-1}))\} \qquad \textit{for odd } r,$$

$$\Delta_2\{e_0\otimes_\pi b^p\} = \{e_0\otimes_\pi(ab^{p-1})\},$$

where Δ_2: $\mathrm{Ker}\,\Delta\to\mathrm{Coker}\,\Delta$ is the secondary Bockstein operation.

We use the following notations of complexes:

(2.5)
$$N^n=S^{n-1}\cup_{p^2}e^n,$$
$$L'_n=M^{np-1}\cup_{\alpha\delta}CM^{np+p-2}, \qquad L_n=M^{np-1}\cup_{\alpha i}e^{np+q-1},$$
$$P_n=(N^{np}\vee M^{np-1})\cup_{(n\lambda\alpha\delta,\alpha)}CM^{np+q-1}$$
$$(=(N^{np}\vee L_n)\cup_{(n\lambda\alpha i,\tilde p)}e^{np+q}),$$

where λ: $M^n\to N^n$ is the map of degree 1 on the top cells [5; §§2–3], and $\tilde p$:

$S^{np+q-1} \to L_n$ is the coextension of p.

From the above discussion of the operations Δ, Δ_2 and P^1_* on $ep^r(M^n)$, we obtain the following three lemmas.

LEMMA 2.6. *$ep^{q-1}(M^n)$ has a* mod p *summand* $N^{np} \vee L'_n$.

In fact, the summands N^{np} and L'_n are obtained from the elements $e_0 \otimes_\pi ab^{p-1}$, $e_0 \otimes_\pi b^p$ and $e_{p-2} \otimes_\pi a^p$, $e_{p-1} \otimes_\pi a^p$, $\partial(e_q \otimes_\pi ab^{p-1})$, $e_{q-1} \otimes_\pi b^p$, respectively.

LEMMA 2.7. *$ep^{q-1}(M^n)$ is p-equivalent to a wedge*

$$L'_n \vee X_n \vee Y_n,$$

where $X_n = M^{np-3} \cup_\alpha CM^{np+q-3}$, and Y_n is $(np-p-1)$-connected and of dimension $np+q-3$.

In fact, X_n is obtained from the elements $e_{p-4} \otimes_\pi a^p$, $e_{p-3} \otimes_\pi a^p$, $e_{q-3} \otimes_\pi b^p$ and $e_{q-2} \otimes_\pi b^p$, and the complementary summand Y_n has the bottom cell corresponding to $e_0 \otimes a^p$ and the top cells corresponding to the elements (2.4) (iii) with $x_i = x_j = a$, $x_k = b$ $(k \neq i, j)$ for some $i \neq j$.

LEMMA 2.8. *$ep^{q+1}(M^n)$ has a* mod p *summand P_n. The inclusion $ep^{q-1}(M^n) \subset ep^{q+1}(M^n)$ is identical on N^{np} and is the following composition on L'_n:*

$$L'_n \xrightarrow{h} L_n \subset P_n,$$

where h is the map smashing the subcomplex S^{np+q-2} of L'_n to the base point (vertex).

In fact P_n is obtained from $N^{np} \vee M^{np-1}$ by removing $\partial(e_q \otimes_\pi ab^{p-1})$ and adding $e_q \otimes_\pi b^p$.

Now we notice that the complex $ep^0(M^n) = (M^n)^{(p)}$ has a mod p summand M^{np} and the map φ^{p-1}_M in (2.2) is the inclusion to this summand. Furthermore, by considering the induced homomorphism of the inclusion $ep^0(M^n) \subset ep^r(M^n)$, we see that the following diagram is commutative for $r = q-1$, $q+1$:

$$\text{(2.9)} \qquad \begin{array}{ccc} M^{np} & \xrightarrow{\lambda} & N^{np} \\ \downarrow \varphi^{p-1}_M & & \downarrow j \\ ep^0(M^n) & \xrightarrow{k} & ep^r(M^n), \end{array}$$

where j and k are the inclusions.

For the complexes L'_m and L_m of (2.5), we have the following commutative diagram of the cofiberings:

$$\begin{array}{ccccc} & & S^{mp+q-2} & = & S^{mp+q-2} \\ & & \big\downarrow & & \big\downarrow{\scriptstyle i} \\ M^{mp-1} & \longrightarrow & L'_m & \longrightarrow & M^{mp+q-1} \\ \Big\| & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle \pi} \\ M^{mp-1} & \xrightarrow{i_L} & L_m & \xrightarrow{q_L} & S^{mp+q-1} \end{array}$$

Applying $[\quad, S^{np}]$ to this diagram, we obtain the following lemma, from the known results on G_* and $\mathscr{A}_*(M^n)$ ([4], [5], [8], [12]).

LEMMA 2.10. *Let $m=n+q$. Then*

$$h^*: [L_m, S^{np}] \longrightarrow [L'_m, S^{np}],$$

$$i_L^*: [L_m, S^{np}] \longrightarrow [M^{mp-1}, S^{np}]$$

are isomorphisms of the p-components, and the p-component of the group $[L_m, S^{np}]$ is isomorphic to Z_p+Z_p, generated by ξ and η satisfying $i_L^*\xi=\pi\beta_{(1)}$ and $i_L^*\eta=\pi\alpha^{p-1}\delta\alpha$.

Also we obtain

LEMMA 2.11. *Let $l=m+pq-2$. Then the p-primary part of $\pi_{lp+q-1}(L_m)$ is isomorphic to Z_p generated by ζ satisfying $q_{L*}\zeta=\beta_1^p$.*

Now we consider the map $ep^{q-1}(\pi\alpha)$. Set $m=n+q$ and

$$\phi' = r\circ ep^{q-1}(\pi\alpha)\circ j': L'_m \longrightarrow ep^{q-1}(M^m) \longrightarrow ep^{q-1}(S^n) \longrightarrow S^{np},$$

where r and j' are the retraction and the inclusion obtained from Lemma 2.3 and Lemma 2.6 respectively. By Lemma 2.10, there exists

$$(2.12) \qquad \phi: L_m \longrightarrow S^{np}$$

such that $\phi=\phi' h$ and we can put

$$\phi = a\xi+b\eta. \qquad a,\ b\in Z_p.$$

LEMMA 2.13. *The coefficient a is ± 1.*

PROOF. The lemma means that the restriction $\phi|S^{mp-2}=\phi'|S^{mp-2}$ represents $\pm\beta_1$. This is proved quite similarly as [10; Lemma 4] by calculating the functional P^p-operation for $ep^{q-1}(\pi\alpha)$. Q.E.D.

For the complex N^n of (2.5), let

$$S^{n-1} \xrightarrow{i'} N^n \xrightarrow{\pi'} S^n$$

be the cofibering. N^n is a Moore space mod p^2, and hence by [5; §4, §7], the group $\mathscr{A}_{pq}(N^n)$ is generated by an element α' of order p^2 satisfying $\alpha'\lambda=\lambda\alpha^p$ and $\rho\alpha'=\alpha^p\rho$, where $\lambda\colon M^n\to N^n$ and $\rho\colon N^n\to M^n$ satisfy $\pi'\lambda=\pi$, $\lambda i=pi'$, $\rho i'=i$ and $\pi\rho=p\pi'$. The element $\alpha'_p=\pi'\alpha' i'$ generates the p-component of G_{pq-1} and satisfies $p\alpha'_p=\alpha_p$, and α' is determined up to $p\alpha'$.

LEMMA 2.14. *Let $m=n+q$. For $t=q-1$, $q+1$, the composition*

$$N^{mp}\xrightarrow{j} ep^t(M^m)\xrightarrow{ep^t(\pi\alpha)} ep^t(S^n)\xrightarrow{r} S^{np}$$

(j is the inclusion and r is the retraction) represents $\pi'\alpha'$ for some choice of α'.

PROOF. Let $k\colon ep^0(M^m)\to ep^t(M^m)$ be the inclusion. Then we have

$$\begin{aligned} r\circ ep^t(\pi\alpha)\circ j\circ\lambda &= r\circ ep^t(\pi\alpha)\circ k\circ\varphi_M^{p-1} && \text{by (2.9)}\\ &= ep^0(\pi\alpha)\circ\varphi_M^{p-1} &&\\ &= \pi\alpha^p && \text{by (2.2)}\\ &= \pi'\alpha'\lambda. && \end{aligned}$$

The sequence

$$[M^{mp}, S^{np}]\xrightarrow{\rho^*}[N^{mp}, S^{np}]\xrightarrow{\lambda^*}[M^{mp}, S^{np}]$$

is exact since $M^{mp}\to N^{mp}\to M^{mp}$ is the cofibering. The groups $[M^{mp}, S^{np}]$ and $[N^{mp}, S^{np}]$ are generated by $\pi\alpha^p$ and $\pi'\alpha'$, and $\rho^*(\pi\alpha^p)=p\pi'\alpha'$. So, replacing α' we obtain the lemma. Q.E.D.

PROPOSITION 2.15. *Let $m=n+q$ and assume that $n\equiv 0 \bmod p$. Then we have $a=\pm 1$ and $b=-2$ in the equality $\phi=a\xi+b\eta$.*

PROOF. By Lemma 2.13, we only prove $b=-2$. By Lemma 2.10, $\phi|M^{mp-1}=\phi'|M^{mp-1}$ represents $\phi''=a\pi\beta_{(1)}+b\pi\alpha^{p-1}\delta\alpha$, and the composition

$$N^{mp}\vee M^{mp-1}\longrightarrow ep^{q-1}(M^m)\xrightarrow{ep^{q-1}(\pi\alpha)} ep^{q-1}(S^n)\longrightarrow S^{np}$$

represents $\pi'\alpha'\vee\phi''$ by Lemma 2.14. $N^{mp}\vee M^{mp-1}$ is the subcomplex of P_m in (2.5), which is the mapping cone of $(m\lambda\alpha\delta, \alpha)$. By Lemma 2.8, $ep^{q+1}(M^m)$ has a summand P_m, and by Lemma 2.3, $ep^{q+1}(S^n)$ has a summand S^{np} if $n\equiv 0 \bmod p$. Let $\bar{\phi}\colon P_m\to S^{np}$ be the component of $ep^{q+1}(\pi\alpha)$ with respect to these summands. By Lemma 2.14, the element $\pi'\alpha'\vee\phi'$ is the component of $ep^{q-1}(\pi\alpha)$, and so we have the commutative diagram:

$$\begin{array}{ccc} N^{mp} \vee L'_m & \xrightarrow{1\vee h} & N^{mp} \vee L_m \\ \downarrow{\scriptstyle \pi'\alpha' \vee \phi'} & & \downarrow \\ S^{np} & \xleftarrow{\bar{\phi}} & P_m \end{array}$$

by Lemma 2.8, where the right vertical arrow is the inclusion. Since $\pi'\alpha' \vee \phi' = (1 \vee h)^*(\pi'\alpha' \vee \phi)$ and $(1 \vee h)^*$ is isomorphic by Lemma 2.10, we see that the element $\pi'\alpha' \vee \phi$ has an extension $\bar{\phi}$. Therefore $\pi'\alpha' \vee \phi''$ is extensible to P_m if $n \equiv 0 \bmod p$, and so

$$0 = (\pi'\alpha' \vee \phi'')(-2\lambda\alpha\delta, \alpha) = -2\pi'\alpha'\lambda\alpha\delta + (a\pi\beta_{(1)} + b\pi\alpha^{p-1}\delta\alpha)\alpha$$
$$= -(2+b)\pi\alpha^{p+1}\delta .$$

Since $\pi\alpha^{p+1}\delta \neq 0$, we obtain $b = -2$ as desired. Q.E.D.

Finally we consider $ep^{q-1}(\beta_{(1)}i)$. Set $l = m + pq - 2$ (m: large), and put

$$\psi' = r' \circ ep^{q-1}(\beta_{(1)}i) \circ j \colon S^{lp+q-1} \to ep^{q-1}(S^l) \to ep^{q-1}(M^m) \to L'_m ,$$

$$\psi = h\psi' \colon S^{lp+q-1} \longrightarrow L'_m \longrightarrow L_m. \tag{2.16}$$

PROPOSITION 2.17. *The element ψ represents $\pm\zeta$.*

PROOF. By Lemma 2.11, ψ is a multiple of ζ. We see easily that $q_L\psi$ is the component of $ep^{q-1}(\pi)ep^{q-1}(\beta_{(1)}i) = ep^{q-1}(\beta_1)$ between the top cells. Hence $q_L\psi$ is a suspension of $ep^0(\beta_1) = (\beta_1)^{(p)}$, which is equal to β_1^p up to sign (cf. [9: Prop. 3.1]). Thus, $\psi = \pm\zeta$ by Lemma 2.11. Q.E.D.

§3. Proof of the main theorem

Henceforward, we put

$$m = n + q, \qquad l = m + pq - 2 .$$

These integers are large so that one can work in the stable range. Since $(\pi\alpha)(\beta_{(1)}i) = 0 \colon S^l \to M^m \to S^n$, we have

$$FG = 0 \colon S^{lp+q-1} \longrightarrow ep^{q-1}(M^m) \longrightarrow S^{np} ,$$

where

$$F = r \circ ep^{q-1}(\pi\alpha), \qquad G = ep^{q-1}(\beta_{(1)}i) \circ j ,$$

for the retraction $r \colon ep^{q-1}(S^n) \to S^{np}$ and the inclusion $j \colon S^{lp+q-1} \to ep^{q-1}(S^l)$.

LEMMA 3.1. *For the elements ϕ of* (2.12) *and ψ of* (2.16), *their composition*

$$\phi\psi\colon S^{lp+q-1} \longrightarrow L_m \longrightarrow S^{np}$$

is trivial.

PROOF. By using the decomposition in Lemma 2.7, we can write $F=\phi'+F_1+F_2$ and $G=\psi'+G_1+G_2$, where $F_1\in[X_m, S^{np}]$, $F_2\in[Y_m, S^{np}]$, $G_1\in\pi_{lp+q-1}(X_m)$ and $G_2\in\pi_{lp+q-1}(Y_m)$, and we have

$$\phi'\psi'+F_1G_1+F_2G_2 = FG = 0.$$

From the results on $\pi_*(M^n)$, we have $\pi_{lp+q-1}(X_m)=0$ and so $G_1=0$. Since the p-component of G_k is trivial for $pq-p\leqq k\leqq pq-3$ and for $(p^2-1)q\leqq k\leqq p^2q-2$, F_2G_2 is homotopic to a composition $S^{lp+q-1}\to Y_m^{mp-2}/Y_m^{mp-3}\to S^{np}$, where Y_m^k denotes the k-skeleton of Y_m and so Y_m^{mp-2}/Y_m^{mp-3} is a wedge of copies of S^{mp-2}. The p-components of G_{pq-2} and G_{p^2q-1} are generated by β_1 and the element α''_{p^2}, which lies in the image of the J-homomorphism. Hence $\beta_1\alpha''_{p^2}=0$ and so $F_2G_2=0$. Therefore we have $\phi\psi=\phi'\psi'=0$. Q.E.D.

Now we shall prove our main theorem.

PROOF OF MAIN THEOREM. By the definition of the secondary composition, we have

$$\xi\zeta = \{\pi\beta_{(1)}, \alpha i, \beta_1^p\} \quad \text{mod zero},$$

$$\eta\zeta = \{\pi\alpha^{p-1}\delta\alpha, \alpha i, \beta_1^p\} \quad \text{mod zero},$$

for the elements ξ, η and ζ in Lemmas 2.10–2.11. The second composition is equal to $\{\alpha'_p, \alpha_1, \beta_1^p\}=\{\beta_1^p, \alpha_1, \alpha'_p\}$ up to sign by the relation $\pi\alpha^{p-1}\delta\alpha=\pm\alpha'_p\pi$ and the formula [9; (3.9), i)]. By [5; Prop. 8.1], we have $\eta\zeta=\pm\alpha_1\varepsilon_{p-1}\neq 0$, where ε_{p-1} is a non zero multiple of β_p and generates the p-component of $G_{(p^2+p-1)q-2}$.

By Propositions 2.15, 2.17 and Lemma 3.1, there is a relation $(\pm\xi-2\eta)\zeta=0$. Hence,

$$\{\pi\beta_{(1)}, \alpha i, \beta_1^p\} = \xi\zeta = \pm 2\eta\zeta = \pm 2\alpha_1\varepsilon_{p-1} \neq 0.$$

Thus, $\gamma_1\neq 0$ follows from Proposition 1.2. Q.E.D.

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