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Hyperpolynomial Approximation of Solutions of Hereditary Systems

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1. Introduction

Consider an operator L on $C[0, \tau]$, where $C[0, \tau]=\{\phi|\phi: [0, \tau]\to R^n$, continuous$\}$ with norm $\|\cdot\|$. Suppose that x is a solution of the equation $L(x)=h$, subject to the initial condition $x(0)=\alpha$. Then a problem in approximation theory is whether there are hyperpolynomials $S_n^*\in\Pi_n^*$ (Π_n^* is the set of all hyperpolynomials S_n^* of degree less than or equal to n, which satisfy the condition $S_n^*(0)=\alpha$, [5]) such that $\|L(x)-L(S_n^*)\|=\inf_{S\in\Pi_n^*}\|L(x)-L(S)\|$, $n=1, 2,\dots$ and $\lim_{n\to\infty} S_n^*=x$, uniformly on $[0, \tau]$.

The above problem has been studied in the following cases:

i) $L(x)\equiv x'+B(t, x)$, $\|\cdot\|=\|\cdot\|_p$ (L_p-norm), $1\leqq p\leqq\infty$. ([1], [3], [4].)

ii) $L(x)\equiv x'+B(t, x)+\int_0^t F(t, s, x(s))ds$, $\|\cdot\|=\|\cdot\|_p$, $1<p\leqq\infty$. ([5].)

The purpose of this paper is to study the same problem when L is an operator, which gives a hereditary system [2] and $\|\cdot\|=\|\cdot\|_p$, $1\leqq p\leqq\infty$. The results here generalize those of [1], [3], [4], [5] not only for the case of the L_p-norm, $1<p\leqq\infty$ but also for the L_1-norm.

2. Preliminaries

Let I be an interval of R, $A\subseteqq R$ be compact with $\max A=0$, $\alpha: I\times A\to R$ be a continuous function, nondecreasing with respect to the second variable and $\alpha(t, 0)=t$, $t\in I$. If $x: \alpha(I, A)\to R^n$ is continuous and $C(A)=\{f|f: A\to R^n$, continuous$\}$, we define an operator $Q_t x: I\to C(A)$ by the relation

$$(Q_t x)(\theta)=x(\alpha(t, \theta)), \qquad t\in I, \quad \theta\in A.$$

An hereditary differential system is a relation of the form

$$(x-g(t, Q_t x))'=f(t, Q_t x)$$

where $f, g: I\times C(A)\to R^n$ are continuous.

Suppose $U\subseteqq C(A)$ is open. We say that a continuous function $g: U\to R^n$

is nonatomic at zero if for every $(t, \phi) \in U$ there exist $s_0 = s_0(t, \phi)$, $\mu_0 = \mu_0(t, \phi)$, continuous, $\rho(t, \phi, \mu, s)$ nondecreasing in μ, s and continuous such that

$$\rho(t, \phi, \mu, s) < 1, \qquad |g(t, \psi) - g(t, \phi)| \leqq \rho(t, \phi, \mu, s) \|\psi - \phi\|_\infty$$

for every $(t, \psi) \in U$, $\psi \in M$, $s \in [0, s_0]$, $\mu \in [0, \mu_0]$, where

$$M = \{\psi \in C(A) \colon (t, \psi) \in U, \|\psi - \phi\|_\infty \leqq \mu, \psi(\theta) = \phi(\theta), \theta \in A \cap (-\infty, -s]\}.$$

Also the function g is said to be of type T if

i) g is uniformly continuous on closed and bounded sets,
ii) g is nonatomic at zero, and
iii) when $\lim_{n\to\infty} S_n = x$, uniformly on a closed interval J, then $\lim_{n\to\infty} g'(t, Q_t S_n) = g'(t, Q_t x)$, uniformly on J.

In what follows we consider the system

(1) $$L(x) \equiv (x - g(t, Q_t x))' + f(t, Q_t x) = h(t), \qquad t \in [0, \tau],$$

subject to the initial condition

(2) $$Q_0 x = \phi, \qquad \phi \in C(A),$$

where $g \colon [0, \tau] \times C(A) \to R^n$ is of type T, $f \colon [0, \tau] \times C(A) \to R^n$ is uniformly continuous on closed and bounded sets and $h \colon [0, \tau] \to R^n$ is continuous. By $\|\cdot\|_p$, $1 \leqq p \leqq \infty$, we denote the L_p-norm and by Π_n, $n = 1, 2, \dots$, the set of all functions defined on $A \cup [0, \tau]$, which coincide with ϕ on A and a certain $S_n^* \in \Pi_n^*$ on $[0, \tau]$.

3. Main results

THEOREM. *Let $x(t)$, $t \in [0, \tau]$ be a unique solution of the system* ((1), (2)). *Then there exist an integer n_0 and $S_n \in \Pi_n$ such that*

$$\|L(x) - L(S_n)\|_p = \inf_{S \in \Pi_n} \|L(x) - L(S)\|_p, \qquad n \geqq n_0, \quad 1 \leqq p \leqq \infty,$$

and $\lim_{n\to\infty} \|x - S_n\|_\infty = 0$.

The proof of this theorem requires the following lemmas.

LEMMA 1. *Consider the systems*

(H_n) $$(x - g(t, Q_t x))' + f(t, Q_t x) + f_n(t) = 0, \quad t \in [0, \tau],$$
$$Q_0 x = \phi, \quad \phi \in C(A)$$

$n=0, 1,\ldots,$ *where* $f_n\colon [0,\tau]\to R^n$ *are continuous and* $f_0=0$. *Suppose that* $\lim_{n\to\infty}\|f_n\|_p=0$ *and* $x_0(t)$, $t\in[0,\tau]$ *is a unique solution of* (H_0). *Then there exist an integer* n_0 *and solutions* $x_n(t)$, $t\in[0,\tau]$, *of* (H_n), $n\geqq n_0$, *such that* $\lim_{n\to\infty}\|x_n-x_0\|_\infty=0$.

The proof of this lemma is exactly analogous to that of Theorem 5.1 in [2].

LEMMA 2. *Let* $\mu_{p,n}=\inf_{S\in\Pi_n}\|L(x)-L(S)\|_p$, $n=1, 2,\ldots$, $1\leqq p\leqq\infty$. *Then* $\lim_{n\to\infty}\mu_{p,n}=0$.

PROOF. According to Lemma 2 in [5] there exist $S_n^*\in\Pi_n^*$, $n=1, 2,\ldots$, such that $\lim_{n\to\infty}\|S_n^*-x\|_\infty=0$ and $\lim_{n\to\infty}\|S_n^{*\prime}-x'\|_\infty=0$. Hence

$$\begin{aligned}\mu_{\infty,n} &= \inf_{S\in\Pi_n}\|L(x)-L(S)\|_\infty\\ &\leqq \|L(x)-L(S_n)\|_\infty\\ &\leqq \|x'-S_n'\|_\infty+\|g'(t, Q_tx)-g'(t, Q_tS_n)\|_\infty+\|f(t, Q_tx)-f(t, Q_tS_n)\|_\infty .\end{aligned}$$

Since g is of type T and f uniformly continuous on closed and bounded sets, we conclude $\lim_{n\to\infty}\mu_{\infty,n}=0$ and consequently $\lim_{n\to\infty}\mu_{p,n}=0$, $1\leqq p\leqq\infty$.

LEMMA 3. *Suppose that* $x(t)$, $t\in[0,\tau]$, *is a unique solution of* ((1), (2)) *and* $S_n\in\Pi_n$, $n=1, 2,\ldots$, *which satisfy* $\lim_{n\to\infty}\|L(x)-L(S_n)\|_p=0$. *Then* $\lim_{n\to\infty}\|x-S_n\|_\infty=0$.

PROOF. If we put $w_n(t)=x(t)-S_n(t)$ and $k_n(t)=L(x(t))-L(S_n(t))$, $t\in[0,\tau]$, then

$$\begin{aligned}k_n(t) &= L(x(t))-L(x(t)-w_n(t))\\ &= w_n'(t)-g'(t, Q_tx)+g'(t, Q_t(x-w_n))+f(t, Q_tx)-f(t, Q_t(x-w_n)) .\end{aligned}$$

Therefore, the functions w_n are solutions of

$$(W_n)\qquad \begin{aligned}&(w-g(t, Q_tx)+g(t, Q_t(x-w)))'+f(t, Q_tx)-f(t, Q_t(x-w))-k_n(t) = 0,\\ &Q_0w = 0,\end{aligned}$$

where $k_0=0$.

Since $\lim_{n\to\infty}\|k_n\|_p=0$ and zero is the only solution of (W_0), by Lemma 1, we get $\lim_{n\to\infty}\|w_n\|_\infty=0$.

LEMMA 4. *If* $x(t)$, $t\in[0,\tau]$, *is a solution of* ((1), (2)) *and* $\min_{S\in\Pi_k}\|L(x)-L(S)\|_p$ *does not exist, then there exist* $S_{k,n}\in\Pi_k$, $n=1, 2,\ldots$, *such that* $\lim_{n\to\infty}\|L(x)-L(S_{k,n})\|_p=0$ *and* $\|S_{k,n}\|_\infty>k$, $n=1, 2,\ldots$.

PROOF. There exist $S_{k,n} \in \Pi_k$, $n=1, 2,\ldots$, which satisfy

$$\mu_{p,k} = \inf_{S\in\Pi_k} \|L(x)-L(S)\|_p = \lim_{n\to\infty} \|L(x)-L(S_{k,n})\|_p.$$

The sequence $S_{k,n}$, $n=1, 2,\ldots$, is unbounded with respect to $\|\cdot\|_\infty$ since, in contrary, we have that $S_{k,n}$, $n=1, 2,\ldots$, is bounded, $S_{k,n} \in \Pi_k$, $n=1, 2,\ldots$, Π_n^* is finite dimensional and consequently there exists a subsequence S_{k,k_n}, $n=1, 2,\ldots$ such that $\lim_{n\to\infty} S_{k,k_n}=S_k \in \Pi_k$, $\lim_{n\to\infty} S'_{k,k_n}=x'$, uniformly. Thus $\mu_{p,k}=\lim_{n\to\infty} \|L(x)-L(S_{k,k_n})\|_p=\|L(x)-L(S_k)\|_p$, which is a contradiction.

PROOF OF THEOREM. If the first result of the theorem does not hold, then there exists an increasing sequence λ_n of integers such that $\min_{S\in\Pi_{\lambda_n}} \|L(x)-L(S)\|_p$ do not exist. Thus, by Lemma 4, for every λ_n there exist $S_{\lambda_n} \in \Pi_{\lambda_n}$, $n=1, 2,\ldots$, which satisfy the relations

$$\|L(x)-L(S_{\lambda_n})\|_p \leqq \mu_{p,\lambda_n}+\frac{1}{\lambda_n} \tag{3}$$

$$\|S_{\lambda_n}\|_\infty > \lambda_n, \qquad n = 1, 2,\ldots. \tag{4}$$

From (3), Lemma 2 and Lemma 3 we get $\lim_{n\to\infty} \|x-S_{\lambda_n}\|_\infty=0$, which is a contradiction to (4).

Now, the second result of the theorem is obvious.

The above theorem leads to the following corollary.

COROLLARY. *Suppose* $x(t)$, $t\in[0, \tau]$, *is a unique solution of the system*

$$L(x) \equiv x'+B(t, x)+\int_0^t F(t, s, x(s))ds = h(t), \qquad t\in[0, \tau],$$

$$x(0) = \phi(0), \quad \phi\in C(A)$$

where $B\colon [0, \tau]\times R^n\to R^n$, $F\colon [0, \tau]\times[0, \tau]\times R^n\to R^n$ *and* $h\colon [0, \tau]\to R^n$ *are continuous. Then there exist* $S_n^* \in \Pi_n^*$ *and an integer* n_0 *such that*

$$\|L(x)-L(S_n^*)\|_p = \inf_{S\in\Pi_n^*} \|L(x)-L(S)\|_p, \qquad n \geqq n_0, \quad 1 \leqq p \leqq \infty,$$

and $\lim_{n\to\infty} \|x-S_n^*\|_\infty=0$.

PROOF. By considering in the theorem $A=[-1, 0]$, $I=[0, \tau]$, $\alpha(t, \theta)=t(1+\theta)$, $t\in I$, $\theta\in A$, $g=0$ and $f(t, \phi)=B(t, \phi(0))+\int_{-1}^0 F(t, t(1+\theta), \phi(\theta))d\theta$, $(t, \phi)\in I\times C(A)$, it follows that g is of type T and f is uniformly continuous on closed and bounded sets. Thus the corollary is an immediate consequence of the theorem.

REMARK. The above corollary is the main result of [5] for $1<p\leqq\infty$ and moreover extends it for the L_1-norm.

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