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S^3 Actions on 4 Dimensional Cohomology Complex Projective Spaces

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§1. Introduction

Recently, F. Uchida [5] has determined smooth $SU(3)$ actions on homotopy complex projective spaces $hP_3(C)$.

The purpose of this note is to study smooth S^3 $(=SU(2))$ actions on cohomology complex projective planes by the analogous methods.

Let C and H be the complex and quaternion fields. Regard the complex projective plane as

$$P_2(C) = P(H \times C)$$

by the right complex multiplication. Then the smooth S^3 $(\subset H)$ action on $P_2(C)$ is given by

$$q\cdot[p, a] = [qp, a] \qquad (q \in S^3, p \in H, a \in C). \tag{1.1}$$

Also, regard H as the right complex vector space, set

$$P_2(C) = P(C^3) = P(H \otimes_C H/\sim)$$

where $p \otimes q \sim q \otimes p$ $(p, q \in H)$, and consider the smooth S^3 action on $P_2(C)$ given by

$$r\cdot[p \otimes q] = [rp \otimes rq] \qquad (r \in S^3, p, q \in H). \tag{1.2}$$

Now consider a 4 dimensional orientable closed smooth manifold

$$M = CHP_2(C),$$

having the same cohomology ring as $P_2(C)$, and assume that M admits a non-trivial smooth S^3 action.

Then, we obtain the following main theorem.

Theorem 1.3. *If M satisfies the above conditions, then M is S^3 equivariantly diffeomorphic to the complex projective plane $P_2(C)$ with the S^3 action given by* (1.1) *or* (1.2). *In each case, the principal isotropy subgroup is the unit group* $\{1\}$ *or the cyclic group Z_4 of order* 4, *and the fixed point set $F(S^3, M)$ consists of* *a single point or is empty.*

We recall in § 2 the basic facts about the smooth actions. After preparing in § 3 some known results on closed subgroups and real representations of S^3, we prove Theorem 1.3 in § 4 by showing several propositions concerning the latter half of Theorem 1.3.

§ 2. Smooth actions

Let G be a Lie group and M be a smooth manifold. A smooth G action α on M is a smooth map

$$\alpha: G \times M \longrightarrow M, \qquad \alpha(g, x) = gx,$$

satisfying the conditions

$$(g_1 g_2)x = g_1(g_2 x), \quad ex = x, \qquad (g_1, g_2 \in G, x \in M),$$

where e is the identity of G.

Assume that a smooth G action on M is given. For any $x \in M$, denote by

$$G_x = \{g \in G; gx = x\}$$

the isotropy subgroup of G at x, which is a closed subgroup of G, and by

$$G \cdot x = \{gx; g \in G\}$$

the orbit of x, which is a G invariant submanifold of M. Then the following basic facts hold.

(2.1) Let $x \in M$, and ν be the normal bundle of the orbit $G \cdot x = G/G_x$ in M. Then the given G action on M induces naturally the G action on ν as bundle maps, and we obtain the orthogonal action of the isotropy subgroup G_x on the fibre ν_x over x. It is called the *normal representation* of G_x and denoted by ρ_x.

(2.2) (*The differentiable slice theorem*) Assume that G is a compact Lie group. Then the normal bundle ν of (2.1) is G equivalent to the G bundle $G \times_{G_x} \nu_x \to G/G_x$, where G_x acts on ν_x via ρ_x, and also the orbit $G \cdot x = G/G_x$ has an open tubular neighborhood in M, which is G equivariantly diffeomorphic to $G \times_{G_x} \nu_x$. (Cf. [3, (3.1)].)

(2.3) If G is a compact Lie group and M is connected, then there exists the conjugate class

$$(H) = \{\text{conjugate subgroups of } H \text{ in } G\}$$

of a closed subgroup $H \subset G$ such that the set

$$M_{(H)} = \{x \in M; G_x \in (H)\}$$

is a dense open submanifold of M. The conjugate class (H) is called the *type of principal isotropy subgroups*, and $H' \in (H)$ is called a principal isotropy subgroup. (Cf. [1, IV, Th. 3.1].)

By (2.2) and (2.3), we see easily the following fact.

(2.4) If M is connected, the normal representation ρ_x of G_x at x is trivial if and only if G_x is a principal isotropy subgroup.

§3. Closed subgroups and real representations of S^3.

In this section, we prepare some results on the Lie group $S^3 = SU(2)$.

LEMMA 3.1. *Any closed connected proper subgroup $H \neq \{1\}$ of S^3 is conjugate to a maximal torus S^1 of S^3.*

PROOF. The Lie algebra $\mathfrak{su}(2)$ of the Lie group $S^3 = SU(2)$ is given by

$$\mathfrak{su}(2) = \{X \in GL(2, C); \operatorname{trace} X = 0, {}^t\bar{X} + X = 0\}$$

$$= \left\{ \begin{pmatrix} ix & a \\ -\bar{a} & -ix \end{pmatrix}; x \in R, a \in C \right\}.$$

Let $\mathfrak{h}$ be a Lie subalgebra of $SU(2)$, which is the Lie algebra of H. It is clear that $H = \{1\}$ or S^3 if $\dim \mathfrak{h} = 0$ or 3. If $\dim \mathfrak{h} = 1$, then $\mathfrak{h}$ is commutative and so H is conjugate to S^1.

Assume that $\dim \mathfrak{h} = 2$, and consider the bracket

$$r = [p, q]$$

for a base $\{p, q\}$ of $\mathfrak{h}$. Then the element $t = x_1 p + y_1 q$ with $r = x_0 p + y_0 q$ and $x_0 y_1 - x_1 y_0 = 1$ satisfies $[r, t] = r$. Set

$$r = \begin{pmatrix} ix & a \\ -\bar{a} & -ix \end{pmatrix}, \quad t = \begin{pmatrix} iy & b \\ -\bar{b} & -iy \end{pmatrix}, \quad (x, y \in R, a, b \in C).$$

Then the equality $r = [r, t] = rt - tr$ implies

$$ix = -a\bar{b} + b\bar{a} \qquad \text{and} \quad a = 2i(xb - ya).$$

By adding $\bar{a} \times$(the second equality) to its conjugate, and using the first equality, we see that $2a\bar{a} = -2x^2$, and so $a = x = 0$, i.e., $r = 0$. Therefore $\mathfrak{h}$ is commutative, and its Lie group H is a 2 dimensional torus. But this is a contradiction since $H \subset S^3$, and so $\dim \mathfrak{h} \neq 2$. *q.e.d.*

Corollary 3.2. *If H is a closed proper subgroup of S^3 and* $\dim H \geqq 1$, *then H is conjugate to a maximal torus S^1 or its normalizer NS^1 in S^3.*

Now, we consider the representations of $S^3 = SU(2)$.

Let $C^{(n-1)}[X_1, X_2]$ be the n dimensional complex vector space of all complex polynomials on X_1, X_2 of degree $n-1$, and define the n dimensional complex representation ρ_n $(n \geqq 2)$ of S^3 as follows:

$$(3.3) \qquad (\rho_n(p))f(X_1, X_2) = f(aX_1 + cX_2, bX_1 + dX_2),$$

for $p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S^3$ and $f(X_1, X_2) \in C^{(n-1)}[X_1, X_2]$.

Then, it is well known that any irreducible complex representation of S^3 is equivalent to ρ_n for some $n \geqq 2$, and so any 4 dimensional complex representation of S^3 is equivalent to

$$(3.4) \qquad \rho_4, \ 1 \oplus \rho_3, \ \rho_2 \oplus \rho_2 \quad \text{or} \quad 1 \oplus \rho_2,$$

where 1 means the trivial representation.

Lemma 3.5. *ρ_4 is not the complexification of a real representation of S^3.*

Proof. Assume that ρ_4 is a complexification. Then the S^3 module $C^{(3)}[X_1, X_2]$ has an S^3 invariant non-degenerate symmetric form β, where the action of S^3 is given by ρ_4 (cf. [2, Th. 11.4, p. 191]). Thus, by (3.3), we have

$$\begin{aligned} \beta(X_1^3, X_2^3) &= \beta\left(\rho_4\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(X_1^3), \rho_4\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(X_2^3)\right) \\ &= \beta(-X_2^3, X_1^3) = -\beta(X_1^3, X_2^3), \end{aligned}$$

which shows $\beta(X_1^3, X_2^3) = 0$. By operating $\rho_4\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$ $(a\bar{a} = 1)$, we have in the same way

$$\beta(X_1^3, X_1^i X_2^j) = \beta(a^3 X_1^3, a^i \bar{a}^j X_1^i X_2^j) = a^{3+i-j} \beta(X_1^3, X_1^i X_2^j) \ (i+j = 3),$$

and so $\beta(X_1^3, X_1^i X_2^j) = 0$ for $1 \leqq i \leqq 3$ by taking $a^{3+i-j} = -1$. These show that β is degenerate, which contradicts the condition of β. *q.e.d.*

Lemma 3.6. *$1 \oplus \rho_2$ is not also the complexification of a real representation of S^3.*

Proof. Assume that $1 \oplus \rho_2$ is a complexification. Then $C^{(3)}[X_1, X_2] = C^{(1)}[Y_1, Y_2] \oplus C^{(1)}[Z_1, Z_2]$ has an S^3 invariant non-degenerate symmetric form β, where the action of S^3 is given by $1 \oplus \rho_2$, and we see $\beta(Z_1, Z_2) = \beta(Z_1, Z_1) = 0$, by the same way as in the above proof. Also,

$$\beta(Z_1, Y_i) = \beta(\rho_2\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}(Z_1), Y_i) = -\beta(Z_1, Y_i),$$

i.e., $\beta(Z_1, Y_i)=0$ ($i=1, 2$), and we have a contradiction. *q.e.d.*

PROPOSITION 3.7. *Consider the real representations*

$$\eta_i\colon S^3 = SU(2) \longrightarrow SO(4) \qquad (i = 1, 2),$$

defined by using the quaternion field H as follows:

$$\eta_1(p)q = pq, \quad \eta_2(p)q = pqp^{-1} \qquad (p \in S^3 \subset H, q \in H).$$

Then, any non-trivial representation $\eta\colon S^3 \to O(4)$ is equivalent to η_1 or η_2.

PROOF. Since the complexification $\tilde{\eta}$ of η is equivalent to $1\oplus\rho_3$ or $\rho_2\oplus\rho_2$ of (3.4) by Lemmas 3.5 and 3.6. On the other hand, we see easily that the traces of $1\oplus\rho_3$ and $\rho_2\oplus\rho_2$ are equal to those of η_2 and η_1, respectively, and so we have the desired results. *q.e.d.*

Finally, we notice real representations of the normalizer NS^1 of S^1 in S^3.

PROPOSITION 3.8. *If $\gamma\colon NS^1 \to O(2)$ is a representation such that $\gamma|S^1$ is is non-trivial, then γ is equivalent to*

$$\gamma_n\colon NS^1 \longrightarrow O(2) \text{ for some even } n > 0,$$

which is defined by

$$\gamma_n(a)b = a^n b, \ \gamma_n(j)b = -\bar{b} \qquad (a \in S^1 \subset C, j \in NS^1 - S^1, b \in C).$$

PROOF. By the assumption, we can take γ up to equivalence so that

$$\gamma(a)b = a^n b \qquad (a \in S^1, b \in C)$$

for some integer $n>0$. Set $\gamma(j)=(x_{kl}) \in O(2)$. Then, we see immediately by the relation $ja=\bar{a}j$ that

$$x_{00}^2 + x_{01}^2 = 1, \quad x_{01} = x_{10}, \quad x_{11} = -x_{00}.$$

Therefore, it is easy to see that γ is equivalent to γ' given by

$$\gamma'(a)b = a^n b, \qquad \gamma'(j)b = -\bar{b},$$

where n must be even since $j^2=-1$. *q.e.d.*

§4. Smooth S^3 actions on $CHP_2(C)$.

In the rest of this note, assume that a 4 dimensional orientable closed smooth manifold

$$M = CHP_2(C)$$

has the same cohomology ring as the complex projective plane $P_2(C)$ and that M admits a non-trivial smooth S^3 action. It is clear that this action preserves the orientation on M.

Consider a fixed maximal torus S^1 of S^3, and the fixed point set

$$F(S^1, M) = \{x \in M; ax = x \text{ for all } a \in S^1\}$$

of the restricted S^1 action of the given S^3 action. Then, by the result of J. C. Su [4, Th. 7.2] (cf. also [1; IV, Prop. 1.2, Th. 2.1]),

$$F(S^1, M) = F_1 \cup \cdots \cup F_l \tag{4.1}$$

is the disjoint union of connected orientable $2k_i$ dimensional submanifolds F_i of M, where F_i has the same cohomology ring as the complex projective k_i space $P_{k_i}(C)$, and

$$k_1 + \cdots + k_l = 3 - l .$$

PROPOSITION 4.2. *In our case,* $l=3$ *in* (4.1), *that is,* $F(S^1, M)$ *consists of three points*:

$$F(S^1, M) = \{x_1, x_2, x_3\} .$$

PROOF. If $l=1$, then $F(S^1, M)=M$, i.e., the restricted action of a maximal torus S^1 of S^3 is trivial, and so the action of S^3 is also trivial, which contradicts the assumption.

Assume $l=2$. Then by (4.1)

$$F(S^1, M) = F_1 \cup F_2, \quad F_1 \text{ is a point}, \ \dim F_2 = 2 .$$

Therefore $\dim M - \max \dim F_i = 2$. Also the maximum of the dimensions of proper subgroups of S^3 is equal to 1 by Corollary 3.2. Therefore, by the result of F. Uchida [6, Th. 2], we see that $\dim F_1$ is also 2, which contradicts $\dim F_1 = 0$. *q.e.d.*

Now, we consider the isotropy subgroups S^3_x $(x \in M)$ and the type of principal isotropy subgroups (H) of (2.3) for a given S^3 action on M.

LEMMA 4.3. (i) *If* $\dim S_x^3 \geqq 1$, *then* $x \in S^3 \cdot F(S^1, M)$.
(ii) $\dim H = 0$, *and* $S_x^3 \in (H)$ *if* $\dim S_x^3 = 0$.

PROOF. (i) is clear by Corollary 3.2.
(ii) We notice that there is a point $x \in M$ such that $\dim S_x^3 = 0$ by (i).
If $\dim S_x^3 = 0$, then the orbit $S^3 \cdot x = S^3 / S_x^3$ is an orientable 3 dimensional manifold. Therefore the normal bundle ν of $S^3 \cdot x$ in M is orientable, and so the line bundle ν is trivial. Then the normal representation ρ_x is trivial by (2.2), that is, $S_x^3 \in (H)$ by (2.4). *q.e.d.*

PROPOSITION 4.4. *We have only the following two cases* (I) *and* (II), *for the isotropy subgroups* $S_{x_i}^3$ *of* $x_i \in F(S^1, M)$ *and the fixed point set*

$$F(S^3, M) = \{x \in M;\, px = x \text{ for all } p \in S^3\}:$$

$$\text{(I)}\quad S_{x_1}^3 = S^3, \quad S_{x_i}^3 = S^1 \ (i = 2, 3), \quad F(S^3, M) = \{x_1\},$$

$$\text{(II)}\quad S_{x_1}^3 = NS^1, \quad S_{x_i}^3 = S^1 \ (i = 2, 3), \quad F(S^3, M) = \phi.$$

PROOF. Assume that $j \in NS^1 - S^1$ acts trivially on $F(S^1, M) = \{x_1, x_2, x_3\}$. Then it is clear that the orbits $S^3 \cdot x_i$ $(i = 1, 2, 3)$ are disjoint. Choose disjoint closed S^3 invariant tubular neighborhoods V_i of $S^3 \cdot x_i$ $(i = 1, 2, 3)$. Then, S^3 acts on the submanifold $M' = M - \cup_{i=1}^3 \operatorname{Int} V_i$, and the orbit space M'/S^3 is a compact 1 dimensional manifold, by the above lemma. Since each component ∂V_i of $\partial M'$ is S^3 invariant, $\partial(M'/S^3)$ consists of three points and we have a contradiction.

Therefore, j acts non-trivially on $\{x_1, x_2, x_3\}$, and so

$$jx_1 = x_1, \quad jx_2 = x_3, \quad jx_3 = x_2.$$

Then $S_{x_1}^3 \subset NS^1$ and $S_{x_i}^3 = S^1$ $(i = 2, 3)$. The desired result follows from Corollary 3.2. *q.e.d.*

PROPOSITION 4.5. *Let* (H) *be the type of principal isotropy subgroups for a given* S^3 *action on* M, *and* ρ_x *be the normal representation of* (2.1). *Then, according to the case* (I) *or* (II) *of the above proposition, we have*

(I) $\{1\} \in (H)$, *and* $\rho_{x_1}: S^3 \to O(4)$ *is equivalent to* η_1 *of Proposition* 3.7, *and* $\rho_{x_i}: S^1 \to O(2)$ $(i = 2, 3)$ *is so to* δ_1.

(II) $Z_4 \in (H)$, *and* $\rho_{x_1}: NS^1 \to O(2)$ *is equivalent to* γ_2 *of Proposition* 3.8, *and* $\rho_{x_i}: S^1 \to O(2)$ $(i = 2, 3)$ *is so to* δ_4.

Here, $\delta_n: S^1 \to O(2)$ *is given by*

$$\delta_n(a) \cdot b = a^n \cdot b \qquad (a \in S^1 \subset C, b \in C).$$

PROOF. We notice that ρ_{x_i} is non-trivial by (2.4).

(I) ρ_{x_1} is equivalent to η_1 or η_2 by Proposition 3.7. By the definition of η_i, it is easy to see that $\{1\}$ or S^1 is a principal isotropy subgroup of the S^3 action on R^4 via η_1 or η_2. Also, by (2.2), x_1 has a tubular neighborhood in M, which is S^3 equivariantly diffeomorphic to R^4 with S^3 action via ρ_x. Therefore we see $H=\{1\}$ by Lemma 4.3 (ii).

It is clear that ρ_{x_i} $(i=2, 3)$ is equivalent to δ_n for some $n>0$, and the principal isotropy subgroup for S^1 action on R^2 via δ_n is Z_n. Therefore we have $n=1$ by the above result.

(II) If $\rho_{x_1}|S^1$ is non-trivial, then ρ_{x_1} is equivalent to γ_n for some positive even integer n, by Proposition 3.8. Therefore we can see that the principal isotropy subgroup for the NS^1 action on R^2 via ρ_{x_1} is S^1 or

$$Q_n = <j, \exp(2\pi i/n)> \qquad (\text{even } n>0),$$

the subgroup of S^3 generated by j and $\exp(2\pi i/n)$. Also, the principal isotropy subgroup for the S^1 action on R^2 via ρ_{x_2} is Z_m for some m.

By (2.2), choose a tubular neighborhood

$$U_i = S^3 \times_{S_i} R^2, \quad S_i = S^3_{x_i} \qquad (i=1, 2)$$

of the orbit $S^3 \cdot x_i$. Then the principal isotropy subgroup for the S^3 action on U_i coincides with that for the S_i action on R^2 via ρ_{x_i}, since $S^3_{[p,v]} = p(S_i)_v p^{-1}$ for $[p, v] \in U_i$. Therefore, the principal isotropy subgroup is $Q_n = Z_m$ by the above consideration, which implies $m=4$ and $n=2$. *q.e.d.*

Now, consider the smooth S^3 action on the complex projective plane $P_2(C) = P(H \times C)$ given by (1.1). Let

$$D^2(t) = \{a \in C; |a| \leqq t\}, \quad D^4 = \{p \in H; |p| \leqq 1\}$$

be the unit disks. Then, we have easily the S^3 equivariant embeddings

$$D^4 = D^4 \times 1 \longrightarrow P(H \times C), \qquad S^3 \times_{S^1} D^2(1) \longrightarrow P(H \times C)$$

by sending $(p, a) \in H \times C$ to $[p, \bar{a}] \in P(H \times C)$, and so the S^3 equivariant decomposition

$$P_2(C) = P(H \times C) = S^3 \times_{S^1} D^2 \cup D^4, \quad (D^2 = D^2(1)). \tag{4.6}$$

Next, consider the smooth S^3 action on $P_2(C) = P(H \otimes_C H/\sim)$ given by (1.2). Then we have the S^3 equivariant embeddings

$$S^3 \times_{NS^1} D^2(r) \longrightarrow P(H \otimes_C H/\sim),$$

$$S^3 \times_{S^1} D^2(s) \longrightarrow P(H \otimes_C H/\sim)$$

$(0<s=(1-2r)/(1+2r)<1)$, by sending $[p, a] \in S^3 \times_{NS^1} D^2(r)$ or $S^3 \times_{S^1} D^2(s)$ to

$$[(p \otimes p)a+(p \otimes pj)+(pj \otimes pj)\bar{a}] \qquad \text{or} \quad [p \otimes p+(p \otimes p)\bar{a}],$$

respectively, where NS^1 acts on $D^2(r)$ via γ_2 and S^1 acts on $D^2(s)$ via δ_4 (cf. Proposition 4.5 (II)). Then, we have easily the S^3 equivariant decomposition

$$P_2(C) = S^3 \times_{NS^1} D^2(r) \cup S^3 \times_{S^1} D^2(s). \tag{4.7}$$

PROOF OF THEOREM 1.3. The case (I) of Proposition 4.4. By (2.2), we can choose a closed tubular neighborhood $U=S^3 \times_{S^1} D^2$ of the orbit $S^3 \cdot x_2 = S^3 \cdot x_3$ and a closed S^3 invariant neighborhood $V=D^4$ of x_1, such that $U \cap V=\emptyset$ and S^1 acts on D^2 via δ_1 and S^3 acts on D^4 via η_1.

Then, S^3 acts on $N=M-\mathrm{Int}\,U-\mathrm{Int}\,V$ and the orbit space N/S^3 is a compact 1 dimensional manifold by Lemma 4.3. Therefore N/S^3 is diffeomorphic to a closed interval $[0, 1]$, and hence N is equivariantly diffeomorphic to $S^3 \times [0, 1]$, where S^3 acts on the first factor. These show that M has an equivariant decomposition

$$M = U \cup N \cup V \cong S^3 \times_{S^1} D^2 \cup D^4.$$

Thus, M is equivariantly diffeomorphic to $P_2(C)$ of (4.6), as desired.

The case (II) of Proposition 4.4. We can prove this by the same way as above. We choose closed tubular neighborhoods $U=S^3 \times_{S^1} D^2$ of $S^3 \cdot x_2 = S^3 \cdot x_3$ and $V=S^3 \times_{NS^1} D^2$ of $S^3 \cdot x_1$ so that $U \cap V=\emptyset$, where S^1 and NS^1 act on D^2 by δ_4 and γ_2 respectively, by Proposition 4.5 (II). Then, we see that $N/S^3 \cong [0, 1]$ $(N=M-\mathrm{Int}\,U-\mathrm{Int}\,V)$ by the same way as above, and so N is equivariantly diffeomorphic to $(S^3/Z_4) \times [0, 1]$. These show that M has an equivariant decomposition

$$M = U \cup N \cup V \cong S^3 \times_{S^1} D^2 \cup S^3 \times_{NS^1} D^2.$$

Thus, M is equivariantly diffeomorphic to $P_2(C)$ of (4.7), as desired. *q.e.d.*

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