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Oscillations of Differential Inequalities with Retarded Arguments

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In this paper we consider the following differential inequalities with retarded arguments:

$$\text{(A)} \qquad (-1)^n x^{(n)}(t)\operatorname{sgn} x(t) \geqq \sum_{i=1}^{N} p_i(t) f_i(x(g_i(t))),$$

$$\text{(B)} \qquad (-1)^n x^{(n)}(t)\operatorname{sgn} x(t) \geqq p_0(t) \prod_{i=1}^{N} \phi_i(x(g_i(t))).$$

For these inequalities the following conditions will be assumed without further mention:

(a) The functions $p_i(t)$ $(i=0, 1,\dots, N)$ are continuous and nonnegative on $[0, \infty)$.

(b) The functions $f_i(y)$ and $\phi_i(y)$ $(i=1,\dots, N)$ are continuous and positive on $(-\infty, 0) \cup (0, \infty)$ and $f_i(y)\operatorname{sgn} y$ and $\phi_i(y)\operatorname{sgn} y$ are nondecreasing in y.

(c) The functions $g_i(t)$ $(i=1,\dots, N)$ are continuous and nondecreasing on $[0, \infty)$ and

$$g_i(t) \leqq t \quad \text{for} \quad t \geqq 0 \quad \text{and} \quad \lim_{t\to\infty} g_i(t) = \infty.$$

We shall restrict our attention to solutions $x(t)$ of (A) or (B) which exist on a half-line $[t_x, \infty)$. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise a solution is called nonoscillatory.

The object of this paper is to obtain sufficient conditions under which all bounded solutions of the differential inequalities (A) and (B) are oscillatory. Our results generalize the results due to Gustafson [1] and Shreve [8]. For related results we refer the reader to the papers by Koplatadze [2], Kusano and Onose [3], Ladas [4], Ladas, Lakshmikantham and Papadakis [5], and Sficas and Staikos [6, 7].

THEOREM 1. *Assume that*

$$\text{(1)} \qquad \limsup_{t\to\infty} \int_{g(t)}^{t} [s-g(t)]^{n-1} \sum_{i=1}^{N} p_i(s)ds > (n-1)! \limsup_{y\to 0} \frac{|y|}{f(y)},$$

where $f(y) = \min_{1\leqq i\leqq N} f_i(y)$ *and* $g(t) = \max_{1\leqq i\leqq N} g_i(t)$.

Then, if $n\geqq 2$ *all bounded solutions of* (A) *are oscillatory, while if* $n=1$ *all solutions of* (A) *are oscillatory.*

Proof. Suppose there exists a bounded nonoscillatory solution $x(t)$ of (A). Without loss of generality we may suppose that $x(t)$ is eventually positive. There is a $t_1>0$ such that $x(g_i(t))>0$ for $t\geqq t_1$, $i=1,\dots, N$. From (A) it follows that $(-1)^n x^{(n)}(t)\geqq 0$ for $t\geqq t_1$. Hence in view of the boundedness of $x(t)$ there exists a $t_2\geqq t_1$ such that

$$(2)\qquad (-1)^j x^{(j)}(t)\geqq 0 \quad \text{for} \quad t\geqq t_2, \qquad j=1,\dots, n.$$

Combining Taylor's formula with remainder

$$(3)\qquad x(s)=\sum_{j=0}^{n-1}\frac{x^{(j)}(t)}{j!}(s-t)^j+\frac{1}{(n-1)!}\int_t^s (s-u)^{n-1}x^{(n)}(u)\,du$$

with the inequality (A), we get for $t\geqq s\geqq t_2$

$$(4)\qquad x(s)\geqq \sum_{j=0}^{n-1}\frac{(-1)^j x^{(j)}(t)}{j!}(t-s)^j$$
$$+\frac{1}{(n-1)!}\int_s^t (u-s)^{n-1}\sum_{i=1}^{N}p_i(u)f_i(x(g_i(u)))du.$$

From (2), (4) and the monotonicity of x, f_i, g_i it follows that

$$x(s)\geqq x(t)+\frac{f(x(g(t)))}{(n-1)!}\int_s^t (u-s)^{n-1}\sum_{i=1}^{N}p_i(u)du.$$

Therefore

$$(5)\qquad x(g(t))\geqq x(t)+\frac{f(x(g(t)))}{(n-1)!}\int_{g(t)}^t [u-g(t)]^{n-1}\sum_{i=1}^{N}p_i(u)\,du$$

for $t\geqq t_3$, where t_3 is chosen so large that $g(t)\geqq t_2$ for $t\geqq t_3$.

Now, by (2), $x'(t)\leqq 0$ for $t\geqq t_2$, so that $x(t)$ decreases to a limit $c\geqq 0$ as $t\to\infty$. From (5) we see that $c=0$. Again from (5) we find

$$(6)\qquad (n-1)!\frac{x(g(t))}{f(x(g(t)))}\geqq \int_{g(t)}^t [u-g(t)]^{n-1}\sum_{i=1}^{N}p_i(u)\,du$$

for $t\geqq t_3$. Taking the limit superior as $t\to\infty$ of both sides of (6) we obtain a contradiction to the hypothesis (1). This contradiction establishes the desired result for the case $n\geqq 2$.

To complete the proof it is sufficient to observe that when $n=1$ every nonoscillatory solution of (A) is necessarily bounded.

In exactly the same way we can prove the following theorem.

THEOREM 2. *Assume that*

$$(7)\qquad \limsup_{t\to\infty}\int_{g(t)}^{t}[s-g(t)]^{n-1}p_0(s)\,ds>(n-1)!\limsup_{y\to 0}\frac{|y|}{\phi(y)}$$

where $\phi(y)=\prod_{i=1}^{N}\phi_i(y)$ *and* $g(t)=\max_{1\leqq i\leqq N} g_i(t)$.

Then, if $n\geqq 2$ *all bounded solutions of* (B) *are oscillatory, and if* $n=1$ *all solutions of* (B) *are oscillatory.*

REMARK 1. Theorems 1 and 2 generalize the results of Gustafson [1, Theorems 3.1, 4.1] and Shreve [8, Theorem 1]. An important special case of (A) and (B) to which the above theorems apply is the retarded differential equation

$$x^{(n)}(t)+(-1)^{n+1}p(t)|x(g(t))|^{\alpha}\operatorname{sgn}x(g(t))=0,\qquad 0<\alpha\leqq 1.$$

REMARK 2. Oscillation criteria of similar nature have been obtained by Koplatadze [2], Ladas, Lakshmikatham and Papadakis [5] and Sficas and Stakios [6, 7]. For example, according to Theorem 3 of [7], all bounded solutions of (A) are oscillatory if

$$(8)\qquad \limsup_{t\to\infty}\int_{g(t)}^{t}[g(t)-g(s)]^{n-1}\sum_{i=1}^{N}p_i(s)ds>(n-1)!\limsup_{y\to 0}\frac{|y|}{f(y)},$$

where $g(t)$ and $f(y)$ are the same as in Theorem 1.

The following example shows that in some cases (1) is better than (8).

EXAMPLE 1. The retarded differential equation

$$(9)\qquad x^{(4)}(t)-\frac{a}{t^4}x(\sqrt{t})=0,$$

a being a positive constant, has no bounded nonoscillatory solutions. This follows from Theorem 1 (or 2) since

$$\limsup_{t\to\infty}\int_{\sqrt{t}}^{t}\frac{a}{s^4}[s-\sqrt{t}]^3\,ds=\infty.$$

On the other hand,

$$\limsup_{t\to\infty}\int_{\sqrt{t}}^{t}\frac{a}{s^4}[\sqrt{t}-\sqrt{s}]^3\,ds=\frac{a}{3},$$

so that the criterion (8) is applicable to (9) only when $a>18$.

We note that the ordinary differential equation

$$x^{(4)}(t)-\frac{a}{t^4}x(t)=0$$

associated with (9) has a bounded nonoscillatory solution of the form $x(t)=t^{-\lambda}$ ($\lambda>0$). This shows that the oscillation of bounded solutions of (9) is caused by the presence of the delay.

We shall state oscillation criteria of slightly different kind for the differential inequalities (A) and (B).

THEOREM 3. *Let*

$$f(y)=\min_{1\leqq i\leqq N} f_i(y) \quad \textit{and} \quad g(t)=\max_{1\leqq i\leqq N} g_i(t)$$

and suppose that $g'(t)\geqq 0$,

$$\int_{+0}^{+a}\frac{dy}{f(y)}<\infty\,,\quad \int_{-a}^{-0}\frac{dy}{f(y)}<\infty \text{ for some } a>0\,, \tag{10}$$

and

$$\int^{\infty} g'(t)\Big(\int_{g(t)}^{t}[s-g(t)]^{n-2}\sum_{i=1}^{N} p_i(s)ds\Big)dt=\infty\,. \tag{11}$$

Then, all bounded solutions of (A) *are oscillatory.*

PROOF. Let $x(t)$ be a bounded nonoscillatory solution of (A) which is eventually positive. It follows from (A) that $(-1)^n x^{(n)}(t)\geqq 0$ for $t\geqq t_1$, t_1 being sufficiently large, and hence that there is a $t_2\geqq t_1$ such that (2) holds for $t\geqq t_2$.

Applying Taylor's formula to $x'(s)$ and using (A) and (2), we have

$$\begin{aligned} x'(s) &= \sum_{j=1}^{n-1}\frac{x^{(j)}(t)}{(j-1)!}(s-t)^{j-1}+\frac{1}{(n-2)!}\int_t^s (s-u)^{n-2}x^{(n)}(u)\,du \\ &\leqq \frac{-1}{(n-2)!}\int_s^t (u-s)^{n-2}\sum_{i=1}^{N} p_i(u) f_i(x((g_i(u)))\,du \end{aligned} \tag{12}$$

for $t\geqq s\geqq t_2$. Putting $s=g(t)$ in (12) and taking the monotonicity of f_i, g_i, x into account, we obtain

$$-x'(g(t))\geqq \frac{f(x(g(t)))}{(n-2)!}\int_{g(t)}^{t}[u-g(t)]^{n-2}\sum_{i=1}^{N} p_i(u)\,du \tag{13}$$

for $t\geqq t_3$, where t_3 is taken so that $g(t)\geqq t_2$ for $t\geqq t_3$. Multiplying both sides of (13) by $g'(t)/f(x(g(t)))$ and then integrating over $[t_3, t]$, we obtain

$$\begin{aligned} &\int_{x(g(t))}^{x(g(t_3))}\frac{dy}{f(y)} \\ &\geqq \frac{1}{(n-2)!}\int_{t_3}^{t} g'(s)\Big(\int_{g(s)}^{s}[u-g(s)]^{n-2}\sum_{i=1}^{N} p_i(u)du\Big)ds\,. \end{aligned} \tag{14}$$

In view of (2) $x(t)$ tends to a finite limit $c \geqq 0$ as $t \to \infty$. Therefore, by (10), the left hand side of (14) remains bounded, while on account of (11) the right hand side becomes unbounded as $t \to \infty$. This contradiction proves our theorem.

Similarly we can prove the following

THEOREM 4. *Let*

$$\phi(y) = \prod_{i=1}^{N} \phi_i(y) \quad \textit{and} \quad g(t) = \max_{1 \leqq i \leqq N} g_i(t)$$

and suppose that $g'(t) \geqq 0$,

$$\int_{+0}^{+a} \frac{dy}{\phi(y)} < \infty, \quad \int_{-a}^{-0} \frac{dy}{\phi(y)} < \infty \textit{ for some } a > 0,$$

and

$$\int^{\infty} g'(t) \left(\int_{g(t)}^{t} [s - g(t)]^{n-2} p_0(s) ds \right) dt = \infty.$$

Then, all bounded solutions of (B) *are oscillatory.*

REMARK 3. An important special case to which Theorems 3 and 4 are applicable is the equation

$$x^{(n)}(t) + (-1)^{n+1} p(t) |x(g(t))|^{\alpha} \operatorname{sgn} x(g(t)) = 0, \qquad 0 < \alpha < 1.$$

EXAMPLE 2. Consider the retarded differential equation

$$x''(t) - \left| x\left(t - \frac{1}{t}\right) \right|^{\alpha} \operatorname{sgn} x\left(t - \frac{1}{t}\right) = 0, \qquad 0 < \alpha < 1. \tag{15}$$

From Theorem 3 (or 4) it follows that all bounded solutions of (15) are oscillatory, since

$$\int^{\infty} \left(1 + \frac{1}{t^2}\right) \left(\int_{t-t^{-1}}^{t} ds \right) dt = \int^{\infty} \left(1 + \frac{1}{t^2}\right) \frac{1}{t} dt = \infty.$$

However, neither (1) nor (8) is applicable to (15). In fact, as is easily verified,

$$\limsup_{t \to \infty} \int_{t-t^{-1}}^{t} \left[s - t + \frac{1}{t} \right] ds = 0,$$

$$\limsup_{t \to \infty} \int_{t-t^{-1}}^{t} \left[t - \frac{1}{t} - s + \frac{1}{s} \right] ds = 0.$$

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