HIROSHIMA MATH. J.
5(1975), 181-185

Infinite Dimensional Noetherian Hilbert Domains

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(Received January 11, 1975)

1. As we know, examples of Hilbert domains are to a large extent the so-called equicodimensional rings, namely rings in which the height of every maximal ideal coincides with the dimension of R. As for a noetherian ring R, it is well known that R is an equicodimensional Hilbert ring if and only if the polynomial ring $R[X]$ in an indeterminate X over R is equicodimensional.

Examples of Hilbert domains with maximal ideals of different height were given by some authors (cf. [3], [6] and [7]). In particular, Heinzer constructed noetherian Hilbert domains with maximal ideals of various preassigned height. The purpose of this note is to give an alternative way of construction of such an example and to construct noetherian Hilbert regular domains with an infinite dimension.

The author wishes to express his hearty thanks to Prof. M. Nishi for his suggestions, and to Prof. H. Yanagihara for his comments in writing this paper.

2. All rings considered are commutative with identity. By a locally noetherian ring we mean a ring in which every localization by a maximal ideal is a noetherian ring, and for a prime ideal $\mathfrak{p}$ in a ring A, *depth*$(\mathfrak{p})$ means the dimension of $A/\mathfrak{p}$.

LEMMA 1. *Let A be a locally noetherian ring and $\mathfrak{p}$ be a prime ideal with depth$(\mathfrak{p})\geq 2$. Then the following statements a), b), and c) hold:*

a) $U=\{\mathfrak{P}\in Spec(A);\, \mathfrak{P}\supset\mathfrak{p},\, ht(\mathfrak{P}/\mathfrak{p})=1\}$ *is an infinite set.*

b) *If s is a non unit of A but not contained in $\mathfrak{p}$, then the subset of U consisting of $\mathfrak{P}$ which does not contain s is infinite.*

c) *Let U' be an infinite subset of U. Then we have* $\mathfrak{p}=\bigcap_{\mathfrak{P}\in U'}\mathfrak{P}$.

PROOF. If A is noetherian, it is clear that each statement holds. Otherwise, by taking a maximal ideal $\mathfrak{m}$ in A such that $\mathfrak{m}\supset\mathfrak{p}$ and $ht(\mathfrak{m}/\mathfrak{p})\geq 2$, it suffices to consider the ring $A_{\mathfrak{m}}$ in place of A.

LEMMA 2. *Let A be a locally noetherian ring. Then A is a Hilbert ring if and only if every prime ideal $\mathfrak{p}$ with depth$(\mathfrak{p})=1$ is the intersection of maximal ideals containing $\mathfrak{p}$.*

PROOF. The "only if" part is obvious. We assume that for every prime ideal $\mathfrak{p}$ with *depth*$(\mathfrak{p})=1$, $\mathfrak{p}$ is the intersection of maximal ideals containing $\mathfrak{p}$. As every prime ideal $\mathfrak{p}$ with $depth(\mathfrak{p})\geq 2$ is the intersection of prime ideals properly containing $\mathfrak{p}$ by a) and c) of Lemma 1, A is a Hilbert ring by assumption.

In [2], p. 106, Grothendieck proved that for a noetherian ring A, A_s is a Hilbert ring for every non nilpotent element s of $Rad(A)$. The following proposition shows that for locally noetherian case, the same statement holds.

PROPOSITION 3. *Let A be a locally noetherian ring and s be a non nilpotent element of $Rad(A)$. Then A_s is a Hilbert ring.*

PROOF. By Lemma 2, it suffices to show that for any prime ideal $\mathfrak{p}_s$ in A_s with $depth(\mathfrak{p}_s)=1$, $\mathfrak{p}_s$ is the intersection of the maximal ideals containing $\mathfrak{p}_s$. As s is an element of $Rad(A)$, $depth(\mathfrak{p})\geq 2$. Hence b) of Lemma 1 implies that $U'=\{\mathfrak{P}\in Spec(A);\ \mathfrak{P}\supset\mathfrak{p},\ ht(\mathfrak{P}/\mathfrak{p})=1,\ \mathfrak{P}\not\ni s\}$ is an infinite set; hence $\mathfrak{p}=\bigcap_{\mathfrak{P}\in U'}\mathfrak{P}$ by c) of Lemma 1; therefore $\mathfrak{p}_s=\bigcap_{\mathfrak{P}\in U'}\mathfrak{P}_s$.

The following lemma is obvious by Theorem 105 in [4].

LEMMA 4. *Let A be an integral domain and $\mathfrak{p}_1, \mathfrak{p}_2,\dots, \mathfrak{p}_n$ prime ideals in A, no two of which are comparable. Then $R=A_{\mathfrak{p}_1}\cap A_{\mathfrak{p}_2}\cap\cdots\cap A_{\mathfrak{p}_n}$ is a semi local ring and $R\cap\mathfrak{p}_iA_{\mathfrak{p}_i}$, $i=1, 2,\dots, n$ are the maximal ideals in R.*

LEMMA 5. *Let A be a noetherian catenary domain and $\mathfrak{P}$ be a prime ideal in A. Let s be a non zero element of $\mathfrak{P}$. Let $\mathfrak{p}$ be a maximal element with respect to the inclusion relation in the family $\{\mathfrak{p}\in Spec(A);\ \mathfrak{p}\subset\mathfrak{P},\ s\notin\mathfrak{p}\}$. Then we have $ht(\mathfrak{p})=ht(\mathfrak{P})-1$.*

The proof is almost clear and is omitted.

COROLLARY 6 (Heinzer). *For any given positive integers $r_1>r_2>\cdots>r_n$, there exists a noetherian integrally closed Hilbert domain R such that $r_1, r_2,\dots, r_n$ are precisely the integers which occur as the height of a maximal ideal in R.*

PROOF. Let k be a field, $m=r_1+1$ and $A=k[X_1, X_2,\dots, X_m]$. Let $\mathfrak{p}_1, \mathfrak{p}_2,\dots, \mathfrak{p}_n$ be prime ideals in A, no two of which are comparable, and $ht(\mathfrak{p}_i)=r_i+1$. Then $B=A_{\mathfrak{p}_1}\cap A_{\mathfrak{p}_2}\cap\cdots\cap A_{\mathfrak{p}_n}$ is a semi local ring with maximal ideals $\mathfrak{p}_iA_{\mathfrak{p}_i}\cap B$, $i=1, 2,\dots, n$. Let s be a non zero element of $\mathfrak{p}_1\cap\mathfrak{p}_2\cap\cdots\cap\mathfrak{p}_n$. Proposition 3 implies that $R=B_s$ is a Hilbert ring, and it is clear that the height of each maximal ideal in R is r_i for some i by Lemma 5 and R is integrally closed.

3. In this section, we construct a noetherian Hilbert regular domain with an infinite dimension by applying a similar method to Nagata's construction of the infinite dimensional noetherian domains ([6], p. 203).

LEMMA 7. *Let k be a field, and $A=k[X_1, X_2, X_3,\dots]$. For positive integers*

$e_1<e_2<e_3<\cdots$, let $\mathfrak{p}_i$ be the prime ideal in A generated by $X_{e_{i-1}+1},\dots, X_{e_i}$, where $e_0=1$. Then the following statements hold.

a) *If f is a non zero element, then f is contained in only a finite number of $\mathfrak{p}_i$.*

b) *If $\mathfrak{a}$ is an ideal in A contained in $\bigcup_{i=1}^{\infty} \mathfrak{p}_i$, $\mathfrak{a}$ is contained in $\mathfrak{p}_i$ for some i.*

PROOF. The statement *a*) is trivial.

b) At first, we assume that $\mathfrak{a}$ is finitely generated, say $\mathfrak{a}=(f_1,\dots,f_r)$. For a sufficiently large n, we have $f_1,\dots,f_r\in k[X_1,\dots, X_n]$. As $\mathfrak{p}_i \cap k[X_1,\dots, X_n]=0$ for i satisfying $n<e_{i-1}$, $\mathfrak{a}\cap k[X_1,\dots, X_n]$ is contained in a finite union of $\mathfrak{p}_i \cap k[X_1,\dots, X_n]$; therefore $\mathfrak{a}\subset \mathfrak{p}_i$ for some i. In general case, we put $\mathfrak{a}_n=(f_1,\dots,f_n)$, where $\mathfrak{a}=(f_1,f_2,\dots)$ (Note that $\mathfrak{a}$ is generated by a countable number of elements of A). As $\mathfrak{a}_n$ is finitely generated, $\mathfrak{a}_n$ is contained in $\mathfrak{p}_{i(n)}$ for some $i(n)$. The set $\{i(n); n=1,2,\dots\}$ is finite because f_1 is contained in only a finite number of $\mathfrak{p}_i$ by a), where $\mathfrak{a}$ is contained in a finite union of $\mathfrak{p}_i$. Thus $\mathfrak{a}\subset \mathfrak{p}_i$ for some i.

LEMMA 8. *Let k be a field and $A=k[X_1, X_2, X_3,\dots]$. For positive integers $e_1<e_2<e_3<\cdots$, let $\mathfrak{p}_i$ be the prime ideal in A generated by X_1, $X_{e_{i-1}+2},\dots, X_{e_i+1}$. Let $B=S^{-1}A$ and $\mathfrak{P}_i=S^{-1}\mathfrak{p}_i$, where $S=A-\bigcup_{i=1}^{\infty} \mathfrak{p}_i$. Then the following statements hold:*

a) *The maximal ideals in B are $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3,\dots$.*

b) *B_{X_1} is a noetherian regular domain.*

PROOF. a) It suffices to show that for any ideal $\mathfrak{a}$ in A such that $\mathfrak{a}\subseteq\bigcup_{i=1}^{\infty}\mathfrak{p}_i$ we have $\mathfrak{a}\subseteq \mathfrak{p}_i$ for some i. Let $\varphi\colon A\to k[X_2, X_3,\dots]=A/X_1A$ be the canonical homomorphism. As $\varphi(\mathfrak{a})\subset\bigcup_{i=1}^{\infty}\varphi(\mathfrak{p}_i)$ and $\varphi(\mathfrak{p}_i)=(X_{e_{i-1}+2},\dots, X_{e_i+1})$, $\varphi(\mathfrak{a})$ is contained in $\varphi(\mathfrak{p}_i)$ for some i by b) of Lemma 7, whence $\mathfrak{a}\subset \mathfrak{p}_i$.

b) Since $B_{\mathfrak{p}_i}$ is the localization of $k(\dots, X_j,\dots)[X_1, X_{e_{i-1}+2},\dots, X_{e_i+1}]$, $j\neq 1$, $e_{i-1}+2,\dots, e_i+1$, by the prime ideal $(X_1, X_{e_{i-1}+2},\dots, X_{e_i+1})$, B is regular and locally noetherian. We will show that B_{X_1} is noetherian. Let $\mathfrak{a}_{X_1}$ be any non zero ideal of B_{X_1}, where $\mathfrak{a}$ is an ideal in B, and let f be an element of $\mathfrak{a}\cap A$ such that $\varphi(f)\neq 0$. Lemma 7 implies that there exists only a finite number of maximal ideals in $S^{-1}\varphi(A)$ which contain $\varphi(f)$; hence f is contained in only a finite number of maximal ideals in B; hence also $\mathfrak{a}$ is; therefore $\mathfrak{a}$ is finitely generated because B is locally noetherian; therefore B_{X_1} is noetherian.

PROPOSITION 9. *For any given positive integers $r_1<r_2<r_3<\cdots$, there exists a noetherian Hilbert regular domain R such that r_1, r_2, $r_3,\dots$ are precisely the integers which occur as the height of maximal ideals in R.*

PROOF. Let $e_i=r_1+r_2+\cdots+r_i$. Let $R=B_{X_1}$, where B_{X_1} is just the same as in Lemma 8. As B is locally noetherian and X_1 is an element of $Rad(B)$, R

is a Hilbert ring by Proposition 3. Lemma 5 implies that for every maximal ideal $\mathfrak{m}$, $ht(\mathfrak{m})=r_i$ for some i.

REMARK TO PROPOSITION 9. a) $dim(R)=\infty$.

b) *Let* $\mathfrak{a}$ *be a cardinal number not less than* $\aleph_0$. *If* $card(k)>\mathfrak{a}$, *then* R *is an* $H(\mathfrak{a})$*-ring. (As for the definition of an* $H(\mathfrak{a})$*-ring, see* [1])

Before proving the statement b), we need the following Lemma.

LEMMA. *Let* k *be a field with cardinality greater than* $\mathfrak{a}$, *and let* $\mathfrak{p}$, $\mathfrak{P}$ *be prime ideals in* $A=k[X_1,\dots,X_n]$ *such that* $\mathfrak{p}\subset\mathfrak{P}$ *and* $ht(\mathfrak{P}/\mathfrak{p})=2$. *Then the cardinality of the set* $\{\mathfrak{q}\in Spec(A);\ \mathfrak{p}\subset\mathfrak{q}\subset\mathfrak{P}\}$ *is greater than* $\mathfrak{a}$.

PROOF. By applying normalization theorem to $k[X_1,\dots,X_n]/\mathfrak{p}$, we may assume that $\mathfrak{p}=0$ and $ht(\mathfrak{P})=2$. Since $A_{\mathfrak{P}}$ is a regular local ring of dimension 2, $\mathfrak{P}A_{\mathfrak{P}}$ is of the form $(t_1, t_2)A_{\mathfrak{P}}$. As $(t_1+at_2, t_2)A_{\mathfrak{P}}=\mathfrak{P}A_{\mathfrak{P}}$ for every $a\in k$, t_1+at_2, t_2 is a regular system of parameter of $A_{\mathfrak{P}}$; therefore, $\mathfrak{p}_a=(t_1+at_2)A_{\mathfrak{P}}$ is prime in $A_{\mathfrak{P}}$. It is trivial that $\mathfrak{p}_a=\mathfrak{p}_b$ implies $a=b$; hence the cardinality of the set of prime ideals of height 1 which are contained in $\mathfrak{P}$ is greater than $\mathfrak{a}$.

PROOF OF b). Let $\mathfrak{P}_{X_1}$ be a $G(\mathfrak{a})$-ideal in R. Suppose that $\mathfrak{P}_{X_1}$ is not maximal in R. As $R_0=R/\mathfrak{P}_{X_1}$ is a $G(\mathfrak{a})$-domain and noetherian, R_0 has not the property $J(\mathfrak{a})$ by Proposition 1 in [1]; hence for some $J(\mathfrak{a})$-subset D of R_0, we have $Ht_1(R_0)=H_R(D)$. Since R_0 is noetherian, there is only a finite number of prime ideals of height one which contain a for each element a of D, whence we have $card(Ht_1(R_0))\leq\mathfrak{a}$. On the other hand, we have $\mathfrak{P}\subset\mathfrak{P}_i$ for some i by a) of Lemma 8. Since P_{X_1} is not maximal in B_{X_1}, $ht(\mathfrak{p}_i/\mathfrak{p})\geq 2$, where $\mathfrak{P}=S^{-1}\mathfrak{p}$. By applying Lemma to $k(\dots, X_j,\dots)[X_1, X_{e_{i-1}+2},\dots, X_{e_i+1}]$, $j\neq 1$, $e_{i-1}+2,\dots, e_i+1$, we see that the cardinality of the set $U=\{\mathfrak{q}\in Spec(B);\ \mathfrak{P}\subset\mathfrak{q}\subset\mathfrak{P}_i,\ ht(\mathfrak{q}/\mathfrak{P})=1\}$ is greater than $\mathfrak{a}$. There exists only a finite number of the elements of U which contain X_1; therefore $card(Ht_1(R_0))>\mathfrak{a}$, which leads to a contradiction. Hence $\mathfrak{P}_{X_1}$ is maximal in R.

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