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Geometry of Homogeneous Lie Loops

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Introduction

Let M be a differentiable manifold with a given linear connection ∇ and $e \in M$ be a fixed point. Then we have considered in [5] the local multiplication μ at e compatible with ∇, which is given by

$$\mu(x, y) = \mathrm{Exp}_x \circ \tau_{e,x} \circ \mathrm{Exp}_e^{-1}(y),$$

where Exp_x denotes the exponential mapping at x and $\tau_{e,x}$ denotes the parallel displacement of tangent vectors along the geodesic joining e to x in a normal neighborhood of e.

If M is a reductive homogeneous space A/K with the canonical connection, due to K. Nomizu, then the local multiplication μ given above satisfies

$$\mu(x, y) = (\exp X)\cdot y;\ X = \mathrm{Exp}_e^{-1}(x) \in \mathfrak{M} \subset \mathfrak{A},$$

where $\mathfrak{A} = \mathfrak{M} + \mathfrak{K}$ is the decomposition of the Lie algebra of A such that ad $(K)\mathfrak{M} \subset \mathfrak{M}$. (Cf. [15, Theorem 10.2].) Therefore, if M is reduced to a Lie group A itself, then the canonical connection is reduced to the $(-)$-connection of [3] and the local multiplication μ coincides with the multiplication of A in local.

These facts suggest us a problem of the existence of a global differentiable binary system on a reductive homogeneous space A/K, which coincides locally with the above geodesic local multiplication μ. We have been interested in this problem and in the question how such a multiplication relates to the canonical connection and to the general Lie triple system defined on the tangent space $\mathfrak{M}$, which will be called the Lie triple algebra in this paper (cf. [5–8]).

The main purpose of the present paper is to investigate the above problem and to provide the basic concepts to construct the global theory of differentiable binary systems, as an analogy and also as a generalization of the theory of Lie groups and Lie algebras.

Our considerations are based on the purely algebraic concept of a *homogeneous loop* (Definition 1.4) and the concept of a *homogeneous Lie loop* (Definition 3.1), a homogeneous loop admitting a natural differentiable structure. We shall prove that any homogeneous Lie loop has the canonical connection and is a reductive homogeneous space. We shall investigate the condition for the multiplication of such a loop G to coincide in local with the geodesic local multiplication μ given above. Especially, if G has in addition the *symmetric property*, i.e.,

$$(xy)^{-1} = x^{-1}y^{-1} \qquad (x, y \in G),$$

then we shall prove that G is a symmetric homogeneous space and satisfies the above condition.

We study in § 1 various properties of homogeneous loops used in the subsequent sections.

We shall consider in § 2 the *semi-direct product* $A = G \times K$ of a homogeneous loop G by some group K of automorphisms of G, and show that A is a group and G can be regarded as the factor set A/K by the left coset decomposition of A modulo its subgroup K.

In § 3, we shall investigate the various properties of the *canonical connection* of a homogeneous Lie loop G. Above all, we shall prove that G can be identified with the reductive homogeneous space $A(G)/K(G)$ with the canonical decomposition $\mathfrak{A} = \mathfrak{G} + \mathfrak{K}$ of the Lie algebra of $A(G)$, $A(G)$ being the semi-direct product $G \times K(G)$ (Theorem 3.7).

In § 4, we shall show that the geodesic local multiplication μ of a locally reductive space G defines the *geodesic homogeneous local Lie loop*, which is a basic example of a homogeneous Lie loop in local.

In § 5, we shall investigate the conditions for a homogeneous Lie loop G to be coincident with any geodesic local Lie loop and we shall say such a loop to be *geodesic*. Connected Lie groups are examples of geodesic homogeneous Lie loops.

§ 6 will be devoted to studying the *symmetric Lie loop* G, i.e., a homogeneous Lie loop with the symmetric property. After proving that G is a symmetric homogeneous space (Theorem 6.1) we shall show that G is geodesic (Theorem 6.4).

Finally, in § 7, we shall consider the *Lie triple algebra* $\mathfrak{G}$ defined on the tangent space of a geodesic homogeneous Lie loop G by

$$XY = [X, Y]_{\mathfrak{G}}; \ [X, Y, Z] = [[X, Y]_{\mathfrak{K}}, Z] \qquad (X, Y, Z \in \mathfrak{G}),$$

with respect to the canonical decomposition $\mathfrak{A} = \mathfrak{G} + \mathfrak{K}$, and show that G can be characterized locally by its Lie triple algebra $\mathfrak{G}$ (Theorems 7.3 and 7.8).

The correspondence between the subloops of G and the subsystems of $\mathfrak{G}$ is going to be discussed in another article.

Recently, A. Sagle and others are studying a local multiplication on a reductive homogeneous space with the same interest as ours in its relations to linear connections and to non-associative algebras. (Cf. [17, Appendix] and [18].)

For the terminologies used in this paper, we refer mainly to R. H. Bruck

[1] in the theory of loops, and S. Kobayashi–K. Nomizu [9] in differential geometry.

§ 1. Homogeneous loops

In this section, we shall introduce the notion of homogeneous loops and investigate their basic properties.

Let $G=(G, \mu)$ be a binary system with a binary operation

$$\mu: G\times G \longrightarrow G.$$

The multiplication $\mu(x, y)$ $(x, y\in G)$ is denoted by xy when no confusion occurs. The left and right translations by an element $x\in G$ are denoted by

$$L_x, R_x: G \longrightarrow G;\ L_x(y) = xy,\ R_x(y) = yx \qquad (y\in G). \tag{1.1}$$

DEFINITION 1.1. A binary system $G=(G,\mu)$ is a *quasigroup* if all left and right translations (1.1) of G are permutations of G.

A quasigroup G is a *loop* if there is a (two-sided) identity $e\in G$, $xe=ex=x$ $(x\in G)$. The concepts of subloops, homomorphisms and isomorphisms of loops and quasigroups are defined in a natural manner.

For the systematic theory of loops, we refer to R. H. Bruck [1].

DEFINITION 1.2. Let G be a loop. The *left translation group* $L(G)$ of G is the transformation group of G generated by all left translations of G. Also, the *left inner mapping group* $L_0(G)$ of G is the subgroup of $L(G)$ generated by all *left inner mappings*

$$L_{x,y} = L_{xy}^{-1}\circ L_x\circ L_y\,^{1)} \qquad (x, y\in G). \tag{1.2}$$

PROPOSITION 1.1. *The left inner mapping group $L_0(G)$ of a loop G is the isotropy subgroup of the left translation group $L(G)$ at the identity $e\in G$.*

PROOF. Consider the subset

$$H = \{\alpha\in L(G);\ L_{\alpha(e)}^{-1}\circ\alpha\in L_0(G)\}\subset L(G).$$

It is clear that $L_0(G)$ is contained in H since $L_{x,y}(e)=e$. Conversely, if $\alpha\in H$ leaves e fixed, then $\alpha\in L_0(G)$. Therefore, it is sufficient to show that $H=L(G)$.

For any $\alpha\in H$, put $\alpha(e)=a$ and $\theta=L_a^{-1}\circ\alpha\in L_0(G)$. Then, for any $x\in G$ we see that $L_x\circ\alpha\in H$, since $(L_x\circ\alpha)(e)=xa$ and

1) In this paper, $g\circ f$ means the composition of mappings f and g such as $g\circ f(x)=g(f(x))$. Therefore, we notice that our notations differ slightly to those in [1].

$$L_{xa}^{-1}\circ L_x\circ\alpha = L_{xa}^{-1}\circ L_x\circ L_a\circ\theta = L_{x,a}\circ\theta \in L_0(G).$$

Furthermore, we see $\beta = L_x^{-1}\circ\alpha \in H$, since $a = xb$ ($b = \beta(e)$) and

$$L_b^{-1}\circ\beta = (L_{x,b})^{-1}\circ L_{xb}^{-1}\circ\alpha = (L_{x,b})^{-1}\circ\theta \in L_0(G).$$

Thus we see that $L(G) \subset L(G)\circ H \subset H$ as desired. *q.e.d.*

REMARK 1.1. The *multiplication group* $M(G)$ of a loop G is the transformation group of G generated by all left and right translations of G. Also, the *inner mapping group* $I(G)$ is the subgroup of $M(G)$ generated by all left inner mappings $L_{x,y}$, *right inner mappings* $R_{x,y}$ and *proper inner mappings* T_x ($x, y \in G$), where

$$R_{x,y} = R_{xy}^{-1}\circ R_y\circ R_x; \qquad T_x = L_x^{-1}\circ R_x. \tag{1.3}$$

Then, the above proof is on the same line as that of [1, IV Lemma 1.2]:

$I(G)$ is the isotropy subgroup of $M(G)$ at $e \in G$.

DEFINITION 1.3. A loop G is said to have the *left inverse property*, or to be a *left I.P. loop*, if, for any $x \in G$, there corresponds an element $x^{-1} \in G$ such that

$$x^{-1}(xy) = y \qquad (y \in G), \tag{1.4}$$

or equivalently,

$$L_{x^{-1},x} = \mathrm{id}^{2)} \quad \text{or} \quad L_x^{-1} = L_{x^{-1}}. \tag{1.4$'$}$$

This element x^{-1} is uniquely determined by x, and x^{-1} is the inverse of x.

Now, let G be a left I.P. loop with the identity e. For any fixed element $a \in G$, consider a binary operation μ_a: $G \times G \to G$ defined by

$$\mu_a(x, y) = a((a^{-1}x)(a^{-1}y)) \qquad (x, y \in G), \tag{1.5}$$

where the multiplications on the right hand side are the original ones of G.

Then it is easy to see the following

LEMMA 1.2. *$G^{(a)} = (G, \mu_a)$ is a loop with the identity a, and $G^{(e)}$ is the same as the original loop G.*

This loop $G^{(a)}$ will be called the *transposed loop of G centered at a*. In the following, we denote the multiplication (1.5) by

$$x \overset{(a)}{\cdot} y = \mu_a(x, y).$$

THEOREM 1.3. *Let G be a left I.P. loop. For any $a, b \in G$, the left inner*

2) id means the identity transformation.

mapping $L_{a,b}$ of (1.2) *is an automorphism of G if and only if the transposed loop $G^{(b)}$ is isomorphic to $G^{(ab)}$ under the left translation L_a of G.*

PROOF. By (1.5) and (1.2), we have

$$L_a(x \overset{(b)}{\cdot} y) = (L_{ab} \circ L_{a,b})((b^{-1}x)(b^{-1}y)),$$
$$L_a(x) \overset{(ab)}{\cdot} L_a(y) = L_{ab}(L_{a,b}(b^{-1}x)L_{a,b}(b^{-1}y)).$$

Since L_{ab} and $L_{b^{-1}}$ are permutations of G, these are equal if and only if $L_{a,b}$ is an automorphism of G. *q.e.d.*

Since $L_{a^{-1},a}=\mathrm{id}$ by (1.4)′, we have

COROLLARY 1.4. *Any transposed loop $G^{(a)}$ of a left I.P. loop G is isomorphic to G by $L_{a^{-1}}$, and is thereby a left I.P. loop.*

LEMMA 1.5. *The inverse of x in $G^{(a)}$ is expressed by $a(a^{-1}x)^{-1}$ in the original loop G.*

PROOF. Put $x'=a(a^{-1}x)^{-1}$. Then we see easily $\mu_a(x', x)=a$. Since a is the identity of the left I.P. loop $G^{(a)}$, the element x' must be the inverse of x in $G^{(a)}$. *q.e.d.*

By the condition of Theorem 1.3, we give the following

DEFINITION 1.4. A loop G is called a *left A-loop*, if the left inner mapping group $L_0(G)$ of Definition 1.2 is a subgroup of the automorphism group $AUT(G)$ of G. By a *homogeneous loop*, we mean a left A-loop with the left inverse property.

REMARK 1.2. A loop G is an *A-loop*, if $I(G)$ in Remark 1.1 is a subgroup of $AUT(G)$. Various properties of A-loops have been investigated by R. H. Bruck and L. J. Paige [2].

THEOREM 1.6. *Let G be a homogeneous loop. Then, for any $a, b \in G$, the transposed loops $G^{(a)}$ and $G^{(b)}$ are isomorphic under the left translation L_c of G, where c is the element determined by $b=ca$. Moreover, the transposed loop $(G^{(a)})^{(b)}$ of $G^{(a)}$ coincides with $G^{(b)}$.*

PROOF. The first half of the theorem is an immediate consequence of Theorem 1.3.

Let $(G', \bullet)$ denote $(G^{(a)}, \overset{(a)}{\cdot})$, which is a left I.P. loop by Corollary 1.4. Consider the multiplication

$$\mu'_b(x, y) = b\bullet((b'\bullet x)\bullet(b'\bullet y)) \qquad \text{in } G'^{(b)},$$

where $b'=a(a^{-1}b)^{-1}$ is the inverse of b in G' by Lemma 1.5. By (1.5) and (1.4), we see that $b'\bullet z=a((a^{-1}b)^{-1}(a^{-1}z))$ and

$$\mu'_b(x, y) = (L_a \circ L_d)((d^{-1}(a^{-1}x))(d^{-1}(a^{-1}y))) \qquad (d = a^{-1}b).$$

Here, $L_a \circ L_d = L_{ad} \circ L_{a,d} = L_b \circ L_{a,d}$ and $L_{a,d}$ is an automorphism of G by the assumption. Also

$$L_{a,d}(d^{-1}(a^{-1}z)) = (L_b^{-1} \circ L_a \circ L_d)(d^{-1}(a^{-1}z)) = b^{-1}z.$$

These show that $\mu'_b(x, y)=L_b((b^{-1}x)(b^{-1}y))=\mu_b(x, y)$ as desired. *q.e.d.*

By this theorem, we can give the following

DEFINITION 1.5. A class $\{G_a; a \in G\}$ of homogeneous loops on the same underlying set G will be called a *homogeneous structure* on G, if G_b is the transposed loop $G_a^{(b)}$ of G_a centered at b for any a, $b \in G$. A *homomorphism*

$$\phi\colon \{G_a; a \in G\} \longrightarrow \{H_b; b \in H\}$$

of homogeneous structures is a mapping $\phi\colon G \to H$ such that $\phi\colon G_a \to H_{\phi(a)}$ is a homomorphism for any $a \in G$.

COROLLARY 1.7. *For any homogeneous loop G, the class*

$$\mathscr{H}_G = \{G^{(a)}; a \in G\}$$

of all transposed loops of G is a homogeneous structure on G, and conversely any homogeneous structure on G is given by this manner. Also, any automorphism $\phi\colon \mathscr{H}_G \to \mathscr{H}_G$ is a composition

$$\phi = L_a \circ \alpha, \quad a = \phi(e), \quad \alpha \in AUT(G),$$

where e is the identity of G and L_a is the left translation (1.1) *of G.*

LEMMA 1.8. *The following identities are valid in a homogeneous loop G:*

(1.6) $$L_{x,y} = L_{y,y^{-1}x^{-1}}.$$

(1.7) $$L_{x,y}^{-1} = L_{y^{-1},x^{-1}} = L_{u,v} \qquad (u = xy, uv = x).$$

(1.8) $$(xy)^{-1} = x(y(y^{-1}x^{-1})^2).$$

PROOF. Since $L_{x,y} \in AUT(G)$ and $L_{x,y}(y^{-1}x^{-1})=(xy)^{-1}$, we have easily $L_{x,y}((y^{-1}x^{-1})z)=(xy)^{-1}L_{x,y}(z)$ and so

$$L_{x,y} = L_{xy} \circ L_{x,y} \circ L_{y^{-1}x^{-1}} = L_{y,y^{-1}x^{-1}},$$

$$L_{x,y}^{-1} = L_{y^{-1}x^{-1}}^{-1} \circ L_{y^{-1}} \circ L_{x^{-1}} = L_{y^{-1},x^{-1}} = L_{x^{-1},xy}$$
$$= L_{x^{-1},u} = L_{u,u^{-1}x} = L_{u,v}.$$

These show (1.6) and (1.7). Also, (1.8) is obtained by

$$(xy)^{-1} = (xy)((xy)^{-1})^2 = (xy)L_{x,y}((y^{-1}x^{-1})^2) = x(y(y^{-1}x^{-1})^2).$$

q.e.d.

DEFINITION 1.6. A loop G is said to be *left alternative* if

(1.9) $\quad x(xy) = (xx)y, \qquad$ for any $\quad x, y \in G,$

or equivalently $L_{x,x} = \mathrm{id}$ in G. A left I.P. loop G is said to be *left power alternative*[3] if

(1.10) $\quad L_{x^p,x^q} = \mathrm{id}$, for any $x \in G$ and any integers p, q,

where $x^0 = e$, $x^n = xx^{n-1}$ and $x^{-n} = (x^n)^{-1}$ for any positive integer n. Also, a loop G is said to be *power associative* (resp. *di-associative*) if any element of G generates (resp. any two elements of G generate) an associative subloop, i.e., a subgroup of G.

The following is evident:

PROPOSITION 1.9. *A power alternative loop is power associative, and a di-associative loop is left power alternative.*

As a corollary to Theorem 1.3, we have the following

PROPOSITION 1.10. *Assume that a left I.P. loop is left power alternative. Then, the transposed loop $G^{(a^p)}$ is isomorphic to G for any $a \in G$ and any integer p.*

PROPOSITION 1.11. *Let G be a homogeneous loop. Then the following three conditions are equivalent to each other*:

(1) *G is left power alternative.*

(2) $L_{x^n,x} = \mathrm{id}$, *for any $x \in G$ and any positive integer n.*

(3) $L_{x,x^n} = \mathrm{id}$, *for any $x \in G$ and any positive integer n.*

PROOF. From the definition (1.10), it is clear that (1) implies (2) and (3).

Assume (2), which is equivalent to $(x^n x)z = x^n(xz)$ $(x, z \in G, n > 0)$. Then we see easily by induction that $x^n x^m = x^{n+m}$ and $(x^{-1})^n = x^{-n}$ for any positive integers m, n. So we see also that

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3) In [10, II], we called G with this property *left di-associative in the strong sense.*

$$\mathrm{id} = (L_{x^n,x})^{-1} = L_{x^{-1},(x^{-1})^n}$$

for any $x \in G$ and $n>0$ by (1.7). Therefore we get (3).

Now assume (3). Then we see by induction

$$(*) \qquad x^{n+m}z = x^n(x^m z) \qquad (x, z \in G, n, m \geqq 0),$$

which shows

$$(**) \qquad L_{x^n,x^m} = \mathrm{id}, \qquad \text{for} \quad n, m \geqq 0.$$

For $n \geqq m \geqq 0$, since $x^{n-m} = x^{-m}(x^m x^{n-m}) = x^{-m}x^n$ by (1.4) and $(*)$, we have $L_{x^{-n},x^m} = L_{x^m,x^{n-m}} = \mathrm{id}$ by (1.6) and $(**)$. This and (1.7) show $L_{x^{-n},x^m} = L^{-1}_{x^{-m},x^n} = \mathrm{id}$ for $m \geqq n \geqq 0$. Therefore, $(**)$ holds also for $m \geqq 0 \geqq n$. Finally, we have $(**)$ for $m<0$ by these results and the first equality of (1.7), and so (1) is valid. *q.e.d.*

PROPOSITION 1.12. *Let G be a homogeneous loop, satisfying the condition*

$$(1.11) \qquad (xy)^{-1} = y^{-1}x^{-1}, \qquad \textit{for any} \quad x, y \in G.$$

Then

(1) *G has the* right inverse property, i.e., $(yx)x^{-1} = y$ *for any $x, y \in G$.*

(2) *$R_{x,y} = L_{y^{-1},x^{-1}}$ and thereby $R_{x,y}$ is an automorphism of G.*

(3) *If G is left alternative in addition, then G is left power alternative and hence power associative.*

PROOF. (1) and (2) are easily seen by (1.11) and the left inverse property.

(3): By (2), we see $R_{x,y}(x^{-1}) = y(xy)^{-1} = y(y^{-1}x^{-1}) = x^{-1}$ and so $R_{x,y}(x^n) = x^n$ for $n>0$. The latter equation is equivalent to $L_{x^n,x}(y) = y$, and hence G is left power alternative by the above proposition. *q.e.d.*

DEFINITION 1.7. A homogeneous loop G will be said to have the *symmetric property* if the *inverse mapping* $J: G \to G$, $J(x) = x^{-1}$, is an automorphism of G, that is, the identity

$$(1.12) \qquad (xy)^{-1} = x^{-1}y^{-1} \qquad (x, y \in G)$$

holds for G.

PROPOSITION 1.13. *Let G be a homogeneous loop. Then G has the symmetric property if and only if any one of the following identities holds for G $(x, y, z \in G)$:*

$$(1) \qquad x(y(yz^{-1})) = (xy)((xy)(xz)^{-1}).$$

$$(2) \qquad x(y(yz)) = (xy)((xy)(x^{-1}z)).$$

$$(3) \qquad L_{xy} \circ L_{xy} = L_x \circ L_y \circ L_y \circ L_x.$$

Moreover the following is valid if G has the symmetric property:

$$(4) \qquad L_{x^{-1},y^{-1}} = L_{x,y} \qquad (x, y \in G).$$

PROOF. It is clear that (1.12) is obtained by substituting x^{-1} for y in (1), or y^{-1} for z in (2). Also, (3) is a restatement of (2).

Assume that G has the symmetric property. By replacing x with x^{-1} in (1.8), we get

$$(*) \qquad x(xy^{-1}) = y(y^{-1}x)^2 .$$

Further, by substituting zx for x and zy for y in $(*)$, we get

$$(**) \qquad (zx)((zx)(zy)^{-1}) = (zy)((zy)^{-1}(zx))^2 .$$

Since $L_{z,y}(y^{-1}x)=(zy)^{-1}(zx)$ and $L_{z,y}$ is an automorphism, we have

$$((zy)^{-1}(zx))^2 = L_{z,y}((y^{-1}x)^2) = (zy)^{-1}(z(y(y^{-1}x)^2)).$$

Thus, by $(*)$ and $(**)$, (1) is shown as follows:

$$z(x(xy^{-1})) = (zy)((zy)^{-1}(zx))^2 = (zx)((zx)(zy)^{-1}).$$

(2) follows immediately from (1.12) and (1).

Also, (4) follows from (1.12) and (3). *q.e.d.*

REMARK 1.3. From (1.7) and the above proposition it follows easily that the symmetric property implies the identity $L_{x,x}\circ L_{x,x}=\mathrm{id}$. In §6 it will be shown that if G is a homogeneous Lie loop with the symmetric property, $L_{x,x}=\mathrm{id}$ holds, that is, G is left alternative.

Here, we shall give some examples of homogeneous loops.

EXAMPLE 1.1. A group G is a homogeneous loop with the trivial left inner mapping group $L_0(G)$.

EXAMPLE 1.2. An A-loop G (cf. Remark 1.2) is a homogeneous loop, if G has the (left and right) inverse property, or equivalently, if G is di-associative (cf. [2, Theorem 3.1]).

EXAMPLE 1.3. A loop G is a *Moufang loop* if G satisfies the following condition for any $x, y, z \in G$:

$$(1.13) \qquad x(y(xz)) = ((xy)x)z .$$

Then a Moufang loop has the inverse property, and is an A-loop if it is commutative. (Cf., e.g., [1, VII Lemmas 3.1, 3.3].) Therefore, a commutative Moufang loop is a homogeneous loop.

EXAMPLE 1.4. In general, Moufang loops are not always (left) A-loops. For instance, the sphere $S^7=\{z\in\mathfrak{C};\ |z|=1\}$ in the Cayley number system $\mathfrak{C}$ is a Moufang loop with the multiplication of $\mathfrak{C}$. But it is not a homogeneous loop. Nevertheless, S^7 contains a non-trivial homogeneous subloop $G=\{\pm e_i;\ i=0,\dots,7\}$ consisting of all generators of $\mathfrak{C}$ and their inverses. The left inner mappings act on G by $L_{e_i,e_j}(e_k)=e_k$ or $-e_k$ according as $e_ie_j=\pm e_k$ or not.

EXAMPLE 1.5. In [8, II Theorems 3, 5] we have proved the following: *Let G be a subset of a group A satisfying the conditions*

(1) *$e\in G$ and $G^{-1}=G$, where e is the identity of A.*

(2) *If $x, y\in G$, then $xyx\in G$.*

(3) *Any element $x\in G$ has a unique square root $x^{1/2}$ in G. Then G is a homogeneous loop with the symmetric property, under the multiplication*

$$\mu(x, y) = x^{1/2}yx^{1/2} \qquad (x, y\in G). \tag{1.14}$$

Let P_n denote the set of all positive definite symmetric real $n\times n$ matrices. Then P_n satisfies the above conditions (1)–(3) as a subset of the group of all non-singular $n\times n$ matrices and so P_n is a homogeneous loop with the symmetric property under the multiplication $\mu(X, Y)=X^{1/2}YX^{1/2}$ for $X, Y\in P_n$.

Also, the set H_n of all positive definite Hermitian matrices is a homogeneous loop under the multiplication as above.

REMARK 1.4. More generally, it can be shown that, if G is a Moufang loop in which each element has a unique square root, the loop $G(1/2)=(G,\mu)$ of [1, VII Theorem 5.2] is a homogeneous loop with the symmetric property, where μ is defined by (1.14).

§2. Semi-direct products

In this section we shall study the semi-direct product of a homogeneous loop G by a subgroup K of the automorphism group $AUT(G)$ of G.

DEFINITION 2.1. Let G be a homogeneous loop and K be a transformation group of G such that $L_0(G)\subset K\subset AUT(G)$, where $L_0(G)$ is the left inner mapping group of Definition 1.2. The *semi-direct product* $G\times K$ of G by K is the Cartesian product $G\times K$ together with the binary operation

$$(x, \alpha)(y, \beta) = (x\alpha(y), L_{x,\alpha(y)}\circ\alpha\circ\beta) \tag{2.1}$$

for $(x,\alpha), (y,\beta)\in G\times K$, where $L_{x,\alpha(y)}\in L_0(G)$ is the left inner mapping of (1.2).

THEOREM 2.1. *The semi-direct product* $G\times K$ *of the above definition is a group with the identity* (e, id) *(*e *is the identity of* G*) and the inverse*

$$(2.2)\qquad (x,\alpha)^{-1}=(\alpha^{-1}(x^{-1}),\alpha^{-1})\qquad for\quad (x,\alpha)\in G\times K.$$

Moreover, by using the left translation L_x *of* (1.1), *we have an isomorphism*

$$(2.3)\qquad \rho\colon G\times K\longrightarrow AUT(\mathscr{H}_G),\quad \rho(x,\alpha)=L_x\circ\alpha$$

for $(x,\alpha)\in G\times K$, *of* $G\times K$ *into the automorphism group* $AUT(\mathscr{H}_G)$ *of the homogeneous structure* $\mathscr{H}_G$ *of* G *of* Corollary 1.7.

PROOF. From Corollary 1.7 it follows that $\rho(x,\alpha)$ is an automorphism of $\mathscr{H}_G$. Moreover, for any $x, y\in G$ and $\alpha, \beta\in K$, the equality $L_x\circ\alpha=L_y\circ\beta$ implies $x=L_x(e)=L_x(\alpha(e))=L_y(\beta(e))=y$ and so $\alpha=\beta$. Therefore ρ is injective.
Also

$$\begin{aligned}\rho(x,\alpha)\circ\rho(y,\beta)&=L_x\circ\alpha\circ L_y\circ\beta=L_x\circ L_{\alpha(y)}\circ\alpha\circ\beta\\&=L_{x\alpha(y)}\circ L_{x,\alpha(y)}\circ\alpha\circ\beta=\rho((x,\alpha)(y,\beta)).\end{aligned}$$

Thus ρ preserves the multiplication, and so $G\times K$ is associative. It is easily seen that (e, id) is the identity and the inverse is given by (2.2). *q.e.d.*

The automorphism $L_x\circ\alpha=\rho(x,\alpha)$ of $\mathscr{H}_G$ is called the *representation* of $(x,\alpha)\in G\times K$. Then the semi-direct product $G\times K$ can be identified with the subgroup $\rho(G\times K)$ of $AUT(\mathscr{H}_G)$ by the above theorem. In particular, by Corollary 1.7, we have the following

COROLLARY 2.2. $\rho\colon G\times AUT(G)\to AUT(\mathscr{H}_G)$ *is an onto isomorphism. Therefore, the semi-direct product* $G\times AUT(G)$ *can be identified with* $AUT(\mathscr{H}_G)$ *and* $AUT(G)$ *is the isotropy subgroup of* $AUT(\mathscr{H}_G)$ *at* $e\in G$.

The following lemma is also proved easily by definition.

LEMMA 2.3. *The notations being the same as in* Theorem 2.1;

(1) *The group* K *is the representation group of the subgroup* $e\times K$ *of* $G\times K$, *and we can identify* $K=e\times K$.

(2) $(x,\alpha)=(x,\mathrm{id})(e,\alpha)$, $(e,\alpha)(x,\mathrm{id})(e,\alpha)^{-1}=(\alpha(x),\mathrm{id})$ *for any* $x\in G$ *and* $\alpha\in K$.

(3) $G\times K=(G_{\mathrm{id}})K$ *(uniquely factored), where* $G_{\mathrm{id}}=G\times\mathrm{id}$.

(4) G_{id} *is an* $\mathrm{ad}(K)$*-invariant subset of* $G\times K$.

Here we give the definitions of normal subloops and quotient loops.

DEFINITION 2.2. A subloop H of a loop G is *normal* if H is invariant under the inner mapping group $I(G)$ of Remark 1.1, i.e., if the followings are valid for any $x, y \in G$:

$$(1)\ x(yH) = (xy)H, \quad (2)\ (Hx)y = H(xy), \quad (3)\ Hx = xH.$$

Then the *quotient loop* G/H of G modulo H can be defined naturally by

$$G/H = \{xH; x \in G\}, \qquad (xH)(yH) = (xy)H,$$

and the projection $p: G \to G/H$, $p(x) = xH$, is an onto homomorphism of loops (cf. [1, IV]).

THEOREM 2.4. *Let H be a subloop of a homogeneous loop G. Then*

(1) *H is a homogeneous loop.*

(2) *If H is a normal subloop of G, the quotient loop G/H of G modulo H is also a homogeneous loop.*

PROOF. It is easily seen that x^{-1} is contained in H if $x \in H$. Hence (1) is clear.

Since the projection $p: G \to G/H$ is an onto homomorphism, (2) is obtained also immediately. *q.e.d.*

THEOREM 2.5. *Let H be a normal subloop of a homogeneous loop G and suppose that a transformation group K, $L_0(G) \subset K \subset AUT(G)$, leaves H invariant. Then:*

(1) *The subset $H \times K$ is a subgroup of the semi-direct product $G \times K$.*

(2) *Any left coset of $H \times K$ is $(x, \alpha)(H \times K) = xH \times K$ $(x \in G, \alpha \in K)$.*

(3) *The factor set $G \times K/H \times K$ by the left coset decomposition of $G \times K$ modulo $H \times K$, with the multiplication*

$$(2.4) \qquad (xH \times K)\cdot(yH \times K) = (xy)H \times K \qquad (x, y \in G),$$

is a homogeneous loop isomorphic to the quotient loop G/H.

PROOF. (1), (2): These are clear by (2.1), (2.2) and the assumption for H.

(3): By (2), the mapping $j: G/H \to G \times K/H \times K$, $j(xH) = xH \times K$, is a bijection. Therefore we have (3) by the definition of the quotient loop and Theorem 3.4. *q.e.d.*

REMARK 2.1. In the above theorem the kernel K_1 of the restriction homomorphism $r: K \to AUT(H)$, $r(\alpha) = \alpha|_H$ $(\alpha \in K)$, is a normal subgroup of $H \times K$, and $H \times K/K_1$ is isomorphic to the semi-direct product $H \times K_2$, where K_2 is the image of r. In fact, the mapping r induces a homomorphism $(h, \alpha) \to (h, \alpha|_H)$ of the group $H \times K$ onto the semi-direct product $H \times K_2$, since $L_0(H) \subset K_2 \subset AUT(H)$ follows from $L_0(G) \subset K \subset AUT(G)$.

COROLLARY 2.6. *The factor set $G\times K/K$ of left cosets of K, with the multiplication*

$$(2.5) \qquad (x\times K)\cdot(y\times K) = (xy)\times K \qquad (x, y\in G),$$

is a homogeneous loop isomorphic to G itself.

By the above corollary and Corollary 2.2, we get

COROLLARY 2.7. *Any homogeneous loop G can be identified with the set $AUT(\mathscr{H}_G)/AUT(G)$ of all cosets of $AUT(G)$ in the automorphism group $AUT(\mathscr{H}_G)$ of the homogeneous structure of G, under the mapping $\bar{j}$: $G\to AUT(\mathscr{H}_G)/AUT(G)$, $\bar{j}(x)=L_x\circ AUT(G)$, $x\in G$.*

THEOREM 2.8. *Let H be a normal subloop of a homogeneous loop G, and consider the subgroup*

$$L_0(G, H) = \{\alpha\in L_0(G); \alpha(xH) = xH \textit{ for any } x\in G\}\subset L_0(G).$$

Then (1) *$L_0(G, H)$ is a normal subgroup of $L_0(G)$ and*

$$L_0(G/H)\cong L_0(G)/L_0(G, H).$$

(2) *The subset $H\times L_0(G, H)$ is a normal subgroup of the semi-direct product $G\times L_0(G)$ and*

$$G/H\times L_0(G/H)\cong G\times L_0(G)/H\times L_0(G, H).$$

PROOF. (1): Any $\alpha\in L_0(G)$ is an automorphism of G and it leaves H invariant. Hence α induces the automorphism α' of G/H by $\alpha'(xH)=(\alpha x)H$, and we obtain the homomorphism

$$f\colon L_0(G)\longrightarrow AUT(G/H), \qquad f(\alpha)=\alpha'.$$

Since $\alpha'=L_{xH,yH}\in L_0(G/H)$ for $\alpha=L_{x,y}$, we see that $\mathrm{Im}(f)=L_0(G/H)$. Also, $\mathrm{Ker}(f)=L_0(G, H)$ by definition. Thus (1) is proved.

(2) Let p: $G\to G/H$ be the natural projection. We see easily by definition that the onto-mapping

$$p\times f\colon G\times L_0(G)\longrightarrow G/H\times L_0(G/H)$$

is a homomorphism of the semi-direct products and $\mathrm{Ker}(p\times f)=H\times L_0(G, H)$. Thus (2) is proved. *q.e.d.*

§3. Homogeneous Lie loops

In this section, we shall consider a homogeneous Lie loop, a homogeneous loop admitting a natural differentiable structure, and assert that such a loop is a reductive homogeneous space due to K. Nomizu [15]. For well known terminologies and results on differentiable manifolds with linear connections, we refer to [9].

DEFINITION 3.1. A *homogeneous Lie loop* G is a homogeneous loop of Definition 1.4 and also a (C^∞-) differentiable[4] manifold such that the loop multiplication μ: $G\times G\to G$ is differentiable.

REMARK 3.1. By A. I. Mal'cev [14], the tangent algebras of some analytic loops have been treated.

EXAMPLE 3.1. A Lie group G is a homogeneous Lie loop.

EXAMPLE 3.2. The set P_n of all positive definite real symmetric $n\times n$ matrices in Example 1.5 is a homogeneous Lie loop. In fact, the multiplication $\mu(X, Y)=X^{1/2}YX^{1/2}$ is differentiable with respect to the natural differentiable structure on P_n. (Cf. [10].)

In the same way, the manifold of all positive definite Hermitian matrices is a homogeneous Lie loop. These are examples of homogeneous Lie loops with the symmetric property, that is, symmetric Lie loops (cf. § 6).

PROPOSITION 3.1. *Let G be a homogeneous Lie loop. Then the inverse mapping J: $G\to G$, $J(x)=x^{-1}$ ($x\in G$), is a diffeomorphism of G.*

PROOF. Choose a local coordinates $(u^1, u^2,\dots, u^n)$ ($n=\dim G$) centered at the identity e with a domain U. Then there exists a neighborhood V of e such that $\mu(V\times V)\subset U$. Since $\mu(x, e)=x$ for $x\in U$, it follows that

$$\frac{\partial\mu^i}{\partial x^j}(0, 0)=\delta^i_j \qquad (i, j=1, 2,\dots, n).$$

Thus the implicit function theorem shows that there exists a neighborhood W of e such that J is differentiable on W.

Now, for any fixed element $a\in G$, consider the neighborhood $W_a=L_a(W)$ of a, where L_a denotes the left translation of G. Since $J(x)=a(w(w^{-1}a^{-1})^2)$ (Lemma 1.10), for any $x=aw\in W_a$, we see that J is differentiable at a by the above result and the differentiability of μ. *q.e.d.*

REMARK 3.2. By using Proposition 3.1, we can show that every connected homogeneous Lie loop is generated by any neighborhood of its identity. In fact it is proved by a method similar to that for connected Lie groups.

Let G be a homogeneous Lie loop and consider the module $\mathfrak{X}(G)$ of all

4) In the rest of this paper, the differentiability is always assumed to be of class C^∞.

differentiable vector fields on G over the real algebra $\mathfrak{F}(G)$ of all differentiable real valued functions on G. Denote by $L_x^{(a)}$ the left translation in the transposed loop $G^{(a)}$ of G centered at a (cf. Lemma 1.4). For any two vector fields $X, Y \in \mathfrak{X}(G)$, we construct the vector field $\nabla_X Y$ by

$$(\nabla_X Y)_a = \lim_{t\to\infty} \frac{1}{t}[(dL_{x(t)}^{(a)})^{-1}(Y_{x(t)}) - Y_a], \text{ [5]} \tag{3.1}$$

for $a \in G$, where $x(t)$ is an integral curve of X through $a = x(0)$. Then the following lemma is seen easily:

LEMMA 3.2. *The assignment* $(X, Y) \to \nabla_X Y$ *defines a linear connection* ∇ *on* G, *that is,* ∇_X *is linear and satisfies*

$$\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z\,,$$

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y, \quad (X, Y, Z \in \mathfrak{X}(G) \text{ and } f, g \in \mathfrak{F}(G))\,.$$

DEFINITION 3.2. The linear connection ∇ on G defined by (3.1) will be called the *canonical connection* of the homogeneous Lie loop G.

In this section, we consider a homogeneous Lie loop $G = (G, \nabla)$ together with the canonical connection ∇.

THEOREM 3.3. *Any left translation* L_x $(x \in G)$ *of a homogeneous Lie loop* G *is an affine transformation of the canonical connection* ∇. *Therefore,* ∇ *is invariant under the left translation group* $L(G)$ (Definition 1.2) *of* G.

PROOF. As was shown in the proof of Theorem 1.3, we have

$$L_x(g \overset{(a)}{\cdot} y) = (xg) \overset{(xa)}{\cdot} (xy)\,,$$

where $\overset{(b)}{\cdot}$ denotes the multiplication of the transposed loop $G^{(b)}$. This is equivalent to

$$\phi \circ L_g^{(a)} = L_{\phi(g)}^{(\phi(a))} \circ \phi \qquad (\phi = L_x)\,. \tag{$*$}$$

Then, we see by the definition (3.1) that

$$\begin{aligned} d\phi(\nabla_X Y)_a &= \lim_{t\to 0} \frac{1}{t}\, d\phi[(dL_{g(t)}^{(a)})^{-1}(Y_{g(t)}) - Y_a] \\ &= \lim_{t\to 0} \frac{1}{t}[(dL_{\phi(g(t))}^{(\phi(a))})^{-1}(d\phi(Y)_{\phi(g(t))}) - d\phi(Y)_{\phi(a)}]\,. \end{aligned} \tag{$**$}$$

Here, $g(t)$ is an integral curve through $a = g(0)$ of the vector field X, and so $\phi(g(t))$

5) For a differentiable map ϕ, we denote by $d\phi$ the induced linear mapping of tangent vectors.

is an integral curve of $d\phi(X)$. Therefore, by the definition (3.1) of ∇, (**) implies

$$d\phi(\nabla_X Y) = \nabla_{d\phi(X)} d\phi(Y). \tag{3.2}$$

Hence ϕ is an affine transformation. *q.e.d.*

THEOREM 3.4. *The differentiable automorphism group $Aut(G)$ of a homogeneous Lie loop G is a subgroup of the affine transformation group $Aff(G)$.*

PROOF. Let $\phi \in Aut(G)$. Then we have (*) in the above proof, by applying ϕ to (1.5). Thus we see that ϕ satisfies (3.2) by the above proof. *q.e.d.*

Now, let G be a connected homogeneous Lie loop. Then, since any left inner mapping $L_{x,y} = L_{xy}^{-1} \circ L_x \circ L_y$ $(x, y \in G)$ is a diffeomorphism of G, the left inner mapping group $L_0(G)$ (Definition 1.2) is a subgroup of $Aut(G)$ by Definition 1.4, the latter being a subgroup of $Aff(G)$ by the above theorem. Since $Aff(G)$ acts on G as a Lie transformation group (cf., e.g., [9, IV Theorem 1.5]), it is seen that $Aut(G)$ is a closed subgroup of $Aff(G)$, and we can consider the set

$$K(G) = \text{the closure of } L_0(G) \text{ in } Aut(G), \tag{3.3}$$

which is also a Lie transformation group of G. Moreover, $L_0(G)$ is connected since G is supposed to be so, and consequently the group $K(G)$ is a connected Lie group satisfying $L_0(G) \subset K(G) \subset Aut(G) \subset AUT(G)$.

Furthermore, we can consider the semi-direct product

$$A(G) = G \times K(G) \tag{3.4}$$

of Definition 2.1, which is a group by Theorem 2.1. $A(G)$ is also a connected Lie group with the product manifold structure by definition. Then we have the following theorem by Lemma 2.3 and Corollary 2.6.

THEOREM 3.5. *For a connected homogeneous Lie loop G, the connected Lie group $A(G)$ of* (3.4) *contains $K(G) = e \times K(G)$ as a closed subgroup, and the mapping*

$$i\colon G \longrightarrow A(G)/K(G), \quad i(x) = x \times K(G) \qquad (x \in G), \tag{3.5}$$

is a loop-isomorphism onto the homogeneous space $A(G)/K(G)$ with the multiplication (2.5).

Also, we can show that $A(G)/K(G)$ is a reductive homogeneous space, defined by K. Nomizu [15] as follows:

Let $M = A/K$ be a homogeneous space of a connected Lie group A by a closed subgroup K.

DEFINITION 3.3. A homogeneous space $M=A/K$ is *reductive*[6] if and only if A acts effectively on M and the Lie algebra $\mathfrak{A}$ of A is decomposed into a direct sum such as

$$\mathfrak{A} = \mathfrak{M}+\mathfrak{K} \text{ (direct sum)}, \qquad \mathrm{ad}(K)\mathfrak{M} \subset \mathfrak{M}, \tag{3.6}$$

where $\mathfrak{K}$ is the Lie algebra of K and $\mathfrak{M}$ is a subspace of $\mathfrak{A}$.

With respect to the direct sum decomposition (3.6), we see that the subspace $\mathfrak{M}$ can be identified with the tangent space $\mathfrak{M}_e$ to M at the origin $e=\pi(K)$ under the induced linear map $d\pi$ of the natural projection $\pi\colon A\to M=A/K$. In the Lie group A, we can find a neighborhood V of the identity $1\in A$, a connected submanifold $N\subset\exp\mathfrak{M}$ and a connected submanifold $H\subset K$ such that the mapping $(a, h)\to ah$ $(a\in N, h\in H)$ gives a diffeomorphism of $N\times H$ onto V (cf., e.g., [4, II Lemma 2.4]). For $X\in\mathfrak{M}=\mathfrak{M}_e$, we put a vector field X^* on $\pi(N)=N^*$ as

$$X^*_{\pi(a)} = dt_a(X) \qquad (a\in N), \tag{3.7}$$

where t_a is the natural action of a on M.

THEOREM 3.6. [15, Theorem 8.1] *Let $M=A/K$ be a reductive homogeneous space with a fixed decomposition* (3.6). *Then there exists a one-to-one correspondence between the set of all A-invariant linear connections ∇ on M and the set of all bilinear functions $\Lambda\colon\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}$ satisfying*

$$\mathrm{ad}(k)\Lambda(X, Y) = \Lambda(\mathrm{ad}(k)X, \mathrm{ad}(k)Y) \qquad (X, Y\in\mathfrak{M}, k\in K). \tag{3.8}$$

The correspondence is given by

$$\Lambda(X, Y) = (\nabla_{X^*}Y^*)_e \qquad (X, Y\in\mathfrak{M}), \tag{3.9}$$

where X^* and Y^* are vector fields defined as (3.7).

DEFINITION 3.4. By the above theorem, there corresponds to $\Lambda=0$ an A-invariant connection ∇ on $M=A/K$. This connection is called the *canonical connection* of the reductive homogeneous space M.

Now we prove the following:

THEOREM 3.7. *In* Theorem 3.5, *the homogeneous space $A(G)/K(G)$ is reductive and the isomorphism $i\colon G\to A(G)/K(G)$ of* (3.5) *is an affine isomorphism with respect to the canonical connections of* Definitions 3.2 *and* 3.4.

PROOF. Since the action of any element $a=(x, \alpha)\in A(G)$ on G is the same as its representation $L_x\circ\alpha$, $A(G)$ acts effectively on G. By the product manifold

6) We note that, in the original definition of the reductive homogeneous space of [15], the action of A on M is not assumed to be effective. See also [9, X p. 198 Remark].

structure, the Lie algebra $\mathfrak{A}$ of $A(G)=G\times K(G)$ is decomposed into the direct sum

$$\mathfrak{A} = \mathfrak{G}+\mathfrak{K}, \tag{3.10}$$

where $\mathfrak{G}=\mathfrak{G}_e$ is the tangent space to the submanifold $G_{\mathrm{id}}=G\times\mathrm{id}$ at e and $\mathfrak{K}$ is the Lie algebra of $K(G)$. By Lemma 2.3 (4), the submanifold G_{id} is invariant by $\mathrm{ad}(K(G))$ acting on $A(G)$. Hence we have

$$\mathrm{ad}(K(G))\mathfrak{G} \subset \mathfrak{G} \quad \text{in } \mathfrak{A}. \tag{3.11}$$

Thus $A(G)/K(G)$ is a reductive homogeneous space by (3.10) and (3.11). It is clear that i: $G\to A(G)/K(G)$ of (3.5) is a diffeomorphism, which induces a linear connection ∇' on $A(G)/K(G)$ from the canonical connection ∇ of G. For each $a=(x,\alpha)\in A(G)$, $L_x\circ\alpha$ is an affine transformation by Theorems 3.3 and 3.4. Hence, $t_a=i\circ L_x\circ\alpha\circ i^{-1}$ is an affine transformation of ∇', that is, ∇' is an $A(G)$-invariant connection on $A(G)/K(G)$.

Now, we choose a neighborhood V of $1=(e,\mathrm{id})\in A(G)$ and submanifolds N, H such as $1\in N\subset\exp\mathfrak{G}$, $1\in H\subset K(G)$ and $N\times H\to V$, $(n,h)\to nh$, is a diffeomorphism. For X_e, $Y_e\in\mathfrak{G}$ we define the vector fields X^* and Y^* on $N^*=\pi(N)$ as (3.7). Then we have

$$(\nabla'_{X^*}Y^*)_{\pi(1)} = (\nabla'_{di(\tilde{X})}di(\tilde{Y}))_{\pi(1)}; \tag{3.12}$$

where $\tilde{X}$ and $\tilde{Y}$ are the vector fields on G defined by

$$\tilde{X}_x = dL_x(X_e),\quad \tilde{Y}_x = dL_x(Y_e) \qquad (x\in G). \tag{3.13}$$

In fact, if we consider a differentiable curve $\exp tX=(x(t),\alpha(t))=a(t)$ in N, we have $a(0)=1$, $(d\alpha/dt)_0=0$ and $Y^*_{\pi(a(t))}=di\circ dL_{x(t)}\circ d\alpha(t)(Y_e)$. So, for any connection on N^*, the covariant derivative of Y^* and $di(\tilde{Y})$ in the direction of $di(X_e)$ at $\pi(1)$ must be coincident. Thus by the definition (3.1) of the canonical connection ∇ of G we see $(\nabla_{\tilde{X}}\tilde{Y})_e=0$ and so the right hand side of (3.12) must be equal to zero. Therefore ∇' is the canonical connection of the reductive homogeneous space $A(G)/K(G)$ by Theorem 3.6 and Definition 3.4. *q.e.d.*

Hereafter, we denote by $G=A(G)/K(G)$ the connected homogeneous Lie loop G identified with the reductive homogeneous space $A(G)/K(G)$ together with their canonical connections under the natural isomorphism i: $G\to A(G)/K(G)$. The direct sum decomposition (3.10) of the Lie algebra $\mathfrak{A}$ of $A(G)$ will be called the *canonical decomposition of* $\mathfrak{A}$.

Here we recall some results given in [15], concerning reductive homogeneous spaces. These are valid for any connected homogeneous Lie loop $G=A(G)/K(G)$ by Theorem 3.7.

THEOREM 3.8. [15, Theorem 10.2] *The canonical connection of a reductive homogeneous space $G=A/K$, with the decomposition $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$ of the Lie algebra of A, is the unique A-invariant connection which has the following property:*

For every $X\in\mathfrak{G}$, let $a(t)=\exp tX$ be the 1-parameter subgroup of A generated by X and let $x(t)=\pi(a(t))$ be the curve through $x(0)=e=\pi(1)$. Then the parallel displacement of vectors along the curve $x(t)$ is the same as the translation by the natural action of $a(t)$ on G.

THEOREM 3.9. [15, Theorem 10.3] *In* Theorem 3.8, *denote by S and R the torsion and curvature tensors*[7] *of the canonical connection, respectively. Then*

(1) ∇ *is* locally reductive, *that is,* $\nabla S=0$ *and* $\nabla R=0$.

(2) $S_e(X, Y)=[X, Y]_{\mathfrak{G}}$ *for* $X, Y\in\mathfrak{G}$.

(3) $R_e(X, Y)Z=[[X, Y]_{\mathfrak{K}}, Z]$ *for* $X, Y, Z\in\mathfrak{G}$.

(4) [9, X Corollary 2.5 (3)] ∇ *is complete.*

Here, $[\ ,\ ]_{\mathfrak{G}}$ *(resp.* $[\ ,\ ]_{\mathfrak{K}}$*) denotes the $\mathfrak{G}$-component (resp. $\mathfrak{K}$-component) of the Lie bracket* $[\ ,\ ]$ *in* $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$.

EXAMPLE 3.3. If G is reduced to a connected Lie group then $K(G)=\{\mathrm{id}\}$ and $A(G)=G$. In this case the canonical connection of G is reduced to the $(-)$-connection of É. Cartan [3, § 1].

As was seen in the proof of Theorem 3.7, the action of $A(G)=G\times K(G)$ on G is the same as that of its representation group $\rho(A(G))=\{L_x\circ\alpha;\ (x,\alpha)\in A(G)\}$ of Theorem 2.1. The fact similar to this is valid also for the semi-direct product $G\times AUT(G)$. By Corollary 2.2 and Theorems 3.3, 3.4, its representation group is the differentiable automorphism group $Aut(\mathscr{H}_G)$ of the homogeneous structure $\mathscr{H}_G$ of G, which is a subgroup of $Aff(G)$. By definition, $\mathscr{H}_G$ assigns to each $a\in G$ the transposed loop $G^{(a)}$ (cf. Definition 1.5 and Theorem 1.7).

THEOREM 3.10. *Let G be a connected homogeneous Lie loop. The (differentiable) automorphism group $Aut(\mathscr{H}_G)$ of the homogeneous structure $\mathscr{H}_G$ of G is a subgroup of the affine transformation group $Aff(G)$ of the canonical connection of G. The automorphism group $Aut(G)$ (resp. $Aut(G^{(a)})$) is the isotropy subgroup of $Aut(\mathscr{H}_G)$ at the identity e (resp. at $a\in G$).*

Moreover, G can be regarded as the reductive homogeneous space $Aut(\mathscr{H}_G)/Aut(G)$.

PROOF. These are proved in the same way as that of the proof of Theorem 3.7, by substituting $K(G)$ with $Aut(G)$. *q.e.d.*

7) In this paper, we adopt the signs of S and R opposite to the usual ones, that is, we define

$$S(X, Y)=[X, Y]-\nabla_X Y+\nabla_Y X,\quad R(X,Y)=\nabla_{[X,Y]}-[\nabla_X, \nabla_Y]$$

for $X, Y\in\mathfrak{X}(G)$.

§4. Geodesic local Lie loops

In this section, we shall give a geometric example of homogeneous Lie loop in local. By means of the parallel displacements of geodesics along geodesics, we define a local multiplication for a differentiable manifold with a linear connection, and show that it satisfies the conditions of the homogeneous loop in local, for a locally reductive space.

Let G be a connected differentiable manifold with a given linear connection ∇. For a differentiable curve $\gamma: t \to \gamma(t)$ $(a<t<b)$ in G, by using the parallel displacement $\tau_{t,s}: \mathfrak{G}_{\gamma(t)} \to \mathfrak{G}_{\gamma(s)}$ of tangent vectors[8] along γ and the exponential mapping $\mathrm{Exp}_{\gamma(t)}$ with respect to the connection, we can define the diffeomorphism

$$\phi_{t,s}(\gamma) = \mathrm{Exp}_{\gamma(s)} \circ \tau_{t,s} \circ (\mathrm{Exp}_{\gamma(t)})^{-1} \tag{4.1}$$

of a normal neighborhood $U_{\gamma(t)}$ of $\gamma(t)$ onto a normal neighborhood $U_{\gamma(s)}$ of $\gamma(s)$. We call $\phi_{t,s}(\gamma)$ the *parallel displacement of geodesics along the curve* γ, since it sends each geodesic through $\gamma(t)$ in $U_{\gamma(t)}$ to a geodesic through $\gamma(s)$ in $U_{\gamma(s)}$.

Let U be a normal neighborhood of a fixed point $e \in G$, such that U is a normal neighborhood of each point of U. For the existence of such a neighborhood (said to be *simple and convex*), see, e.g., [4, I Theorem 6.2]. For any $x \in U$, choose the unique geodesic γ in U such that $\gamma(0)=e$, $\gamma(1)=x$, and set

$$\mu(x, y) = \phi_{0,1}(\gamma)(y) \quad \text{for } y \text{ contained in a domain of } \phi_{0,1}(\gamma), \tag{4.2}$$

where $\phi_{0,1}(\gamma)$ is the parallel displacement along γ of (4.1). Then

(4.2.1) μ is a local multiplication defined on a (nonempty) open subset of $U \times U$ containing (e, e) with its values in U, and μ is differentiable on a neighborhood of (e, e).

(4.2.2) For any $x \in U$, both of $\mu(e, x)$ and $\mu(x, e)$ are defined and equal to x, that is, e is the *identity*.

(4.2.3) For $x \in U$, the *left translation* $L_x: y \to \mu(x, y)$ is a local diffeomorphism of a neighborhood of e onto a neighborhood of x, and the *right translation* $R_x: y \to \mu(y, x)$ is also so.

The differentiability of μ and (4.2.3) for R_x have been shown in [5, Theorem 1].[9]

DEFINITION 4.1. A pair (U, μ) defined as above will be called a *geodesic local Lie loop* of G at e.

In general, we give the following

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8) $\mathfrak{G}_x$ denotes the tangent space to G at x.

9) In [5], $\mu(x, y)$ of (4.2) is denoted by $f_e(y, x)$.

DEFINITION 4.2. Let M be a differentiable manifold, and U be a connected neighborhood of a fixed point $e \in M$. If a local multiplication μ satisfies (4.2.1)–(4.2.3), then (U, μ) will be called a *local Lie loop*.

Moreover, if the following conditions (4.2.4) and (4.2.5) are satisfied, the local Lie loop (U, μ) will be said to be *homogeneous*:

(4.2.4) There exists a neighborhood V of e and, for $x \in V$, there exists an *inverse* $x^{-1} \in U$ such that $\mu(x^{-1}, x) = \mu(x, x^{-1}) = e$ and $L_{x^{-1}} \circ L_x$ induces an identity map on a neighborhood of e.

(4.2.5) If $\mu(x, y) = xy$ and its inverse $(xy)^{-1}$ are all defined, then the *left inner mapping*

$$L_{x,y} = L_{(xy)^{-1}} \circ L_x \circ L_y$$

induces a local diffeomorphism commuting with μ, i.e., $L_{x,y} \circ \mu = \mu \circ (L_{x,y} \times L_{x,y})$, in a neighborhood of (e, e).

In the sequel, we consider the geodesic local Lie loops in a locally reductive space (cf. Theorem 3.9 (1)).

PROPOSITION 4.1. *Let G be a differentiable manifold with a linear connection ∇. Then the following conditions* (1)–(3) *are mutually equivalent*:

(1) *∇ is locally reductive*, i.e., *$\nabla S = 0$ and $\nabla R = 0$.*

(2) *The parallel displacement* (4.1) *of geodesics along any differentiable curve in G induces a local affine transformation.*

(3) *The parallel displacement of geodesics along any geodesic induces a local affine transformation.*

PROOF. (1)$\Leftrightarrow$(2) is found in [9, VI Corollary 7.6]. (2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(1): Let S and R denote the torsion and curvature tensors of ∇, respectively. At any point $e \in G$, we take a normal neighborhood U of e. For any tangent vectors $X, Y, Z \in \mathfrak{G}_e$ at e, denote by X^*, Y^* and Z^* the vector fields on U given by the parallel displacement of X, Y, Z, respectively, along the geodesic joining e to each point in U. Let $x(t)$ be the geodesic tangent to X at $e = x(0)$. Since the parallel displacement of geodesics along $x(t)$ is a local affine transformation, $S(t) = S_{x(t)}(Y^*_{x(t)}, Z^*_{x(t)})$ is constant on the geodesic $x(t)$. Hence, we have

$$\begin{aligned}
(\nabla_X S)_e(Y, Z) &= ((\nabla_{X^*} S)(Y^*, Z^*))_e \\
&= [(\nabla_{X^*}(S(Y^*, Z^*))]_e - S_e((\nabla_{X^*} Y^*)_e, Z) \\
&\quad - S_e(Y, (\nabla_{X^*} Z^*)_e) \\
&= 0.
\end{aligned}$$

$(\nabla_X R)_e = 0$ is shown in the same way. *q.e.d.*

As an immediate consequence of the above proposition, we have

LEMMA 4.2. *Let (U, μ) be a geodesic local Lie loop of a locally reductive space. Then any left translation L_x $(x \in U)$ is a local affine transformation.*

Now we prove the following

THEOREM 4.3. *Let G be a locally reductive space. Then any geodesic local Lie loop of G is homogeneous.*

PROOF. Let (U, μ) be a geodesic local Lie loop at $e \in G$, where U is a simple and convex normal neighborhood of e. For any $x = \mathrm{Exp}_e X$ $(X \in \mathfrak{G}_e)$ of U, put $x^{-1} = \mathrm{Exp}_e(-X)$. Let $x(t)$ be the geodesic in U such as $x(0) = e$ and $x(1) = x$. Then, by definition, $x(t)^{-1} = x(-t)$ whenever $x(-t)$ is defined in U. Suppose that x^{-1} and xy belong to U for $x, y \in U$ and consider a geodesic triangle in U constructed by the geodesic segments γ_1, γ_2 and γ_3 joining e to x, x to xy and xy to e, respectively. Then γ_2 is the L_x-image of the geodesic segment joining $e = y(0)$ to $y = y(1)$. Since $L_{x^{-1}}$ is a parallel displacement of geodesics along the geodesic $x(t)$ by the definition (4.2), it sends every geodesic joining e to $\gamma_2(s)$ to a geodesic through x^{-1}, and, in particular, it sends $x(t)$ to $x(t-1)$. So we have $L_{x^{-1}}(x) = e$. By Lemma 4.2, $L_{x^{-1}}$ is a local affine transformation at each point for which it is defined, and so the geodesic segment γ_2 must be mapped onto a geodesic through $e = L_{x^{-1}}(x)$, and the parallelism of vectors along the geodesic $x(t)$ must be preserved by $L_{x^{-1}}$. From these facts it follows that $L_{x^{-1}}$-image of the geodesic segment γ_2 is the geodesic $y(s)$ from e to y, that is, $L_{x^{-1}}(L_x(y(s))) = y(s)$. Thus the condition (4.2.4) is shown.

The left inner mapping $L_{x,y} = L_{xy}^{-1} \circ L_x \circ L_y$ is also a local affine transformation at each point of its domain, if x, y and $(xy)^{-1}$ belongs to U. Hence it commutes with the local multiplication μ, for μ is defined by means of parallel displacements along geodesics. As was proved above, $L_{xy}^{-1}(xy) = e$ and so we have $L_{x,y}(e) = e$. Therefore, $L_{x,y}$ is a local automorphism of (U, μ) in a neighborhood of e. Thus we get (4.2.5). *q.e.d.*

PROPOSITION 4.4. *In any geodesic local Lie loop (U, μ) of a locally reductive space G, any geodesic $x(t)$ through the identity $e = x(0)$ is a local 1-parameter subgroup of (U, μ). Moreover, $L_{x(t)}$ is a local 1-parameter group of local transformations of G, that is, $L_{x(t)} \circ L_{x(s)} = L_{x(t+s)}$ whenever they are defined in U.*

Conversely, any local 1-parameter subgroup of (U, μ) is a geodesic of G.

PROOF. Let $x(t)$ be a geodesic through $e = x(0)$ in G. For a fixed value s such as $x(s) \in U$, $L_{x(s)}$ is a local affine transformation defined as the parallel displacement of geodesics along the geodesic $x(t)$ from e to $x(s)$ by (4.2). Hence, by the uniqueness of the geodesic tangent to $\dot{x}(s)$[10] at $x(s)$, $L_{x(s)}x(t)$ must coincide with $x(s+t)$ as long as they are contained in U. This shows the first assertion.

By the proof of Proposition 4.5, we have also $dL_{x(t),x(s)}=\mathrm{id}$ as the linear transformation of the tangent space $\mathfrak{G}_e$. Since $L_{x(t),x(s)}$ is a local affine transformation by Lemma 4.2, it must be the identity mapping on its domain (cf. [9, VI §6 Lemma 4]).

Conversely, if $x(t)$ is a local 1-parameter subgroup of (U, μ), then it is differentiable in t by definition, and so it is an integral curve of the vector field X, $X_x = dL_x(\dot{x}(0))$ $(x \in U)$, through $e=x(0)$. On the other hand, the geodesic $\tilde{x}(t)$ tangent to $\dot{x}(0)$ at e must satisfy the differential equation

$$\frac{d\tilde{x}}{dt} = dL_{\tilde{x}(t)}(\dot{x}(0))$$

with the initial condition $\tilde{x}(0)=e$, for $dL_{\tilde{x}(t)}$ is the parallel displacement of vectors along $\tilde{x}(t)$. Therefore we get $x(t)=\tilde{x}(t)$. *q.e.d.*

PROPOSITION 4.5. *In* Proposition 4.4, *the induced linear map* $dL_{x,y}\colon \mathfrak{G}_e \to \mathfrak{G}_e$ *of any left inner mapping* $L_{x,y}$, *if it is defined for* $x, y \in U$, *is an element of the restricted holonomy group* Ψ_e^0.

PROOF. Let γ_1, γ_2 and γ_3 be the geodesic segments considered in the proof of Theorem 4.3. Then, by (3) of Proposition 4.1 and by the definition (4.2), we can see that $dL_x \circ dL_y$ induces the parallel displacement of vectors along the piecewise differentiable curve γ_1 followed by γ_2. By Theorem 4.3, $dL_{(xy)^{-1}}$ is the inverse of dL_{xy} and so it must be the parallel displacement along γ_3. Thus we see that $dL_{x,y}=dL_{(xy)^{-1}} \circ dL_x \circ dL_y$ gives the composition of the parallel displacements along the geodesic triangle constructed with γ_1, γ_2 and γ_3. Since the normal neighborhood U is contractible by definition, $dL_{x,y}$ belongs to Ψ_e^0. *q.e.d.*

§5. Geodesic homogeneous Lie loops

Any homogeneous Lie loop G is a locally reductive space by Theorems 3.7 and 3.9, and so a geodesic local Lie loop (U, μ) of Definition 4.1 at the identity e of G is homogeneous by Theorem 4.3. In this section, we investigate the conditions for the given loop G itself to be coincident locally with the geodesic local Lie loop (U, μ) of G.

Let G be a connected homogeneous Lie loop with the canonical connection ∇ and regard it as a reductive homogeneous space $A(G)/K(G)$ by Theorem 3.7, where $A(G)=G \times K(G)$ is the semi-direct product of G by the closure $K(G)$ of the left inner mapping group $L_0(G)$. Let $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$ be the canonical decomposition

10) $\dot{x}(s)$ denotes the tangent vector to the curve $x(t)$ at $x(s)$.

(3.10) of the Lie algebra of $A(G)$, $\mathfrak{G}$ and $\mathfrak{K}$ being identified with the tangent space to G at e and the Lie algebra of $K(G)$, respectively. For each $X_0 \in \mathfrak{G}$, denote $\tilde{X}$ and $\bar{X}$ the vector fields on G defined by the left and right translations of G as follows:

$$\tilde{X}_x = dL_x(X_0),\ \bar{X}_x = dR_x(X_0) \qquad (x \in G). \tag{5.1}$$

PROPOSITION 5.1. *For each $X_0 \in \mathfrak{G} \subset \mathfrak{A}$, denote by*

$$\exp tX_0 = (x(t), \alpha(t)), \quad x(t) \in G, \alpha(t) \in K(G), \tag{5.2}$$

the 1-parameter subgroup of the Lie group $A(G)=G\times K(G)$ generated by X_0. Then,

(1) *$x(t)$ is a geodesic tangent to X_0 at e.*

(2) $x(s+t)=x(s)\alpha(s)(x(t))$,
$\alpha(s+t)=L_{x(s),\alpha(s)(x(t))}\circ\alpha(s)\circ\alpha(t)$.

(3) $x(t)^{-1}=\alpha(t)(x(-t))$, $\alpha(t)^{-1}=\alpha(-t)$.

(4) $\dfrac{dx}{dt}=\bar{X}_{x(t)}$, *that is, $x(t)$ is an integral curve of the vector field $\bar{X}$ of* (5.1), *through $e=x(0)$, and*

$$\frac{d\alpha}{dt} = \frac{d}{ds}\Big|_{s=0} (L_{x(s),x(t)}\circ\alpha(t)).$$

PROOF. Since the natural projection π: $A(G)\to G=A(G)/K(G)$ sends $(x(t), \alpha(t))$ to $x(t)$ by Theorem 3.7, (1) is immediate from Theorem 3.8. (2) and (3) are clear by the definition (2.1) and (2.2) in Theorem 2.1. (4) is obtained by differentiating the equations of (2) with respect to s and evaluating at $s=0$. *q.e.d.*

PROPOSITION 5.2. *For any X_0, $Y_0 \in \mathfrak{G}$, let $\tilde{X}$ and $\tilde{Y}$ be the vector fields on G defined as* (5.1) *by left translations of G. Denote by $\tilde{x}(t)$ and $\tilde{y}(t)$ integral curves of $\tilde{X}$ and $\tilde{Y}$, respectively, through $e=\tilde{x}(0)=\tilde{y}(0)$. Then for each fixed value of t, the curve $\tilde{z}_t(s)=\tilde{x}(t)\tilde{y}(s)$ in G is an integral curve of the vector field*

$$\tilde{Z}_t\colon u \longrightarrow dL_u^{(\tilde{x}(t))}(\tilde{Y}_{\tilde{x}(t)}) \qquad (u \in G),$$

where $L_u^{(\tilde{x}(t))}$ denotes the left translation in the transposed loop $G^{(\tilde{x}(t))}$ of G centered at $\tilde{x}(t)$ (cf. Lemma 1.2).

PROOF. By using the definition (1.5) of the multiplication in $G^{(\tilde{x}(t))}$, we have

$$\begin{aligned}\tilde{Z}_t(\tilde{z}_t(s)) &= dL_{\tilde{x}(t)}\circ dL_{(\tilde{x}(t))^{-1}\tilde{z}(s)}\circ dL^{-1}_{\tilde{x}(t)}(\tilde{Y}_{\tilde{x}(t)})\\ &= dL_{\tilde{x}(t)}\circ dL_{\tilde{y}(s)}(Y_0)\\ &= dL_{\tilde{x}(t)}\left(\frac{d\tilde{y}}{ds}\right)=\frac{d}{ds}(\tilde{z}_t(s)).\end{aligned}$$

q.e.d.

LEMMA 5.3. *For each $z \in G$*

$$dJ \circ dR_z + dL_z^{-1} = 0 \qquad on \quad \mathfrak{G}, \tag{5.3}$$

where J denotes the inverse mapping $J(x) = x^{-1}$ of G.

PROOF. For each $X_0 \in \mathfrak{G}$, let $x(t)$ be a differentiable curve in G satisfying the conditions $x(0) = e$ and $(dx/dt)_0 = X_0$. From (1.8) of Lemma 1.8, we obtain

$$(x(t)z)^{-1} = x(t)(z(z^{-1}x(t)^{-1})^2).$$

Differentiating both sides of this equation with respect to t and evaluating at $t=0$, we get

$$dJ \circ dR_z(X_0) = dR_{z^{-1}}(X_0) + dR_{z^{-1}} \circ dJ(X_0) + dL_{z^{-1}} \circ dJ(X_0).$$

Since $dJ(X_0) = -X_0$, we have the required equation. *q.e.d.*

A 1-*parameter subgroup $x(t)$ of a homogeneous Lie loop G* is an immersion $x: \boldsymbol{R} \to G$ which is a homomorphism of homogeneous Lie loops, where $\boldsymbol{R}$ denotes the additive Lie group of real numbers.

PROPOSITION 5.4. *For $X_0 \in \mathfrak{G}$ let $\tilde{X}$ and $\bar{X}$ be the vector fields of* (5.1). *Denote by $\tilde{x}(t)$ and $\bar{x}(t)$ the maximal integral curves of $\tilde{X}$ and $\bar{X}$, respectively, through $e = \tilde{x}(0) = \bar{x}(0)$. Then the following conditions* (1)–(5) *are mutually equivalent*:

(1) *$\tilde{x}(t)$ is a geodesic in G.*
(2) *$\tilde{x}(t)$ is a 1-parameter subgroup of G.*
(3) $(\tilde{x}(t)\tilde{x}(s))^{-1} = \tilde{x}(t)^{-1}\tilde{x}(s)^{-1} \qquad (s, t \in \boldsymbol{R})$.
(4) *$dT_{\tilde{x}(t)}(X_0) = X_0$, where $T_x = L_x^{-1} \circ R_x$.*
(5) $\tilde{x}(t) = \bar{x}(t) \qquad (t \in \boldsymbol{R})$.
(6) *$\bar{x}(t)$ is the 1-parameter subgroup of G.*

PROOF. (1)$\Leftrightarrow$(5) is clear from (1) and (4) of Proposition 5.1, that is $\bar{x}(t) = x(t)$ in (5.2). Also, (5)$\Leftrightarrow$(4) is clear by definition of $\tilde{x}(t)$ and $\bar{x}(t)$.

(1)$\Rightarrow$(2): If $\tilde{x}(t)$ is a geodesic in G, $\tilde{z}_t: s \to \tilde{x}(t)\tilde{x}(s)$ is also a geodesic because $L_{\tilde{x}(t)}$ is an affine transformation of G by Theorem 3.3. Clearly, $\tilde{z}_t(0) = \tilde{x}(t)$ and $(d\tilde{z}_t/ds)_{s=0} = dL_{\tilde{x}(t)}(X_0) = d\tilde{x}/dt$. On the other hand, the curve $z_t: s \to \tilde{x}(t+s)$ is a geodesic which satisfies the same initial condition as $\tilde{z}_t$ at $t=0$. Hence we get $\tilde{x}(t+s) = \tilde{x}(t)\tilde{x}(s)$ which proves (2) since $\tilde{x}(t)$ is defined for all values of $\boldsymbol{R}$.

(2)$\Rightarrow$(3) is clear.

(3)$\Rightarrow$(4): If (3) holds, by Lemma 5.3, we get

$$dJ\circ dL_{\tilde{x}(t)}(X_0) = \frac{d}{ds}\Big|_{s=0}(\tilde{x}(t)\tilde{x}(s))^{-1} = \frac{d}{ds}\Big|_{s=0}(\tilde{x}(t)^{-1}\tilde{x}(s)^{-1})$$
$$= dL^{-1}_{\tilde{x}(t)}\circ dJ(X_0) = dJ\circ dR_{\tilde{x}(t)}(X_0),$$

and so we have $dL_{\tilde{x}(t)}(X_0)=dR_{\tilde{x}(t)}(X_0)$ which is the same as (4).

(6) follows clearly from (2) and (5).

(6)⇒(5): If the integral curve $\bar{x}(t)$ of the vector field $\bar{X}$ is a 1-parameter subgroup of G, it follows from (2) and (4) of Proposition 5.1 that $x(s)x(t)=x(s)\alpha(s)(x(t))$ and $x(t)=\bar{x}(t)$ hold for all s, $t\in \boldsymbol{R}$ and so we get

$$\alpha(s)(x(t)) = x(t) \qquad (s, t\in \boldsymbol{R}), \tag{5.4}$$

where $\exp tX_0=(x(t), \alpha(t))$, the 1-parameter subgroup of the Lie group $A(G)=G\times K(G)$ generated by $X_0\in\mathfrak{G}$. Hence $d\alpha(t)(X_0)=X_0$ for $t\in\boldsymbol{R}$. Using (2) of Proposition 5.1 again, we have

$$dx/dt = \frac{d}{ds}\Big|_{s=0}x(t+s) = dL_{x(t)}\circ d\alpha(t)(X_0) = dL_{x(t)}(X_0).$$

Thus we get $x(t)=\tilde{x}(t)$. *q.e.d.*

REMARK 5.1. In the above proof, we have the equality (5.4) provided one of the conditions (1)–(6) of the above proposition for G. Hence, by (2) of Proposition 5.1,

$$\alpha(t+s) = L_{x(t),x(s)}\circ\alpha(t)\circ\alpha(s) \tag{5.5}$$

is valid in that case.

In view of (3) in Proposition 5.4, we have

COROLLARY 5.5. *Suppose that a homogeneous Lie loop G has the symmetric property of* Definition 1.7, *then every geodesic $x(t)$ of G through $e=x(0)$ is a 1-parameter subgroup of G.*

PROPOSITION 5.6. *Let G be a connected homogeneous Lie loop and $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$ be the canonical decomposition of the Lie algebra of $A(G)=G\times K(G)$. Then each of the following conditions* (1)–(4) *implies all the other*:

(1) *The 1-parameter subgroup* $\exp tX_0$ *of $A(G)$ generated by an arbitrary element $X_0\in\mathfrak{G}$ is contained in the submanifold $G_{\mathrm{id}}=G\times\mathrm{id}$ of $A(G)$.*

(2) *For each $X_0\in\mathfrak{G}$, the curve $x(t)$ in G defined by* (5.2) *satisfies the following*

$$L_{x(t),x(s)} = \mathrm{id} \qquad (s, t\in\boldsymbol{R}). \tag{5.6}$$

(3) *For each $X_0\in\mathfrak{G}$, the maximal integral curve $\tilde{x}(t)$ ($\tilde{x}(0)=e$) of the vector*

field $\tilde{X}$ *of* (5.1) *satisfies the following*

$$L_{\tilde{x}(t),\tilde{x}(s)} = \mathrm{id} \qquad (s, t \in \boldsymbol{R}).$$

(4) *The left translation* $L_{x(t)}$ $(t \in \boldsymbol{R})$ *of* G *by any* $x(t)$ *on the curve of* (2) *yields the parallel displacement from any point* $x(s)$ *to* $x(t)x(s)$ *along it.*

Moreover, when any one of these conditions is satisfied, all conditions (1)–(6) *of* Proposition 5.4 *are satisfied.*

PROOF. (1)⇔(2): By Definition 2.1 of the multiplication of the semi-direct product, we get

$$(*) \qquad (x(t), \mathrm{id})(x(s), \mathrm{id}) = (x(t)x(s), L_{x(t),x(s)}).$$

If (1) is satisfied then from $(*)$ it follows that $x(t+s)=x(t)x(s)$ and $L_{x(t),x(s)}=\mathrm{id}$ for the curve $x(t)$ of (2). Conversely, if (5.6) is satisfied for $x(t)$, we have

$$x(s)(x(t)x(u)) = (x(s)x(t))(x(u))$$

for all s, t, $u \in \boldsymbol{R}$. So, by using (4) of Proposition 5.1, we have

$$(**) \qquad \begin{aligned} \frac{d}{dt}(x(t)x(u)) &= dR_{x(u)}(dx/dt) \\ &= dR_{x(u)} \circ dR_{x(t)}(X_0) \\ &= \frac{d}{ds}\Big|_{s=0} [(x(s)x(t))x(u)] \\ &= \frac{d}{ds}\Big|_{s=0} [x(s)(x(t)x(u))] \\ &= dR_{x(t)x(u)}(X_0), \end{aligned}$$

for any fixed value of u. On the other hand, (4) of Proposition 5.1 implies also the equation

$$(***) \qquad dx(u+t)/dt = dR_{x(u+t)}(X_0)$$

for any fixed value of u. By comparing the differential equations $(**)$ and $(***)$, we see that $x(u+t)=x(t)x(u)$ for all values of u, $t \in \boldsymbol{R}$. Combining this result with the assumption (5.6), we see easily that $(x(t), \mathrm{id})$ is the 1-parameter subgroup of the Lie group $A(G)$, generated by X_0. Hence (1) is obtained.

(1)⇔(3): If (1) is satisfied, $x(t)$ is a 1-parameter subgroup of G as was shown above. Then by Proposition 5.4, we see $x(t)=\tilde{x}(t)$. Thus (3) is reduced to (2). Conversely, if (3) is satisfied, in the same way as the proof above, merely replacing x and R_x with $\tilde{x}$ and $L_{\tilde{x}}$, respectively, in $(**)$ and $(***)$, we can obtain (1).

(1)$\Rightarrow$(4): If $\exp tX_0=(x(t), \mathrm{id})$ in the Lie group $A(G)$, its action on $G=A(G)/K(G)$ is equal to the left translation $L_{x(t)}$, by (*). Hence, Nomizu's Theorem (Theorem 3.8) asserts that $L_{x(t)}$ yields the parallel displacement along $x(t)$ from e to $x(t)$. Since $L_{x(t)}$ is an affine transformation of G and since $x(t)$ is a geodesic, we see also that $L_{x(t)}$ yields the parallel displacement from any point $x(s)$ to $x(t)x(s)$ along the curve $s\rightarrow x(t)x(s)$. However this curve is equal to $s\rightarrow x(t+s)$ as was seen in the above proof of (1)$\Rightarrow$(2). Hence (4) is obtained.

(4)$\Rightarrow$(2): Let $\exp tX_0=(x(t), \alpha(t))$ as in (5.2). Then the curve $\gamma: t\rightarrow x(t)$ is a geodesic of G by Proposition 5.1. Since $L_{x(t)}$ is an affine transformation of G, the curve $\gamma': s\rightarrow x(t)x(s)$ is also a geodesic of G. If $dL_{x(t)}$ is the parallel displacement along γ, then the geodesic γ' is tangent to γ at $x(t)$ and so we have $x(t)x(s)=x(t+s)$. Since the affine transformation $L_{x(t)}$ preserves the parallel displacement along any curve, so does it for the curve γ. Thus we have

$$dL_{x(t)}\circ dL_{x(s)} = dL_{x(t)x(s)},$$

which is equivalent to (5.6).

In the course of the above proof, $x(t)$ is shown to be a 1-parameter subgroup of the homogeneous Lie loop G. Hence the second half of the proposition is clear. *q.e.d.*

DEFINITION 5.1. A connected homogeneous Lie loop G will be said to be *geodesic* if one of the conditions (1)–(4) of Proposition 5.6 is satisfied.

By this definition we have

THEOREM 5.7. *Any geodesic local Lie loop (U, μ) of* Definition 4.1 *at the identity e of a connected homogeneous Lie loop $G=A(G)/K(G)$ with the canonical connection is homogeneous and satisfies the condition* (5.6) *for any geodesic $x(t)$ through $e=x(0)$ whenever the local affine transformation $L_{x(t),x(s)}$ is defined, that is, the local Lie loop (U, μ) is geodesic by definition.*

Moreover, if G is geodesic, then the multiplication μ of any geodesic local Lie loop (U, μ) of G at e coincides with the given multiplication of G, as far as μ is defined.

PROOF. The first half of the theorem is an immediate consequence of Theorem 4.3 and Proposition 4.4, taking account of Theorems 3.7 and 3.9, and the remaining half is clear from (4) of Proposition 5.6. *q.e.d.*

By combining this to Example 3.3 and Theorem 3.8, we have

COROLLARY 5.8. *Any connected Lie group G is a geodesic homogeneous Lie loop. For every 1-parameter subgroup $x(t)=\exp tX$ of G ($X\in\mathfrak{G}$), the left translation $L_{x(t)}$ is the parallel displacement along $x(t)$ with respect to the $(-)$-connection of G.*

REMARK 5.2. A. A. Sagle and J. R. Schumi [18] have considered also a local multiplication μ on a reductive homogeneous space A/K defined in a neighborhood of the origin e by

$$\mu(\pi \exp X, \pi \exp Y) = \pi \exp F(X, Y) \tag{5.7}$$

for X, Y belonging to a neighborhood of 0 in the subspace $\mathfrak{G}$ of the direct sum decomposition $\mathfrak{A} = \mathfrak{G} + \mathfrak{K}$ of the Lie algebra of A, where $F(X, Y) \in \mathfrak{G}$ and F is analytic at $(0, 0) \in \mathfrak{G} \times \mathfrak{G}$. (Cf. also [17, Appendix].)

§6. Symmetric Lie loops

In this section, we study the connected homogeneous Lie loops with the symmetric property of Definition 1.7. Such loops are called symmetric Lie loops and shown to be in a special class of symmetric spaces. By using the methods and results of O. Loos [13], we show that any symmetric Lie loop is geodesic in the sense of Definition 5.1.

Let G be a connected homogeneous Lie loop and let $A(G) = G \times K(G)$ be the semi-direct product of G by the closure $K(G)$ of the left inner mapping group $L_0(G)$ (cf. Theorem 3.5).

Denote by σ: $A(G) \to A(G)$ the mapping defined by

$$\sigma(x, \alpha) = (x^{-1}, \alpha) \qquad \text{for} \quad (x, \alpha) \in A(G) = G \times K(G). \tag{6.1}$$

σ is differentiable since the inverse mapping J is so by Proposition 3.1.

THEOREM 6.1. *A connected homogeneous Lie loop G has the symmetric property*, i.e.,

$$(xy)^{-1} = x^{-1}y^{-1} \qquad (x, y \in G) \quad \text{(cf. Definition 1.7)}$$

holds, if and only if the mapping σ of (6.1) *is an involutive automorphism of the Lie group $A(G)$.*

Therefore, the reductive homogeneous space $G = A(G)/K(G)$ is a symmetric homogeneous space by the triple $(A(G), K(G), \sigma)$[11] *if and only if G has the symmetric property.*

PROOF. For any elements (x, α), $(y, \beta) \in A(G) = G \times K(G)$, we have

$$\sigma((x, \alpha)(y, \beta)) = ((x\alpha(y))^{-1}, L_{x,\alpha(y)} \circ \alpha \circ \beta),$$

$$\sigma(x, \alpha)\sigma(y, \beta) = (x^{-1}\alpha(y)^{-1}, L_{x^{-1},\alpha(y)^{-1}} \circ \alpha \circ \beta).$$

11) For the terminologies and results about (affine) symmetric spaces, we refer to [9, XI §§ 2–5] and [13].

For the mapping σ to be an automorphism of $A(G)$, it is necessary and sufficient that

$$(*) \qquad (x\alpha(y))^{-1} = x^{-1}\alpha(y)^{-1} \quad \text{and} \quad L_{x,\alpha(y)} = L_{x^{-1},\alpha(y)^{-1}}$$

hold for all $x, y \in G$ and $\alpha \in K$.

If $(*)$ holds, the first equality implies the symmetric property of G. Conversely, if G has the symmetric property, $(*)$ follows from (4) of Proposition 1.13.

If σ is an automorphism of $A(G)$, it is clear that $K(G)$ is the subgroup of $A(G)$ consisting of all elements fixed by σ, and so the effective symmetric homogeneous space $A(G)/K(G)$ is defined by the triple $(A(G), K(G), \sigma)$. *q.e.d.*

DEFINITION 6.1. A connected homogeneous Lie loop with the symmetric property will be called a *symmetric Lie loop.*

In the rest of this section we assume that G is a symmetric Lie loop. Then, by the above theorem, the general theory of symmetric homogeneous spaces is applicable for $G = A(G)/K(G)$.

The canonical decomposition $\mathfrak{A} = \mathfrak{G} + \mathfrak{K}$ (direct sum) of the Lie algebra of $A(G)$ is just the canonical decomposition of $(\mathfrak{A}, \mathfrak{K}, d\sigma)$, that is, $\mathfrak{G}$ is the eigenspace of $d\sigma$ for the eigenvalue -1 and $\mathfrak{K}$ is the one for $+1$. The canonical connection of the symmetric homogeneous space is, by definition, the canonical connection of the corresponding reductive homogeneous space $G = A(G)/K(G)$, which is identified with the canonical connection of the homogeneous Lie loop G by Theorem 3.7.

We define the *symmetry* S_x of G at each point $x \in G$ by

$$S_x = L_x \circ J \circ L_x^{-1} \qquad (x \in G),$$

where J denotes the inverse mapping of G. By Lemma 1.5, S_x is the inverse mapping of the transposed loop $G^{(x)}$ of G centered at x. By Corollary 5.5, every geodesic $x(t)$ $(t \in \boldsymbol{R})$ of G through $e = x(0)$ is a 1-parameter subgroup of the loop G. Hence, by $J(x(t)) = x(-t)$, the symmetry $S_e = J$ at e is the geodesic symmetry and, for each $x \in G$, S_x is also the geodesic symmetry at x, because the left translation L_x is an affine transformation of G (Theorem 3.3). Thus we see that x is an isolated fixed point of the geodesic symmetry S_x at x. (Cf. [13, Annexe].)

In the following we shall show that a symmetric Lie loop G is geodesic.

LEMMA 6.2. *$L_{x,x} = \mathrm{id}$, that is, G is left alternative, and $S_x = L_{x^2} \circ J$ for any $x \in G$.*

PROOF. We put $\theta = L_{x,x}$ for any fixed $x \in G$. Then $\theta \in Aut(G) \subset Aff(G)$ by Theorem 3.4 and $\theta \circ \theta = \mathrm{id}$ by Remark 1.3. Consider the induced linear map

$d\theta$: $\mathfrak{G}\to\mathfrak{G}$, $\mathfrak{G}$ being the tangent space of G at e. Then, for $Y\in\mathfrak{G}$, $d\theta(Y)=\pm Y$ and so, in a normal neighborhood U of e, $\theta(y)=y$ or $\theta(y)=y^{-1}$ for each $y=\mathrm{Exp}_e(Y)\in U$. But the latter case is impossible if $y\neq e$. In fact, if we consider a continuous curve $x(t)$ joining $e=x(0)$ to $x=x(1)$ in G, then $L_{x(t),x(t)}(y)$ is continuous in t for each fixed $y\in U$. If $y\neq e$ then $y\neq y^{-1}$ in U and since $L_{x(0),x(0)}=\mathrm{id}$ we have $L_{x(t),x(t)}(y)=y$ for $y\in U$, $0\leqq t\leqq 1$. Thus we have $\theta=\mathrm{id}$ on U. By Theorems 6.1 and 7.7 of [9, VI], we can conclude that $\theta=\mathrm{id}$ on G.

By using the fact just proved above, we get

$$S_x(y) = x(x^{-1}y)^{-1} = x(xy^{-1}) = L_x\circ L_x\circ J(y) = L_{x^2}\circ J(y)$$

for any $y\in G$. Hence $S_x=L_{x^2}\circ J$. *q.e.d.*

O. Loos has defined in [13, Chapter II] a symmetric space to be a differentiable manifold G with a differentiable multiplication

$$x*y = S_x(y) \qquad (x, y\in G) \tag{6.2}$$

satisfying the following conditions (R.1)–(R.4):

(R.1) $$x*x = x\,.$$

(R.2) $$x*(x*y) = y\,.$$

(R.3) $$x*(y*z) = (x*y)*(x*z)\,.$$

(R.4) Every $x\in G$ has a neighborhood U such that $x*y=y$ implies $y=x$ for all $y\in U$. Cf. [12] also.

By fixing a base point e of G, he has defined also the *quadratic representation* Q of G by

$$Q(x) = S_x\circ S_e \qquad (x\in G)\,. \tag{6.3}$$

If G is a symmetric space in the usual sense, its geodesic symmetries S_x $(x\in G)$ satisfy the conditions (R.1)–(R.3) and (R.4) is assured also for a normal neighborhood U of x. The mapping Q above means the *transvection* along the geodesic passing through e and x (cf. [9, X, p. 236]).

O. Loos has given also the canonical connection of the symmetric space G by means of the tangent algebra of the multiplication (6.2). (Cf. [13, Chapter I §4 and Chapter II §2].)

If we apply his theory to our symmetric Lie loop G, we can translate it in terms of our multiplication of the loop G as follows:

By Lemma 6.2, (6.2) is written as

$$x*y = x(xy^{-1}) = (xx)y^{-1}\,. \tag{6.4}$$

The quadratic representation (6.3) is written also as

$$Q(x) = L_{x^2}, \tag{6.5}$$

since $S_e = J$.

Also, by examining into details of the definition of the canonical connection of [13, II § 2], we can see that it is exactly the same as our canonical connection of the symmetric Lie loop G. If $x(t) = \mathrm{Exp}_e tX$ ($X \in \mathfrak{G}$) is a geodesic through $e = x(0)$, we have seen in Corollary 5.5 that $x(t)$ is a 1-parameter subgroup of the loop G. By putting

$$\phi_t = Q(\mathrm{Exp}_e(t/2)X) \qquad (X \in \mathfrak{G}), \tag{6.6}$$

O. Loos has proved the following theorem for symmetric spaces in general ([12, Satz 5.7] and [13, II Theorem 2.7]).

THEOREM 6.3. *The transformation* ϕ_t $(t \in \boldsymbol{R})$ *of a symmetric space* G *given by* (6.6) *is a* 1*-parameter group of transformations, and* $d\phi_t$ *induces the parallel displacement of vectors along the geodesic* $x(t) = \mathrm{Exp}_e tX$.

Translating these results into our symmetric Lie loop G, we have the following theorems:

THEOREM 6.4. *Any symmetric Lie loop is geodesic.*

PROOF. Let G be a symmetric Lie loop and let $x(t) = \mathrm{Exp}_e tX$ ($X \in \mathfrak{G}$) be a geodesic of G through $e = x(0)$. By Corollary 5.5 $x(t)$ is a 1-parameter subgroup of G. In view of (6.5) and (6.6), we can see $\phi_t = L_{x(t)}$ $(t \in \boldsymbol{R})$. Then the above theorem implies that $L_{x(t),x(s)} = \mathrm{id}$ for any $t, s \in \boldsymbol{R}$. Thus the proof is completed by Proposition 5.6 (2) and Definition 5.1. *q.e.d.*

THEOREM 6.5. *In* Theorem 6.4, *any geodesic local Lie loop* (U, μ) *at* e *has the symmetric property whose multiplication* $\mu(x, y) = xy$ *can be expressed as*

$$xy = x^{1/2} * (e * y), \tag{6.7}$$

where $x^{1/2}$ *denotes the middle point of the geodesic segment joining* e *to* x *in* U.

PROOF. In view of (6.3) and (6.4), we get

$$\phi_t(y) = Q(x(t/2))(y) = x(t/2) * (e * y) \qquad (y \in G).$$

Since $\phi_t = L_{x(t)}$ is the parallel displacement along $x(t)$, we have (6.7). Restricting it to a normal neighborhood U of e, we have a geodesic local Lie loop (U, μ_U) with the symmetric property. *q.e.d.*

REMARK 6.1. It is well known that the torsion tensor of the canonical connection of a symmetric homogeneous space vanishes identically, and hence so does it in any symmetric Lie loop.

REMARK 6.2. Motivated by the relation between the transvection L_x and the symmetry S_x (or *reflection* across x) in (6.4) and (6.7), we have constructed in [8] a homogeneous loop with the symmetric property, called a *symmetric loop*, that is defined as a left alternative homogeneous loop whose element has a unique square root. Then it can be shown that $(G, *)$ given by (6.4) for a symmetric loop G is a quasigroup satisfying (R.1)–(R.3), called a *quasigroup of reflection*, and that any quasigroup of reflection has the canonical homogeneous structure which assigns to each $e \in G$ a symmetric loop $G^{(e)} = (G, \mu)$ by (6.7).

Taking account of Lemma 6.2, these global algebraic theory of symmetric loops and quasigroups of reflection are applicable for any symmetric Lie loop G if each element of G has a unique square root.

EXAMPLE 6.1. The homogeneous Lie loops P_n and H_n given in Example 3.2 are symmetric Lie loops. In fact, they have the symmetric property and each element of them has a unique square root.

§7. Lie triple algebras

In this section, after giving the definition of the Lie triple algebra (i.e. general Lie triple system introduced by K. Yamaguti [19]), we show that a Lie triple algebra $\mathfrak{G}$ is defined on the tangent space of a geodesic homogeneous Lie loop G and that $\mathfrak{G}$ can be regarded as the tangent algebra of G in a certain sense.

DEFINITION 7.1. A *Lie triple algebra* (or a *general Lie triple system* [19, Definition 2.1]) is an anti-commutative algebra $\mathfrak{G}$ (over an arbitrary field) with the trilinear operation $\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, denoted by $[X, Y, Z]$ for $X, Y, Z \in \mathfrak{G}$, satisfying the following conditions for any $X, Y, Z, U, V \in \mathfrak{G}$:

(7.1.1) $$[X, X, Y] = 0.$$

(7.1.2) $$\mathfrak{S}\{[X, Y, Z] + (XY)Z\} = 0.$$

(7.1.3) $$\mathfrak{S}\{[XY, Z, U]\} = 0.$$

(7.1.4) $$[X, Y, UV] = [X, Y, U]V + U[X, Y, V].$$

(7.1.5) $$[U, V, [X, Y, Z]] = [[U, V, X], Y, Z] + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]].$$

Here the symbol $\mathfrak{S}$ in (7.1.2) and (7.1.3) denotes the cyclic sum with respect to the three elements $X, Y, Z \in \mathfrak{G}$.

REMARK 7.1. From the definition above, it is clear that the underlying anti-commutative algebra of a Lie triple algebra $\mathfrak{G}$ is reduced to a *Lie algebra* if the trilinear operation is trivial, i.e., $[X, Y, Z]=0$.

Also, if the underlying anti-commutative algebra of $\mathfrak{G}$ is trivial, $\mathfrak{G}$ is reduced to a *Lie triple system* under the ternary operation $[X, Y, Z]$ of $\mathfrak{G}$.

Homomorphisms and *isomorphisms* of Lie triple algebras should be understood to be the mappings which preserve both of the binary and ternary operations of them. For the terminologies of Lie triple algebras (general Lie triple systems), see also [20, § 1].

In the sequel we are only concerned with finite dimensional real Lie triple algebras.

Let $\mathfrak{G}$ be a Lie triple algebra. For $X, Y \in \mathfrak{G}$, denote by $D(X, Y)$ the endomorphism of $\mathfrak{G}$ defined by

$$D(X, Y)Z = [X, Y, Z] \qquad (X, Y, Z \in \mathfrak{G}).$$

It is called an *inner derivation* of $\mathfrak{G}$. By (7.1.4) and (7.1.5), any inner derivation of $\mathfrak{G}$ is a *derivation* of both of the binary and ternary operations of $\mathfrak{G}$. Let $\mathfrak{K}_0 = D(\mathfrak{G}, \mathfrak{G})$ denote the Lie algebra of endomorphisms of $\mathfrak{G}$ generated by all inner derivations of $\mathfrak{G}$. In fact, from

$$(7.3) \qquad [D(U, V), D(X, Y)] = D(D(U, V)X, Y) + D(X, D(U, V)Y)$$

it follows that $\mathfrak{K}_0$ is closed under the Lie bracket of endomorphisms.

Now, set $\mathfrak{A}_0 = \mathfrak{G} + \mathfrak{K}_0$ (direct sum) and define a new bracket operation in $\mathfrak{A}_0$ as follows:

$$(7.4.1) \qquad [X, Y] = XY + D(X, Y) \qquad (X, Y \in \mathfrak{G}),$$

$$(7.4.2) \qquad [A, X] = -[X, A] = A(X) \qquad (A \in \mathfrak{K}_0, X \in \mathfrak{G}),$$

$$(7.4.3) \qquad [A, B] = AB - BA \qquad (A, B \in \mathfrak{K}_0).$$

THEOREM 7.1. (Cf. [15, the proof of Theorem 18.1] and [19, Proposition 2.1].) *Let $\mathfrak{G}$ be a Lie triple algebra and $\mathfrak{K}_0 = D(\mathfrak{G}, \mathfrak{G})$ be the Lie algebra of all inner derivations of $\mathfrak{G}$. Then $\mathfrak{A}_0 = \mathfrak{G} + \mathfrak{K}_0$ (direct sum) forms a Lie algebra under the bracket operation of* (7.4.1–3), *and $\mathfrak{K}_0$ is a Lie subalgebra of $\mathfrak{A}_0$.*

PROOF. The bracket (7.4) is bilinear by definition, and $[X, X]=0$ by (7.1.1). Jacobi's identity follows from (7.1.2–5). The fact that $\mathfrak{K}_0$ is a subalgebra of $\mathfrak{A}_0$ is clear from (7.3) and (7.4.3). *q.e.d.*

DEFINITION 7.2. The Lie algebra $\mathfrak{A}_0 = \mathfrak{G} + \mathfrak{K}_0$ obtained in the above theorem is called the *standard enveloping Lie algebra* of the Lie triple algebra $\mathfrak{G}$. In general, $\mathfrak{A} = \mathfrak{G} + \mathfrak{K}$ is called an *enveloping Lie algebra* of $\mathfrak{G}$ if $\mathfrak{K}$ is a Lie algebra generated by derivations of $\mathfrak{G}$ and if $\mathfrak{K}$ contains $\mathfrak{K}_0$, the bracket of $\mathfrak{A}$ being defined as (7.4) for $A, B \in \mathfrak{K}$.

THEOREM 7.2. (Cf. [15, the proof of Theorem 18.1].) *Let G be a locally reductive space with the torsion and curvature tensors S and R, respectively* (cf. foot note 7) in §4). *Then, at each point $e \in G$, the tangent space $\mathfrak{G} = \mathfrak{G}_e$ to G is a Lie triple algebra under the operations defined as follows*:

$$XY = S_e(X, Y); \; [X, Y, Z] = R_e(X, Y)Z \qquad (X, Y, Z \in \mathfrak{G}). \tag{7.5}$$

PROOF. It is clear that $\mathfrak{G}$ is an anti-commutative algebra with respect to XY of (7.5). (7.1.1) follows from $R_e(X, X) = 0$. Also, (7.1.2) and (7.1.3) are obtained from Bianchi's first and second identities (cf. Theorem 5.3 of [9, III]), respectively, by using the assumption $\nabla S = 0$ and $\nabla R = 0$. (7.1.4) and (7.1.5) are the immediate consequences of the following identity substituted T with S and R, respectively:

$$R(X, Y)(T(X_1, \dots, X_k)) = \textstyle\sum_{i=1}^{k} T(X_1, \dots, R(X, Y)X_i, \dots, X_k)$$

for any vector fields $X, Y, X_1, \dots, X_k$, if T is a $(1, k)$-tensor satisfying $\nabla T = 0$.
q.e.d.

From the above theorem we have

THEOREM 7.3. *Let $G = A(G)/K(G)$ be a connected homogeneous Lie loop with the canonical decomposition $\mathfrak{A} = \mathfrak{G} + \mathfrak{K}$ of the Lie algebra $\mathfrak{A}$ of $A(G)$. Then $\mathfrak{G}$ is a Lie triple algebra under the operations*

$$XY = [X, Y]_{\mathfrak{G}}; \quad [X, Y, Z] = [[X, Y]_{\mathfrak{K}}, Z] \tag{7.6}$$

for $X, Y, Z \in \mathfrak{G}$, where the brackets on the right hand side are those in the Lie algebra $\mathfrak{A}$.

Moreover, the adjoint representation of $\mathfrak{K}$ on the subspace $\mathfrak{G}$ of $\mathfrak{A}$ is faithful and $\mathfrak{A}$ is regarded as an enveloping Lie algebra of the Lie triple algebra $\mathfrak{G}$.

PROOF. Since the canonical connection of G is locally reductive by Theorem 3.7, from (2), (3) of Theorem 3.9 and from the above theorem, it follows that $\mathfrak{G}$ is a Lie triple algebra under the operations (7.6).

Since $\mathrm{ad}(K)\mathfrak{G} \subset \mathfrak{G}$, we can restrict the adjoint representation of $\mathfrak{K}$ to $\mathfrak{G}$ as

$$\mathrm{ad}_{\mathfrak{G}} : \mathfrak{K} \longrightarrow \mathrm{End}(\mathfrak{G}), \quad \mathrm{ad}_{\mathfrak{G}}(A)X = [A, X] \qquad (A \in \mathfrak{K}, X \in \mathfrak{G}).$$

Let $\mathfrak{K}_1$ denote the kernel of this Lie algebra homomorphism. Then it is easy to see that $\mathfrak{R}_1$ is an ideal of $\mathfrak{A}$. Since $A(G)$ acts on G effectively, we get $\mathfrak{R}_1=0$ and so $\mathrm{ad}_{\mathfrak{G}}$ is an into isomorphism. By identifying $\mathfrak{R}$ with $\mathrm{ad}_{\mathfrak{G}}(\mathfrak{R})$ under this isomorphism, we have an enveloping Lie algebra $\mathfrak{A}=\mathfrak{G}+\mathrm{ad}_{\mathfrak{G}}(\mathfrak{R})$ of the Lie triple algebra $\mathfrak{G}$. In fact, the inner derivation algebra $\mathfrak{R}_0=D(\mathfrak{G}, \mathfrak{G})$ is an ideal of $\mathfrak{R}=\mathrm{ad}_{\mathfrak{G}}(\mathfrak{R})$. *q.e.d.*

PROPOSITION 7.4. *Let G be a connected locally reductive space. For any two points e and e', the Lie triple algebras $\mathfrak{G}_e$ and $\mathfrak{G}_{e'}$ of* Theorem 7.2 *are mutually isomorphic.*

PROOF. Let γ be a piecewise differentiable curve in G joining e to e'. The parallel displacement τ of vectors along γ is a linear isomorphism of $\mathfrak{G}_e$ onto $\mathfrak{G}_{e'}$ and it preserves the torsion and curvature tensors because they are supposed to be parallel on G. Hence $\tau: \mathfrak{G}_e \to \mathfrak{G}_{e'}$ is an isomorphism of the Lie triple algebras by (7.5). *q.e.d.*

PROPOSITION 7.5. *Let G be a connected locally reductive space and let $\mathfrak{G}=\mathfrak{G}_e$ be the Lie triple algebra of* Theorem 7.2 *at $e \in G$. Then the inner derivation algebra $\mathfrak{R}_0=D(\mathfrak{G}, \mathfrak{G})$ is the holonomy algebra of G, i.e., the Lie algebra of the holonomy group Ψ_e.*

PROOF. Since the curvature tensor R is parallel on G,

$$\tau \circ R_e(X, Y)Z = R_x(\tau(X), \tau(Y))\tau(Z) \qquad (X, Y, Z \in \mathfrak{G})$$

holds for any piecewise differentiable curve from e to an arbitrary point x in G, where τ denotes the parallel displacement along the curve. Then, by Theorem 9.1 in [9, III], the holonomy algebra $\mathfrak{H}$ is generated by the set $\{R_e(X, Y); X, Y \in \mathfrak{G}\}$ of linear endomorphisms of $\mathfrak{G}$. By the definition (7.5) of the ternary operation of $\mathfrak{G}$, $R_e(X, Y)=D(X, Y)$ for any $X, Y \in \mathfrak{G}$. Hence we get $\mathfrak{H}=\mathfrak{R}_0$. *q.e.d.*

PROPOSITION 7.6. *Let G and G' be locally reductive spaces. Let (U, μ) and (U', μ') be geodesic local Lie loops of* Definition 4.1 *with the identities $e \in G$ and $e' \in G'$, respectively. Then (U, μ) and (U', μ') are locally isomorphic if and only if the Lie triple algebras $\mathfrak{G}=\mathfrak{G}_e$ and $\mathfrak{G}'=\mathfrak{G}'_{e'}$ are isomorphic.*

PROOF. Suppose that $F: \mathfrak{G} \to \mathfrak{G}'$ is an isomorphism of the Lie triple algebras. Then, by Theorem 7.4 in [9, VI], there exists a local affine diffeomorphism ϕ of a neighborhood V of e onto a neighborhood V' of e' such that $d\phi_e=F$. Hence this map ϕ sends the parallel displacement of geodesics along any geodesic in V to one in V'. Therefore, for a normal neighborhood W of e contained in $U \cap V$, the restriction of μ to W gives a local isomorphism of the geodesic local Lie loop (W, μ_W) at e and the corresponding $(\phi(W), \phi \circ \mu_W)$ at e'.

Conversely, if a local isomorphism $\phi: W \to W'$, $W \subset U$, $W' \subset U'$, of the geodesic local Lie loops (U, μ) and (U', μ') is given, ϕ sends every geodesic through e to a geodesic through e' by Proposition 4.4. Moreover, since ϕ commutes with the left translations by elements under correspondence, ϕ must send all parallel displacements of geodesics along geodesics in W to those in W' by Lemma 4.2. Then it is shown that $\nabla'_{d\phi(X)}d\phi(Y)=d\phi(\nabla_X Y)$ holds for any vector fields X, Y on W, where ∇ and ∇' denote the linear connections of G and G', respectively. Therefore, ϕ is a local affine diffeomorphism and so $d\phi$: $\mathfrak{G}_e \to \mathfrak{G}'_{e'}$ preserves the torsion and curvature, that is, $d\phi$ is an isomorphism of the Lie triple algebras.

q. e. d.

From Propositions 4.5 and 7.5, we have the following

THEOREM 7.7. *Let $G=A(G)/K(G)$ be a geodesic homogeneous Lie loop with the canonical decomposition $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$. Then there exists a neighborhood U of the identity e such that $dL_{x,y}$ belongs to the restricted holonomy group Ψ_e^0 for any left inner mapping $L_{x,y}$ $(x, y \in U)$.*

The holonomy algebra of Ψ_e^0 is the inner derivation algebra $\mathfrak{K}_0$ of the Lie triple algebra $\mathfrak{G}$ of G at e.

Also, the following theorem follows from Theorems 5.7, 7.4 and Proposition 7.6:

THEOREM 7.8. *Two geodesic homogeneous Lie loops G and G' are locally isomorphic if and only if their Lie triple algebras $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic.*

As a corollary, we have the following by Theorems 6.4, 7.2 and Remarks 6.1, 7.1:

COROLLARY 7.9. *Two symmetric Lie loops G and G' are locally isomorphic if and only if their Lie triple systems $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic.*

REMARK 7.2. It is well known that the curvature tensor of the $(-)$-connection of a connected Lie group vanishes identically. Therefore, Theorems 7.2, 7.3, Corollary 5.8 and Remark 7.1 show that, if G and G' are reduced to connected Lie groups, Theorem 7.8 is reduced to the well known theorem in the theory of Lie groups and Lie algebras.

REMARK 7.3. In T. Nôno [16], it has been proved that a finite dimensional space $\mathfrak{G}$ of vector fields on a differentiable manifold M is a Lie triple system under the operation

$$[X, Y, Z] = [[X, Y], Z] \qquad (X, Y, Z \in \mathfrak{G})$$

if and only if the family $G=\{\phi_{a(t)}; a(t) \in U \subset \boldsymbol{R}^n\}$, $t \in \boldsymbol{R}$, of local 1-parameter transformations $\phi_{a(t)}$, $\phi_0=\phi_{a(0)}=\mathrm{id}$, generated by elements of $\mathfrak{G}$ satisfies the condition (2) of Example 1.5 in local, that is, if and only if G can be regarded as a local symmetric Lie loop under the multiplication

$$\mu(\phi_{a(t)}, \psi_{b(s)}) = \phi_{a(t/2)}\circ\psi_{b(s)}\circ\phi_{a(t/2)},$$

for any local 1-parameter subgroups $\phi_{a(t)}, \psi_{b(s)}$ of local transformations of M belonging to G.

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References

[1] R. H. Bruck: *Survey of Binary Systems* (3rd printing), Springer, 1971.

[2] ——— and L. J. Paige: *Loops whose inner mappings are automorphisms,* Ann. of Math. (2) **63** (1956), 308–328.

[3] É. Cartan and J. A. Schouten: *On the geometry of the group manifold of simple and semi-simple groups,* Proc. Acad. Amsterdam **29** (1926), 803–815.

[4] S. Helgason: *Differential Geometry and Symmetric Spaces,* Academic Press, 1962.

[5] M. Kikkawa: *On local loops in affine manifolds,* J. Sci. Hiroshima Univ. A–I, **28** (1964), 199–207.

[6] ———: *On the decomposition of a linearly connected manifold with torsion,* J. Sci. Hiroshima Univ. A–I, **33** (1969), 1–9.

[7] ———: *On locally reductive spaces and tangent algebras,* Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **5** (1972), 1–13.

[8] ———: *On some quasigroups of algebraic models of symmetric spaces,* Mem. Fac. Lit. Sci. Shimane Univ. Nat. Sci. **6** (1973), 9–13, *II,* ibid., **7** (1974), 29–35.

[9] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry I, II,* Interscience, 1963, 1969.

[10] M. Koecher: *Positivitätsbereiche im $\boldsymbol{R}^n$*, Amer. J. Math. **79** (1957), 575–596.

[11] A. Lichnérowicz: *Géometrie des Groupes de Transformations,* Dunod, 1958.

[12] O. Loos: *Spiegelungsräume und homogene symmetrische Räume,* Math. Z. **99** (1967), 141–170.

[13] ———: *Symmetric Spaces I,* Benjamin, 1969.

[14] A. I. Mal'cev: *Analytic loops* (in Russian), Mat. Sub. **36 (78)** (1955), 569–576.

[15] K. Nomizu: *Invariant affine connections on homogeneous spaces,* Amer. J. Math. **76** (1954), 33–65.

[16] T. Nôno: *Sur les familles triples locales de transformations locales de Lie,* J. Sci. Hiroshima Univ. A–I, **25** (1961), 357–366.

[17] A. A. Sagle and R. E. Walde: *Introduction to Lie Groups and Lie Algebras,* Academic Press, 1973.

[18] A. A. Sagle and J. R. Schumi: *Multiplications on homogeneous spaces, non associative algebras and connections,* Pacific J. Math. **48** (1973), 247–266.

[19] K. Yamaguti: *On the Lie triple systems and its generalization,* J. Sci. Hiroshima Univ. A, **21** (1958), 155–160.

[20] ———: *On cohomology groups of general Lie triple systems,* Kumamoto J. Sci. A, **8** (1969), 135–164.

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