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A Remark on Duality of Domination Principle

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In this paper we remark that we obtain a simpler proof of Higuchi's fundamental theorem (Theorem 1) in [1] if we treat four kernels. We stress that (a) and (b) in our theorem are mutually dual.

We shall use the same notation as in [1]. As the space X with which we are concerned we take a locally compact Hausdorff space satisfying the second axiom of countability as in [1].

THEOREM. *Let G, K, L and N be lower semi-continuous function-kernels on $X \times X$. Then the following statements are equivalent:*

(a) *For $\mu \in E_0$ and $\nu \in M_0$, the inequality $G\mu \leqq K\nu$ on $S\mu$ implies the inequality $L\mu \leqq N\nu$ on X.*

(b) *For $\mu \in E_0$ and $\nu \in M_0$, the inequality $\check{G}\mu \leqq \check{L}\nu$ on $S\mu$ implies the inequality $\check{K}\mu \leqq \check{N}\nu$ on X.*

To prove this theorem we need

LEMMA (cf. Lemmas 3 and 4 in [1]). *Let G, K, L and N be as above. Suppose that for any $\mu \in C_{\mathrm{supp}}(G)$ and $\nu \in M_0$, the inequality $G\mu \leqq K\nu$ on $S\mu$ implies the inequality $L\mu \leqq N\nu$ onX. Then, for any $\mu \in E_0$ and $\nu \in M_0$, $G\mu \leqq K\nu$ on $S\mu$ implies $L\mu \leqq N\nu$ on X.*

PROOF. Choose $\{\mu_n\}$ as in Lemma 2 of [1]. We have $G\mu_n \leqq G\mu \leqq K\nu$ on $S\mu_n \subset S\mu$. By our assumption $L\mu_n \leqq N\nu$ holds on X. Therefore $L\mu \leqq \underline{\lim}_{n\to\infty} L\mu_n \leqq N\nu$ on X. This proves our lemma.

The following proof of our theorem is analogous to that of Theorem 1 in [1]. We need only to prove (a)→(b). Let us assume (a). In view of our lemma it suffices to conclude $\check{K}\mu \leqq \check{N}\nu$ on X for $\mu \in C_{\mathrm{supp}}(G)$ and $\nu \in M_0$, assuming that $\check{G}\mu \leqq \check{L}\nu$ holds on $S\mu$. Take a sequence $\{G_n\}$ (resp. $\{K_n\}$) of finite continuous function-kernels on $X \times X$ increasing to G (resp. K). As in [1] there exists a sequence $\{\lambda_{n,p}\}_{p=1}^{\infty}$ of positive measures supported by $S\mu$ such that

$$G_p \lambda_{n,p} \geqq K_n \varepsilon_y \qquad \text{on} \quad S\mu,$$

$$G_p \lambda_{n,p} = K_n \varepsilon_y \qquad \text{on} \quad S\lambda_{n,p},$$

where ε_y means the unit measure at y. Let $\lambda_n \in E_0(G)$ be a vague adherent point of $\{\lambda_{n,p}\}$. Then we have by (4) of [1]

$$G\lambda_n \leqq K_n\varepsilon_y \leqq K\varepsilon_y \qquad \text{on} \quad S\lambda_n \subset S\mu.$$

By (a) we derive $L\lambda_n \leqq N\varepsilon_y$ on X. For any p

$$\check{K}_n\mu(y) = \int K_n\varepsilon_y d\mu \leqq \int G_p\lambda_{n,p} d\mu = \int \check{G}_p\mu d\lambda_{n,p} \leqq \int \check{G}\mu d\lambda_{n,p}.$$

Let $\{\lambda'_{n,q}\}_{q=1}^{\infty}$ be a subsequence of $\{\lambda_{n,p}\}$ which converges to λ_n vaguely. The finite continuity of the restriction of $\check{G}\mu$ to $S\mu$ implies

$$\check{K}_n\mu(y) \leqq \lim_{q\to\infty}\int \check{G}\mu d\lambda'_{n,q} = \int \check{G}\mu d\lambda_n \leqq \int \check{L}\nu d\lambda_n = \int L\lambda_n d\nu \leqq \int N\varepsilon_y d\nu = \check{N}\nu\,(y).$$

We obtain

$$\check{K}\mu(y) = \lim_{n\to\infty} \check{K}_n\mu(y) \leqq \check{N}\nu(y).$$

This is true for any $y \in X$. Thus (b) is concluded.

Reference

[1] I. Higuchi: Duality of domination principle for nonsymmetric lower semi-continuous function-kernels, Hiroshima Math. J. **5** (1975), 551–559.

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