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Duality of Domination Principle for Non-symmetric Lower Semi-continuous Function-kernels

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1. Introduction

Let G be a lower semi-continuous function-kernel on the product space of a locally compact Hausdorff space X, and $\check{G}$ be the adjoint kernel of G defined by $\check{G}(x, y)=G(y, x)$. We say that the duality of the domination principle holds for G when the following statement is true: $\check{G}$ satisfies the domination principle if and only if G does.

M. Kishi first proved in [5] that the duality of the domination principle holds for a lower semi-continuous function-kernel G under the additional condition that G and $\check{G}$ satisfy the continuity principle.

For a continuous (in the extended sense) function-kernel G, M. Itô and the author verified in [2] that both G and $\check{G}$ satisfy the continuity principle when G satisfies the domination principle (cf. [3]). From this fact, follows the duality of the domination principle for continuous function-kernels.

Concerning a lower semi-continuous (but not continuous) function-kernel, the domination principle does not imply the continuity principle. So the analogous argument does not hold.

Let G and N be lower semi-continuous function-kernels on X. In this paper, we shall verify that G satisfies the relative domination principle with respect to N if and only if $\check{G}$ satisfies the transitive domination principle with respect to $\check{N}$. This was first obtained by M. Kishi in [5] under the assumption that G and $\check{G}$ satisfy the continuity principle. For continuous function-kernels, the author [1] can avoid this additional condition.

The duality of the domination principle follows immediately from the above equivalence.

As applications of these results, we shall investigate the relations among the potential theoretical principles and shall show the transitive law for the relative domination principle.

2. Notation

Let X be a locally compact Hausdorff space satisfying the second axiom of countability[1]. A non-negative function $G(x, y)$ on $X \times X$ is called a lower semi-continuous function-kernel (simply an l.s.c. kernel) on X if $G(x, y)$ is lower semi-continuous on $X \times X$ and $0 < G(x, x) \leqq +\infty$ for any $x \in X$, and is called a continuous function-kernel on X if $G(x, y)$ is continuous in the extended sense on $X \times X$, $0 < G(x, x) \leqq +\infty$ for any $x \in X$ and $G(x, y) < +\infty$ at any point $(x, y) \in X \times X$ with $x \neq y$. If, in particular, a continuous function-kernel $G(x, y)$ is finitely valued on $X \times X$, it is called a finite continuous function-kernel. The kernel $\check{G}$ defined by $\check{G}(x, y) = G(y, x)$ is called the adjoint kernel of G.

For a positive Radon measure μ on X, the potential $G\mu(x)$ and the adjoint potential $\check{G}\mu(x)$ are defined by

$$G\mu(x) = \int G(x, y) d\mu(y) \text{ and } \check{G}\mu(x) = \int \check{G}(x, y) d\mu(y)$$

respectively. The G-energy of μ is defined by $\int G\mu(x) d\mu(x)$.

We denote by M_0 the family of all positive measures with compact support and by $E_0 = E_0(G)$ the family of all measures in M_0 with finite G-energy. Evidently $E_0(G) = E_0(\check{G})$. We say that a property holds G–p.p.p. on a subset A of X if it holds on A except for a set of the inner ν-measure 0 for every $\nu \in E_0(G)$.

3. Principles

In this paper we shall consider the following principles concerning l.s.c. kernels:

(I) *Domination principle.* For $\mu \in E_0$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq G\nu(x)$ on the support $S\mu$ of μ implies the same inequality on X.

(II) *Balayage principle.* For any compact set K and any $\mu \in M_0$, there exists a measure μ' in M_0 supported by K such that

$$G\mu'(x) = G\mu(x) \quad G\text{–}p.p.p. \text{ on } K,$$

$$G\mu'(x) \leqq G\mu(x) \quad \text{on } X.$$

We call μ' a balayaged measure of μ on K with respect to G.

(III) *Maximum principle.* For $\mu \in M_0$, an inequality $G\mu(x) \leqq 1$ on $S\mu$ implies the same inequality on X.

(IV) *Positive mass principle.* For $\mu \in E_0$ and $\nu \in M_0$, an inequality $G\mu(x)$

1) All the results in this paper hold with a slight modification when X does not necessarily satisfy the second axiom of countability.

$\leqq G\nu(x)$ on $S\mu$ implies the inequality $\int d\mu \leqq \int d\nu$.

(V) *Equilibrium principle.* For a given compact set K, there exists a positive measure γ, called a equilibrium measure, supported by K satisfying

$$G\gamma(x) = 1 \qquad G\text{-}p.\,p.\,p. \quad \text{on } K,$$

$$G\gamma(x) \leqq 1 \qquad \text{on } X.$$

(VI) *Complete maximum principle.* For $\mu \in E_0$, $\nu \in M_0$ and for a non-negative number a, an inequality $G\mu(x) \leqq G\nu(x) + a$ on $S\mu$ implies the same inequality on X.

(VII) *Continuity principle.* For $\mu \in M_0$, the finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies the finite continuity of $G\mu(x)$ on the whole space X.

(VIII) *Relative domination principle with respect to N* (written simply $G \prec N$). For $\mu \in E_0(G)$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq N\nu(x)$ on $S\mu$ implies the same inequality on X.

(IX) *Transitive domination principle with respect to N* (written simply $G \sqsubset N$). For $\mu \in E_0(G)$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq G\nu(x)$ on $S\mu$ implies the inequality $N\mu(x) \leqq N\nu(x)$ on X.

4. Relative domination principle

Some of the potential theoretical principles are the special ones of the relative domination principle or of the transitive domination principle. In this section we shall discuss the equivalence of these two principles.

Throughout this paper, the following existence theorem plays a fundamental role.

EXISTENCE THEOREM OF KISHI ([6]). *Let K be a compact Hausdorff space, G be a finite continuous function-kernel on K and u(x) be a non-negative finite continuous function on X. Then there exists a positive measure λ on K such that*

$$G\lambda(x) \geqq u(x) \qquad on \quad K,$$

$$G\lambda(x) = u(x) \qquad on \quad S\lambda.$$

The following modification of the above existence theorem was obtained in [1].

LEMMA 1. *Let G be a lower semi-continuous function-kernel on X, K be a compact subset of X and $u(x)$ be a non-negative finite continuous function on X. Take a sequence $\{G_n\}$ of finite continuous function-kernels which increases to G at each point of $X \times X$ with n (written $G_n \nearrow G$). Then there exists a vaguely bounded sequence $\{\lambda_n\}$ of positive measures supported by K satisfying*

$$G_n\lambda_n(x) \geqq u(x) \quad \text{on} \quad K, \tag{1}$$

$$G_n\lambda_n(x) = u(x) \quad \text{on} \quad S\lambda_n. \tag{2}$$

Each vague adherent point λ of $\{\lambda_n\}$ satisfies

$$G\lambda(x) \geqq u(x) \quad \nu\text{–a.e. on } K \text{ for every } \nu \in C(\check{G}; K), \tag{3}$$

$$G\lambda(x) \leqq u(x) \quad \text{on} \quad S\lambda, \tag{4}$$

where $C(\check{G}; K)$ denotes the family of all meaures in M_0 supported by K such that $\check{G}\nu(x)$ is finite continuous as a function on K.

The following lemma is essentially contained in Lemma 2 in [4]. We denote by $C_{\text{supp}}(G)$ the family of all measures in $E_0(G)$ whose potentials are finite continuous as functions on their own supports.

LEMMA 2. *Let G be an l.s.c. kernel on X. Then for any measure μ in $E_0(G)$, there exists a sequence $\{\mu_n\}$ of positive measures supported by $S\mu$ with the following properties: 1° $\mu_n \in C_{\text{supp}}(G)$, 2° $\{\mu_n\}$ converges vaguely to μ and 3° $\{G\mu_n(x)\}$ converges increasingly to $G\mu(x)$ at every point x of X.*

This lemma gives the following two lemmas which assert that $E_0(G)$ may be replaced by $C_{\text{supp}}(G)$ in the argument of our domination principles.

LEMMA 3. *Let G and N be l.s.c. kernels on X. Suppose that for any $\mu \in C_{\text{supp}}(G)$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq N\nu(x)$ on $S\mu$ implies the same inequality on X. Then $G \prec N$.*

PROOF. Assume that for $\mu \in E_0(G)$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq N\nu(x)$ holds on $S\mu$. We can find, by Lemma 2, a sequence $\{\mu_n\}$ of positive measures with the following properties: 1° $\mu_n \in C_{\text{supp}}(G)$, 2° $\mu_n \to \mu$ vaguely as $n \to +\infty$ and $G\mu_n(x) \nearrow G\mu(x)$ as $n \to +\infty$ at every $x \in X$. Then $G\mu_n(x) \leqq G\mu(x) \leqq N\nu(x)$ on $S\mu$ for every n. The assumption of our lemma asserts that $G\mu_n(x) \leqq N\nu(x)$ on X. Therefore we have

$$G\mu(x) = \lim G\mu_n(x) \leqq N\nu(x) \quad \text{on} \quad X.$$

This implies that $G \prec N$.

In the same way, we have

LEMMA 4. *Let G and N be l.s.c. kernels on X. Suppose that for any $\mu \in C_{\mathrm{supp}}(G)$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq G\nu(x)$ on $S\mu$ implies the inequality $N\mu(x) \leqq N\nu(x)$ on X. Then $G \sqsubset N$.*

Now we can prove

THEOREM 1. *Let G and N be l.s.c. kernels on X. Then the following statements are equivalent.*

(a) *G satisfies the relative domination principle with respect to N.*

(b) *$\check{G}$ satisfies the transitive domination principle with respect to $\check{N}$.*

PROOF. (a)→(b). Suppose that $\check{G}\mu(x) \leqq \check{G}\nu(x)$ holds on $S\mu$ for any $\mu \in C_{\mathrm{supp}}(\check{G})$ and $\nu \in M_0$. It suffices, by virtue of Lemma 4, to prove that $\check{N}\mu(x) \leqq \check{N}\nu(x)$ holds on X. Take a sequence $\{G_n\}$ (resp. $\{N_n\}$) of finite continuous function-kernels on X satisfying $G_n \nearrow G$ (resp. $N_n \nearrow N$). Then for any positive integer n and any $y \in X$, there exists, by Lemma 1, a sequence $\{\lambda_{n,p}\}_{p=1}^{+\infty}$ of positive measures supported by $S\mu$ satisfying

$$G_p\lambda_{n,p}(x) \geqq N_n\varepsilon_y(x) \qquad \text{on} \quad S\mu,$$

$$G_p\lambda_{n,p}(x) = N_n\varepsilon_y(x) \qquad \text{on} \quad S\lambda_{n,p},$$

where ε_y denotes the unit measure at y. Let $\lambda_n \in E_0(G)$ be a vague adherent point of $\{\lambda_{n,p}\}_{p=1}^{+\infty}$. We shall denote a subsequence convergent to λ_n still by $\{\lambda_{n,p}\}$. We have by (4)

$$G\lambda_n(x) \leqq N_n\varepsilon_y(x) \leqq N\varepsilon_y(x) \qquad \text{on} \quad S\lambda_n \subset S\mu.$$

By (a), we derive $G\lambda_n(x) \leqq N\varepsilon_y(x)$ on X. Then, for any p,

$$\check{N}_n\mu(y) = \int N_n\varepsilon_y d\mu \leqq \int G_p\lambda_{n,p}d\mu = \int \check{G}_p\mu d\lambda_{n,p}$$
$$\leqq \int \check{G}\mu d\lambda_{n,p}.$$

The finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies

$$\check{N}_n\mu(y) \leqq \varliminf_{p \to +\infty} \int \check{G}\mu d\lambda_{n,p} = \int \check{G}\mu d\lambda_n \leqq \int \check{G}\nu d\lambda_n$$
$$\leqq \int G\lambda_n d\nu \leqq \int N\varepsilon_y d\nu = \check{N}\nu(y).$$

Letting n tend to $+\infty$, we have $\check{N}\mu(y) \leqq \check{N}\nu(y)$ for any $y \in X$. This implies (b).

(b)→(a). We suppose that an inequality $G\mu(x) \leqq N\nu(x)$ holds on $S\mu$ for

$\mu \in C_{\mathrm{supp}}(G)$ and $\nu \in M_0$. By Lemma 3, we have only to prove that the inequality $G\mu(y) \leqq N\nu(y)$ holds for any $y \in CS\mu$. Let $\{G_n\}$ be a sequence of finite continuous function-kernels satisfying $G_n \nearrow G$. Then for any positive integer n and any $y \in CS\mu$, there exists, by Lemma 1, a sequence $\{\tilde{\lambda}_{n,p}\}_{p=1}^{+\infty}$ of positive measures supported by $S\mu$ satisfying

$$\check{G}_p \tilde{\lambda}_{n,p}(x) \geqq \check{G}_n \varepsilon_y(x) \quad \text{on} \quad S\mu,$$

$$\check{G}_p \tilde{\lambda}_{n,p}(x) = \check{G}_n \varepsilon_y(x) \quad \text{on} \quad S\tilde{\lambda}_{n,p}.$$

A vague adherent point $\tilde{\lambda}_n \in E_0(G)$ of $\{\tilde{\lambda}_{n,p}\}_{p=1}^{+\infty}$ fulfils

$$\check{G}\tilde{\lambda}_n(x) \leqq \check{G}_n \varepsilon_y(x) \leqq \check{G}\varepsilon_y(x) \quad \text{on} \quad S\tilde{\lambda}_n \subset S\mu.$$

By (b), we obtain

$$\check{N}\tilde{\lambda}_n(x) \leqq \check{N}\varepsilon_y(x) \quad \text{on} \quad X.$$

Then, for any p,

$$G_n\mu(y) = \int \check{G}_n \varepsilon_y d\mu \leqq \int \check{G}_p \tilde{\lambda}_{n,p} d\mu \leqq \int G\mu d\tilde{\lambda}_{n,p}.$$

By the finite continuity of the restriction of $G\mu(x)$ to $S\mu$, we have the following inequalities:

$$G_n\mu(y) \leqq \varliminf_{p \to +\infty} \int G\mu d\tilde{\lambda}_{n,p} = \int G\mu d\tilde{\lambda}_n \leqq \int N\nu d\tilde{\lambda}_n$$

$$\leqq \int \check{N}\tilde{\lambda}_n d\nu \leqq \int \check{N}\varepsilon_y d\nu = N\nu(y).$$

Letting n tend to $+\infty$, we have $G\mu(y) \leqq N\nu(y)$ for any $y \in CS\mu$.

Remark 1. This theorem was first proved by M. Kishi [5] under the additional condition that G and $\check{G}$ satisfy the continuity principle. For continuous function-kernels, we discussed in [1] the transitive domination principle only for measures with disjoint supports and obtained the analogous result without assuming the continuity principle.

5. Duality of domination principle

Putting $N = G$ in Theorem 1, we obtain the following duality of the domination principle.

Theorem 2. *Let G be an l.s.c. kernel on X. Then $\check{G}$ satisfies the domination principle if and only if G does.*

REMARK 2. Under the additional assumption that G and $\check{G}$ satisfy the continuity principle, M. Kishi [5] showed the duality of the domination principle and obtained the equivalence of the balayage principle and the domination principle. As shown in Theorem 2, the continuity principle is avoidable in the argument of the duality of the domination principle. But we can not assert, without assuming the continuity principle, the equivalence of the balayage principle and the domination principle.

It is easily seen that, if an l.s.c. kernel G satisfies the balayage principle, then $\check{G}$ satisfies the domination principle. Accordingly we have the following

COROLLARY. *Let G be an l.s.c. kernel on X. Then G satisfies the domination principle when G satisfies the balayage principle.*

6. Maximum principle and complete maximum principle

Theorem 1 gives the following characterization of the maximum principle.

THEOREM 3. *Let G be an l.s.c. kernel on X. Then G satisfies the maximum principle if and only if $\check{G}$ satisfies the positive mass principle.*

PROOF. The maximum principle for G implies that $G \prec 1$. From Theorem 1, it follows that $G \prec 1$ if and only if $\check{G} \sqsubset \check{1} = 1$. The transitive domination principle $\check{G} \sqsubset 1$ is just the positive mass principle for $\check{G}$.

COROLLARY 1. *Let G be an l.s.c. kernel on X. Then G satisfies the complete maximum principle if and only if G satisfies the maximum principle and the domination principle.*

REMARK 3. We have the following generalization of this corollary:
If G, N and K are l.s.c. kernels on X such that $G \prec N$ and $G \prec K$, then $G \prec (N+K)$[2].

COROLLARY 2. *An l.s.c. kernel G on X satisfies the maximum principle when G satisfies the equilibrium principle.*

PROOF. Suppose that $\check{G}\mu(X) \leqq \check{G}\nu(x)$ holds on $S\mu$ for $\mu \in E_0$ and $\nu \in M_0$. Let γ be an equilibrium measure of $S\mu$. Then

$$\int d\mu \leqq \int G\gamma d\mu = \int \check{G}\mu d\gamma \leqq \int \check{G}\nu d\gamma = \int G\gamma d\nu \leqq \int d\nu .$$

2) The author first proved this generalization in the same way as in the proof of Theorem 1. Professor F.-Y. Maeda remarked to the author that this is an immediate consequence of Theorem 1.

Therefore $\check{G}$ satisfies the positive mass principle. Theorem 3 asserts that G satisfies the maximum principle.

Remark 4. We can generalize the above corollary into the following form: Let G and N be l.s.c. kernels on X. Then G satisfies the relative domination principle with respect to N when G satisfies the relative balayage principle with respect to N[3].

7. Transitive law for the relative domination principle

Finally we shall prove that the relation $\prec$ fulfils the transitive law under an additional condition.

Theorem 5. *Suppose that G, N and K are l.s.c. kernels on X such that $G\prec N$ and $N\prec K$. If $N(x, y)$ is locally bounded outside the diagonal set of $X\times X$, then $G\prec K$.*

Proof. Suppose that $G\mu(x)\leqq K\nu(x)$ holds on $S\mu$ for $\mu\in\dot{E}_0(G)$ and $\nu\in M_0$. We may assume, by Lemma 3, that $G\mu(x)$ is finite continuous as a function on $S\mu$. It suffices to prove that for any integer $n\geqq 1$ and $y\in CS\mu$, the inequality $G_n\mu(y)\leqq K\nu(y)$ holds, where $\{G_n\}_{n=1}^{+\infty}$ is a sequence of finite continuous function-kernels such that $G_n\nearrow G$. By Lemma 1, there exists a sequence $\{\tilde{\lambda}_{n,p}\}_{p=1}^{+\infty}$ of measures in M_0 supported by $S\mu$ such that

$$\check{G}_p\tilde{\lambda}_{n,p}(x) \geqq \check{G}_n\varepsilon_y(x) \qquad \text{on} \quad S\mu,$$

$$\check{G}_p\tilde{\lambda}_{n,p}(x) = \check{G}_n\varepsilon_y(x) \qquad \text{on} \quad S\tilde{\lambda}_{n,p}.$$

A vague adherent point $\tilde{\lambda}_n\in E_0(G)$ of $\{\tilde{\lambda}_{n,p}\}_{p=1}^{+\infty}$ fulfils

$$\check{G}\tilde{\lambda}_n(x) \leqq \check{G}_n\varepsilon_y(x) \qquad \text{on} \quad S\tilde{\lambda}_n\subset S\mu.$$

The assumption $G\prec N$ implies that $\check{G}\sqsubset\check{N}$ and therefore we have $\check{N}\tilde{\lambda}_n(x)\leqq\check{N}\varepsilon_y(x)$ on X. Then $\tilde{\lambda}_n\in E_0(N)$, because $N(x, y)$ is locally bounded outside the diagonal set of $X\times X$ and $S\tilde{\lambda}_n\cap\{y\}=\phi$. Consequently $\check{K}\tilde{\lambda}_n(x)\leqq\check{K}\varepsilon_y(x)$ on X, because $\check{N}\sqsubset\check{K}$. Since

$$G_n\mu(y) = \int\check{G}_n\varepsilon_y d\mu \leqq\int\check{G}_p\tilde{\lambda}_{n,p}d\mu \leqq\int G\mu d\tilde{\lambda}_{n,p},$$

by the finite continuity of the restriction of $G\mu(x)$ to $S\mu$, we have

3) We say that G satisfies the relative balayage principle with respect to N when for any compact set K and any $\mu\in M_o$, there exists a measure $\tilde{\mu}$ in M_o supported by K such that

$$G\tilde{\mu}(x) = N\mu(x)\ G\text{-}p.p.p. \text{ on } K \quad\text{and}\quad G\tilde{\mu}(x)\leqq N\mu(x) \text{ on } X.$$

$$G_n\mu(y) \leqq \varliminf_{p\to+\infty} \int G\mu d\tilde{\lambda}_{n,p} = \int G\mu d\tilde{\lambda}_n \leqq \int K\nu d\tilde{\lambda}_n$$

$$\leqq \int \check{K}\tilde{\lambda}_n d\nu \leqq \int \check{K}\varepsilon_y d\nu = K\nu(y).$$

Letting n tend to $+\infty$, we have $G\mu(y) \leqq K\nu(y)$ for any $y \in CS\mu$.

Putting $K=1$ in Theorem 5, we obtain the following

COROLLARY. *Let G and N be l.s.c. kernels on X such that $G \prec N$. Suppose that $N(x, y)$ is locally bounded outside the diagonal set of $X \times X$. Then G satisfies the maximum principle if N does.*

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