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Periodic Solutions for Certain Time-dependent Parabolic Variational Inequalities

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Introduction

For a real Banach space V we denote by V^* the dual space of V, by $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$ the norms in V and V^*, respectively, and by $(\cdot,\cdot)_V$ the natural pairing between V^* and V. A (multivalued) operator A from a Banach space V into its dual V^* (i.e., assigning to each $v \in V$ a subset Av of V^*) is called monotone if

$$(v^*-w^*, v-w)_V \geqq 0 \qquad \text{for any} \quad [v, v^*], [w, w^*] \in G(A),$$

where $G(A)$ is the graph of the operator A, i.e.,

$$G(A) = \{[v, v^*] \in V \times V^*: v \in D(A) \text{ and } v^* \in Av\}$$

with $D(A) = \{v \in V: Av \neq \phi\}$. If A is monotone and there is no proper monotone extension of A, then A is called maximal monotone.

Throughout this paper we let H be a Hilbert space and X a Banach space such that $X \subset H$, X is dense in H and the natural injection from X into H is continuous, and suppose that X is uniformly convex and X^* is strictly convex. Identifying H with its dual space by means of the inner product $(\cdot,\cdot)_H$ in H, we have the relation $X \subset H \subset X^*$. By the symbols "$\xrightarrow{s}$" and "$\xrightarrow{w}$" we mean the convergence in the strong and weak topology, respectively.

Let $0<T<\infty$, $2 \leqq p < \infty$ and $1/p + 1/p' = 1$ and let ψ be an extended real-valued function on $[0, T] \times X$ such that for each $t \in [0, T]$, $\psi(t;\cdot)$ is a lower semicontinuous convex function on X with values in $(-\infty, +\infty]$, $\psi(t;\cdot) \not\equiv +\infty$, and such that for each $v \in L^p(0, T; X)$, $t \to \psi(t; v(t))$ is measurable on $[0, T]$. We define a functional Ψ on $L^p(0, T; X)$ by

$$\Psi(v) = \begin{cases} \int_0^T \psi(t; v(t))dt & \text{if} \quad v \in D(\Psi), \\ +\infty & \text{otherwise}, \end{cases}$$

where $D(\Psi) = \{v \in L^p(0, T; X): t \to \psi(t; v(t)) \text{ is integrable on } (0, T)\}$.

We now pose the following problem: Given an $f \in L^{p'}(0, T; X^*)$, find a $u \in D(\Psi) \cap C([0, T]; H)$ such that

(i) $u(0) = u(T)$,

(ii) $u' \,(=(d/dt)u) \in L^{p'}(0, T; X^*)$,

(iii) $\int_0^T (u'-f, u-v)_X dt \leqq \Psi(v)-\Psi(u)$ for every $v \in D(\Psi)$.

This problem is referred to as the problem $P[\psi, f]$. A weak solution of the problem $P[\psi, f]$ is defined to be a function $u \in D(\Psi)$ which satisfies

$$\int_0^T (v'-f, u-v)_X dt \leqq \Psi(v)-\Psi(u)$$

whenever $v \in D(\Psi) \cap C([0, T]; H)$, $v' \in L^{p'}(0, T; X^*)$ and $v(0)=v(T)$.

We consider the following operator M_p(resp. S_p) from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$: $[u, f] \in G(M_p)$ (resp. $G(S_p)$) if and only if u is a weak (resp. strong) solution of the problem $P[\psi, f]$.

The purpose of this paper is to prove under appropriate assumptions on ψ and f the existence of a strong solution of the problem $P[\psi, f]$ and then to investigate the properties of the operators M_p and S_p. In Section 1 we summarize some results concerning the initial value problem for the above inequality (iii) (cf. [1, 2, 8, 9, 10]). In Section 2 we show that the problem $P[\psi, f]$ has a strong solution by using the results of Section 1 and a fixed point theorem of Browder and Petryshyn [7]. In Section 3 we show that M_p is a maximal monotone operator from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ and is a kind of closure of S_p. This result extends a theorem of Brèzis [5, Theorem II.16] to the time-dependent case.

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1. Initial value problem

Let $\{\psi(t; \cdot): 0 \leqq t \leqq T\}$ be a family of functions as described in the introduction. We put

$$D_t = \{z \in X: \psi(t; z) < \infty\} \qquad \text{for each} \quad t \in [0, T]$$

and $D_H = \{\text{the closure of } D_0 \text{ in } H\}$.

We impose the following two hypotheses on ψ.

(ψ.1) There is a positive constant C with the property: For each $t \in [0, T]$, $z \in D_t$ and $s \in [t, T]$, there is $\tilde{z} \in D_s$ such that

$$\|z-\tilde{z}\|_X \leqq C|t-s|, \quad \text{and}$$

$$\psi(s; \tilde{z}) \leqq \psi(t; z)+C|t-s|(1+\|z\|_X^p+|\psi(t; z)|).$$

(ψ.2) There are positive constants b_0, b_1 and b_2 such that

$$\psi(t; z)+b_0\|z\|_X+b_1 \geqq b_2[z]_X^p \quad \text{for any} \quad t \in [0, T] \text{ and } z \in X,$$

where $[\cdot]_X$ is a seminorm on X so that $[\cdot]_X+\|\cdot\|_H$ gives a norm on X which is equivalent to the norm $\|\cdot\|_X$.

Under these hypotheses we have the following

PROPOSITION 1 (*Kenmochi* [8, 9]). (1) *For any given* $u_0 \in D_0$ *and* $f \in L^{p'}(0, T; X^*)$ *with* $f' \in L^{p'}(0, T; X^*)$, *there exists a function* $u \in D(\Psi) \cap C([0, T]; H)$ *such that*

$$(1.1)\qquad \begin{cases} u(0) = u_0, \quad u' \in L^2(0, T; H), \\ t \longrightarrow \psi(t; u(t)) \text{ is bounded on } [0, T], \text{ and} \\ \int_0^T (u'-f, u-v)_X dt \leqq \Psi(v)-\Psi(u) \qquad \text{for every} \quad v \in D(\Psi). \end{cases}$$

(2) *Let* u_i *be a function in* $D(\Psi) \cap C([0, T]; H)$ *which satisfies* (1.1) *for* $u_0 = u_{0,i} \in D_0$ *and* $f = f_i \in L^{p'}(0, T; X^*)$ *with* $f_i' \in L^{p'}(0, T; X^*)$ $(i=1, 2)$. *Then, for* $s, t \in [0, T]$ *with* $s \leqq t$,

$$(1.2)\qquad \|u_1(t)-u_2(t)\|_H^2 - \|u_1(s)-u_2(s)\|_H^2 \leqq 2\int_s^t (f_1-f_2, u_1-u_2)_X dr.$$

Using Proposition 1 and a result in [10] we can prove the following proposition.

PROPOSITION 2. (1) *For any given* $u_0 \in D_H$ *and* $f \in L^{p'}(0, T; X^*)$, *there exists a function* $u \in D(\Psi) \cap C([0, T]; H)$ *such that*

$$(1.3)\qquad \begin{cases} u(0) = u_0, \quad \text{and} \\ \int_0^T (v'-f, u-v)_X dt - \frac{1}{2}\|u_0 - v(0)\|_H^2 \leqq \Psi(v)-\Psi(u) \\ \text{for every} \quad v \in D(\Psi) \cap C([0, T]; H) \text{ with } v' \in L^{p'}(0, T; X^*). \end{cases}$$

(2) *If* u_i *is a function in* $D(\Psi) \cap C([0, T]; H)$ *satisfying* (1.3) *with* $u_0 = u_{0,i} \in D_H$ *and* $f = f_i \in L^{p'}(0, T; X^*)$ $(i=1, 2)$, *then the inequality* (1.2) *holds for any* $s, t \in [0, T]$ *with* $s \leqq t$.

PROOF. The assertion (2) is true by Corollary 1 of [10]. Hence, we need only to verify the assertion (1). For this purpose choose sequences $\{u_{0,n}\} \subset D_0$ and $\{f_n\} \subset L^{p'}(0, T; X^*)$ such that $f_n' \in L^{p'}(0, T; X^*)$, $u_{0,n} \xrightarrow{s} u_0$ in H and $f_n \xrightarrow{s} f$ in $L^{p'}(0, T; X^*)$. By Proposition 1 there exists, for each n, a function $u_n \in D(\Psi) \cap C([0, T]; H)$ satisfying (1.1) with $u_0 = u_{0,n}$ and $f = f_n$. Since

$$\int_0^T (u_n'-f_n, u_n-v)_X dt \leqq \Psi(v)-\Psi(u_n) \qquad \text{for every} \quad v \in D(\Psi),$$

we have by integration by parts

$$(1.4)\qquad \int_0^T (v'-f_n,\, u_n-v)_X dt - \frac{1}{2}\|u_{0,n}-v(0)\|_H^2 \leqq \Psi(v)-\Psi(u_n)$$

for every $v \in D(\Psi) \cap C([0, T]; H)$ with $v' \in L^{p'}(0, T; X^*)$. Taking u_1 as v in (1.4) and using the assumption (ψ.2), we obtain

$$(1.5)\qquad b_2\int_0^T [u_n(t)]_X^p dt \le b_1 T + \Psi(u_1) + \frac{1}{2}\|u_{0,n}-u_{0,1}\|_H^2$$
$$+\int_0^T (f_n-u_1', u_1)_X dt + \int_0^T (b_0 + \|f_n-u_1'\|_{X^*})\|u_n\|_X dt.$$

On the other hand, it follows from the inequality (1.2) that for any $t \in [0, T]$

$$\|u_n(t)-u_1(t)\|_H^p$$
$$\leqq 2^p\|u_{0,n}-u_{0,1}\|_H^p + 2^{p+1}\left(\int_0^T \|f_n-f_1\|_{X^*}\|u_n-u_1\|_X dt\right)^{p/2}$$
$$(1.6)\qquad \leqq 2^p\|u_{0,n}-u_{0,1}\|_H^p + 2^{p+1}\left(\int_0^T \|f_n-f_1\|_{X^*}^{p'} dt\right)^{p/2p'}\left(\int_0^T \|u_n-u_1\|_X^p dt\right)^{1/2}$$
$$\le 2^p\|u_{0,n}-u_{0,1}\|_H^p + \frac{2^p}{\varepsilon}\left(\int_0^T \|f_n-f_1\|_{X^*}^{p'} dt\right)^{p/p'}$$
$$+\frac{\varepsilon}{2}\int_0^T \|u_n-u_1\|_X^p dt,$$

where ε is an arbitrary positive number. Noting that $\|\cdot\|_X$ is equivalent to $[\cdot]_X + \|\cdot\|_H$, we see from (1.5) and (1.6) that $\{u_n\}$ is bounded in $L^p(0, T; X)$. By (1.4) and (ψ.2) it follows that $\{\Psi(u_n)\}$ is bounded.

Now, the inequality (1.2) implies that $\{u_n\}$ converges in H uniformly on $[0, T]$ to a function $u \in C([0, T]; H)$ with $u(0)=u_0$. Then, obviously, $u \in L^p(0, T; X)$, $u_n \xrightarrow{w} u$ in $L^p(0, T; X)$ as $n \to \infty$, and since Ψ is lower semicontinuous on $L^p(0, T; X)$ by (ψ.1) and (ψ.2),

$$-\infty < \Psi(u) \leqq \liminf_{n\to\infty} \Psi(u_n) < +\infty.$$

Letting $n \to \infty$ in (1.4), we see that u is the desired function.

The following is an immediate consequence of Propositions 1 and 2.

PROPOSITION 3. (1) *For any given $u_0 \in D_H$ and $f \in L^{p'}(0, T; X^*)$ with $f' \in L^{p'}(0, T; X^*)$, there is a function $u \in D(\Psi) \cap C([0, T]; H)$ such that $u(0)=u_0$ and the following holds for each $\delta \in (0, T]$:*

$$(1.7)\qquad \begin{cases} u' \in L^2(\delta, T; H), \\ t \longrightarrow \psi(t; u(t)) \textit{ is bounded on } [\delta, T], \textit{ and} \\ \int_\delta^T (u'-f, u-v)_X dt \leqq \int_\delta^T \{\psi(t; v(t)) - \psi(t; u(t))\} dt \\ \textit{for every} \quad v \in D(\Psi). \end{cases}$$

(2) *Let u_i be a function in $D(\Psi) \cap C([0, T]; H)$ which satisfies* (1.7) *for $u_0 = u_{0,i} \in D_H$ and $f = f_i \in L^{p'}(0, T; X^*)$ with $f_i' \in L^{p'}(0, T; X^*)$ $(i=1, 2)$. Then the inequality* (1.2) *holds for any $s, t \in [0, T]$ with $s \leqq t$.*

REMARK 1.1. In case $X = H$ and $p = 2$ the hypothesis (ψ.1) can be replaced by the following weaker one:

(ψ.1)′ There is a positive nondecreasing function $r \to C(r)$ with the following property: For each $r > 0$, each pair $s, t \in [0, T]$, $s \leqq t$, and for each $z \in D_s$ with $\|z\|_H \leqq r$ there is $\tilde{z} \in D_t$ such that

$$\|\tilde{z} - z\|_H \leqq C(r)|t-s|$$

and

$$\psi(t; \tilde{z}) \leqq \psi(s; z) + C(r)|t-s|(1 + |\psi(s; z)|).$$

In this case Propositions 1, 2 and 3 hold without the condition $f' \in L^2(0, T; H)$, and moreover, the function u appearing in the first statement of Proposition 2 (and 3) is such that $t \to t\psi(t; u(t))$ is bounded on $(0, T]$ (cf. [9]).

2. Existence of a strong solution of $P[\psi, f]$

Let $\{\psi(t; \cdot): 0 \leqq t \leqq T\}$ be a family of functions as described in the introduction. Throughout this section it is assumed that this family satisfies in addition to (ψ.1) the assumptions (ψ.2)′ and (ψ.3) given below.

(ψ.2)′ There are positive constants C_1 and C_2 such that

$$\psi(t; z) \geqq C_1\|z\|_X^p - C_2 \qquad \text{for all} \quad t \in [0, T] \text{ and } z \in X.$$

(ψ.3) $D_T \subset D_0$, i.e., $\{z \in X: \psi(T; z) < \infty\} \subset \{z \in X: \psi(0; z) < \infty\}$.

The objective here is to prove the existence of a strong solution of the problem $P[\psi, f]$ using a fixed point theorem of Browder and Petryshyn [7] and techniques similar to those developed in [3] and [4].

LEMMA 1. *Let $f \in L^\infty(0, T; X^*)$ and let $\{u_n\} \subset D(\Psi) \cap C([0, T]; H)$ be a sequence such that $u_n' \in L^2(0, T; H)$ and*

$$\int_0^T (u_n' - f, u_n - v)_X dt \leqq \Psi(v) - \Psi(u) \qquad \textit{for every} \quad v \in D(\Psi).$$

If the sequence $\{\|u_n(0)\|_H - \|u_n(T)\|_H\}$ *is bounded above, then the sequence* $\{u_n(T)\}$ *is bounded in* H *and moreover,* $\{u_n\}$ *is bounded in* $C([0, T]; H)$.

PROOF. In view of (2) of Proposition 1 we see that

$$\|u_n(t) - u_1(t)\|_H \leqq \|u_n(s) - u_1(s)\|_H \tag{2.1}$$

for any $s, t \in [0, T]$ with $s \leqq t$. Now suppose for contradiction that $\{u_n(T)\}$ is not bounded in H. Then we may assume, taking a subsequence if necessary, that $\|u_n(T)\|_H \to \infty$ as $n \to \infty$. Thus it follows from (2.1) that $\inf_{0 \leqq t \leqq T} \|u_n(t)\|_H \to \infty$ as $n \to \infty$.

We choose a Lipschitz continuous function h from $[0, T]$ into X such that $t \to \psi(t; h(t))$ is bounded on $[0, T]$. It is known that under the hypotheses $(\psi.1)$ and $(\psi.2)$ such a function h does indeed exist (cf. [9, Lemma 3.3]). Let L be an arbitrary number such that $L > C = \operatorname{ess\,sup}_{0 \leqq t \leqq T} \|f - h'\|_{X^*}$. Since $\inf_{0 \leqq t \leqq T} \|u_n(t) - h(t)\|_H \to \infty$ as $n \to \infty$, the assumption $(\psi.2)'$ implies that

$$\psi(t; u_n(t)) - \psi(t; h(t)) \geqq L\|u_n(t) - h(t)\|_X \quad \text{for a.a. } t \in [0, T],$$

provided that n is sufficiently large.

Therefore, for each pair $s, t \in [0, T]$ with $s \leqq t$,

$$\begin{aligned}
&\int_s^t (f - h', u_n - h)_X dr \\
&\geqq \int_s^t (u_n' - h', u_n - h)_X dr + \int_s^t \{\psi(r; u_n) - \psi(r; h)\} dr \\
&\geqq \int_s^t (u_n' - h', u_n - h)_X dr + L\int_s^t \|u_n - h\|_X dr \\
&= \frac{1}{2}(\|u_n(t) - h(t)\|_H^2 - \|u_n(s) - h(s)\|_H^2) + L\int_s^t \|u_n - h\|_X dr,
\end{aligned}$$

so that for each pair $s, t \in [0, T]$, $s \leqq t$,

$$\frac{1}{2}(\|u_n(t) - h(t)\|_H^2 - \|u_n(s) - h(s)\|_H^2) + (L - C)C_3^{-1}\int_s^t \|u_n - h\|_H dr \leqq 0,$$

where C_3 is a positive constant such that $\|x\|_H \leqq C_3\|x\|_X$ for every $x \in X$. This inequality implies that

$$\|u_n(t) - h(t)\|_H - \|u_n(s) - h(s)\|_H + (L - C)C_3^{-1}(t - s) \leqq 0$$

for any $t, s \in [0, T]$ with $s \leqq t$. Therefore, taking $s = 0$ and $t = T$, we have

$$(L-C)C_3^{-1} \leqq \|u_n(0)-h(0)\|_H - \|u_n(T)-h(T)\|_H,$$

which contradicts the hypothesis that $\{\|u_n(0)\|_H - \|u_n(T)\|_H\}$ is bounded above. Hence, it must be true that $\{u_n(T)\}$ is bounded in H.

Combining the inequality (2.1) with the fact that $\{u_n(0)\}$ is bounded in H, we readily conclude that $\{u_n\}$ is bounded in $C([0, T]; H)$. This completes the proof.

One of the main results of this paper is the following existence theorem.

THEOREM 1. *For a given* $f \in L^{p'}(0, T; X^*)$ *with* $f' \in L^{p'}(0, T; X^*)$, *there exists a function* $u \in D(\Psi) \cap C([0, T]; H)$ *such that*

(i) $u(0) = u(T)$;

(ii) $u' \in L^2(0, T; H)$;

(iii) $\int_0^T (u'-f, u-v)_X dt \leqq \Psi(v) - \Psi(u)$ *for every* $v \in D(\Psi)$.

PROOF. Let x be any element of D_H. According to (1) of Proposition 3 there exists a unique function $u \in D(\Psi) \cap C([0, T]; H)$ with initial value $u_0 = x$ and satisfying (1.7). Then we put $Sx = u(T)$. In this manner we can define a (singlevalued) operator S from the closed convex set D_H in H into itself. From Proposition 3 and the assumption (ψ.3) it follows that the range of S is contained in D_0 and that S is contractive on D_H, i.e.,

$$\|Sx - Sy\|_H \leqq \|x-y\|_H \qquad \text{for all} \quad x, y \in D_H.$$

Now we form the sequence of iterates $\{S^n x\}$ for an arbitrary but fixed $x \in D_0$. By definition, $S^n x$ is the value at $t = T$ of the function $u_n(t)$ which satisfies (1.1) with $u_0 = S^{n-1}x$. Then,

$$\|u_n(0)\|_H - \|u_n(T)\|_H \leqq \|S^{n-1}x - S^n x\|_H \leqq \|Sx - x\|_H,$$

which shows that $\{\|u_n(0)\|_H - \|u_n(T)\|_H\}$ is bounded above. Since $\{S^n x\}$ is bounded in H by Lemma 1, we can apply a fixed point theorem of Browder and Petryshyn [7] to conclude that S has a fixed point $\tilde{u}$: $S\tilde{u} = \tilde{u}$. Let u be the function which satisfies (1.1) with $u_0 = \tilde{u}$. Then it is easy to see that this u is the required solution of our problem. Thus the proof is complete.

REMARK 2.1. In case $X = H$ and $p = 2$, modifying slightly the proof of Lemma 1, we see that the conclusion of Lemma 1 is valid if $f \in L^2(0, T; H)$. Hence, if (ψ.1) is replaced by (ψ.1)$'$, then the conclusion of Theorem 1 holds for $f \in L^2(0, T; H)$ without the condition $f' \in L^2(0, T; H)$. See Remark 1.1.

REMARK 2.2. If we replace the assumption (ψ.2)$'$ by (ψ.2), the problem

$P[\psi, f]$ does not necessarily have a strong solution. For example, let $X=H=R^1$ (1-dimensional Euclidean space), $p=2$ and $\psi(t; x)=|x|$ for $x\in R^1$. Then the initial value problem

$$\begin{cases} \dfrac{du}{dt}+\partial\psi(t;\, u(t))\ni 2, & t\in(0,\,\infty)\,, \\ u(0)=x_0\in R^1\,, \end{cases}$$

where $\partial\psi(t;\cdot)$ denotes the subdifferential of $\psi(t;\cdot)$, has the following solutions:

$$u(t) = t+x_0 \qquad\qquad \text{if} \quad x_0 \geqq 0\,,$$

$$u(t)=\begin{cases} t+\dfrac{1}{3}x_0 & \text{for } t\geqq -\dfrac{1}{3}x_0\,, \\ 3t+x_0 & \text{for } 0\leqq t\leqq -\dfrac{1}{3}x_0\,, \end{cases} \qquad \text{if } x_0<0\,.$$

Clearly, $u(0)\not\equiv u(T)$ for any $x_0\in R^1$.

3. Properties of the operators M_p and S_p

Let ψ and Ψ be as in the introduction and let D_H be as in Section 1.

We denote by $\mathscr{F}$ the duality mapping of X into X^* associated with gauge function $\mu(r)=r^{p-1}$. By definition, $\mathscr{F}$ assigns to each $z\in X$ a $z^*\in X^*$ such that $(z^*, z)_X=\|z\|_X^p$ and $\|z^*\|_{X^*}=\|z\|_X^{p-1}$. (Such a z^* is uniquely determined by z because of the strict convexity of X^*.) Then the mapping F from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ defined by $(Fu)(t)=\mathscr{F}[u(t)]$ is also the duality mapping of $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ associated with the same gauge function μ.

The purpose of this section is to prove the following theorem.

THEOREM 2. *Suppose that the following conditions hold*:

(a) *Ψ is lower semicontinuous, $\Psi\not\equiv\infty$ and $\Psi>-\infty$ on $L^p(0, T; X)$.*

(b) *There exists a subset $\mathscr{D}$ of $L^{p'}(0, T; X^*)$ with the property: $\mathscr{D}$ is dense in $L^{p'}(0, T; X^*)$ and for each $g\in\mathscr{D}$ there is $u\in D(S_p)$ such that $g\in u+Fu+S_p(u)$.*

Then we have:

(I) *If $u\in D(M_p)$, then $u\in C([0, T]; H)$ and $u(0)=u(T)$.*

(II) *$[u,f]\in G(M_p)$ if and only if there is a sequence $\{[u_n,f_n]\}\subset L^p(0, T; X)\times L^{p'}(0, T; X^*)$ such that $[u_n,f_n]\in G(S_p)$ for each n, $f_n \xrightarrow{w} f$ in $L^{p'}(0, T; X^*)$, $u_n \xrightarrow{s} u$ in $L^p(0, T; X)$ and in H uniformly on $[0, T]$ as $n\to\infty$.*

(III) *M_p is a maximal monotone operator from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$.*

REMARK 3.1. If X^* is uniformly convex, then "$f_n \xrightarrow{w} f$" in (II) can be replaced by "$f_n \xrightarrow{s} f$".

COROLLARY. *Suppose that the family* $\{\psi(t;\cdot): 0\leqq t\leqq T\}$ *satisfies the assumptions* (ψ.1) *and* (ψ.3). *Then the statements* (I), (II) *and* (III) *of* Theorem 2 *hold. In particular, in case* $X=H$ *and* $p=2$ *the assumption* (ψ.1) *can be replaced by* (ψ.1)$'$.

PROOF OF COROLLARY. Under (ψ.1) (or (ψ.1)$'$) we see that there are positive numbers a_0 and a_1 such that

$$\psi(t;z)+a_0\|z\|_X+a_1 \geqq 0 \qquad \text{for every} \quad z\in X.$$

(Cf. [9, Lemma 3.2].) Using this property and Theorem 1 we can easily verify that (a) and (b) of Theorem 2 are satisfied.

In order to prove Theorem 2 we introduce an operator $\tilde{S}_p$ from $L^p(0,T;X)$ into $L^{p'}(0,T;X^*)$ as follows: $[u,f]\in G(\tilde{S}_p)$ if and only if there is a sequence $\{[u_n,f_n]\}\subset G(S_p)$ such that $f_n \xrightarrow{w} f$ in $L^{p'}(0,T;X^*)$ and $u_n \xrightarrow{s} u$ in $L^p(0,T;X)$.

Proceeding as in the proofs of Lemmas 2 and 3 in [10] we can prove the following two lemmas regarding $\tilde{S}_p$.

LEMMA 2. *Suppose that* (a) *and* (b) *of* Theorem 2 *are satisfied. Then*:
(1) *If* $u\in D(\tilde{S}_p)$, *then* $u\in D(\Psi)\cap C([0,T];H)$ *and* $u(0)=u(T)$.
(2) M_p *is an extension of* $\tilde{S}_p$, *i.e.*, $G(\tilde{S}_p)\subset G(M_p)$.

LEMMA 3. *If* $[u_1,f_1]\in G(M_p)$ *and* $[u_2,f_2]\in G(\tilde{S}_p)$, *then*

$$\int_0^T (f_1-f_2,\, u_1-u_2)_X dt \geqq 0. \tag{3.1}$$

COROLLARY. $\tilde{S}_p$ *is a monotone operator from* $L^p(0,T;X)$ *into* $L^{p'}(0,T;X^*)$.

We now prove the following

LEMMA 4. *If* $u_i\in D(S_p)$ *and* $f_i\in u_i+Fu_i+S_p(u_i)$ $(i=1,2)$, *then for any* $t\in[0,T]$

$$\|u_1(t)-u_2(t)\|_H^2 \leqq \frac{2}{e^T-1}\int_0^T e^r(f_1-f_2,\, u_1-u_2)_X dr + 2\int_0^t (f_1-f_2,\, u_1-u_2)_X dr. \tag{3.2}$$

PROOF. The relation $f_i\in u_i+Fu_i+S_p(u_i)$ $(i=1,2)$ implies that

$$\int_0^T (u_i' + u_i + Fu_i - f_i, u_i - v)_X dt \leqq \Psi(v) - \Psi(u_i) \tag{3.3}$$

for every $v \in D(\Psi)$. For any measurable set $E \subset [0, T]$ we set

$$v_1(t) \text{ (resp. } v_2(t)) = \begin{cases} u_2(t) \text{ (resp. } u_1(t)) & \text{if } t \in E, \\ u_1(t) \text{ (resp. } u_2(t)) & \text{if } t \in [0, T] \backslash E. \end{cases}$$

Since $v_i \in D(\Psi)$ $(i=1, 2)$, we have by (3.3)

$$\int_E (u_i' + u_i + Fu_i - f_i, u_i - u_j)_X dt + \int_E \{\psi(t; u_i) - \psi(t; u_j)\} dt \leqq 0$$

for $i, j=1, 2$, which implies that for $i, j=1, 2$,

$$\begin{aligned} &(u_i'(t) + u_i(t) + (Fu_i)(t) - f_i(t), u_i(t) - u_j(t))_X \\ &\quad + \psi(t; u_i(t)) - \psi(t; u_j(t)) \leqq 0 \qquad \text{for a.a. } t \in [0, T]. \end{aligned} \tag{3.4}$$

Adding inequalities (3.4) with pairs $(i, j)=(1, 2), (2, 1)$ and using the monotonicity of F, we obtain

$$\begin{aligned} &(u_1'(t) - u_2'(t), u_1(t) - u_2(t))_X + \|u_1(t) - u_2(t)\|_H^2 \\ &\quad \leqq (f_1(t) - f_2(t), u_1(t) - u_2(t))_X \qquad \text{for a.a. } t \in [0, T]. \end{aligned}$$

Multiplying both sides of this inequality by e^t, integrating them on $[0, T]$ and noting that $u_i(0) = u_i(T)$ $(i=1, 2)$, we get

$$\begin{aligned} &\frac{1}{2}(e^T - 1)\|u_1(0) - u_2(0)\|_H^2 \\ &\leqq -\frac{1}{2}\int_0^T e^t \|u_1(t) - u_2(t)\|_H^2 dt + \int_0^T e^t (f_1 - f_2, u_1 - u_2)_X dt \\ &\leqq \int_0^T e^t (f_1 - f_2, u_1 - u_2)_X dt. \end{aligned} \tag{3.5}$$

On the other hand, from (2) of Proposition 1 and the monotonicity of the mapping $v \to v + Fv$ from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ it follows that

$$\|u_1(t) - u_2(t)\|_H^2 \leqq \|u_1(0) - u_2(0)\|_H^2 + 2\int_0^t (f_1 - f_2, u_1 - u_2)_X dr \tag{3.6}$$

for any $t \in [0, T]$. Now the required inequality (3.2) follows from (3.5) and (3.6).

We also need the following lemma. Since the proof is easy, we omit it.

LEMMA 5. *Let A be a monotone operator from a (real) Banach space V*

into its dual V^* and let B be a singlevalued strictly monotone operator from V into V^*, that is, $(Bv-Bw, v-w)_V>0$ for any v, $w\in D(B)$ with $v\neq w$. If the range of $A+B$ is all of V^*, then A is maximal monotone.

PROOF OF THEOREM 2. According to Corollary to Lemma 3, $\tilde{S}_p$ is a monotone operator from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$. If the maximal monotonicity of $\tilde{S}_p$ is shown, then Theorem 2 follows readily from Lemmas 2 and 3. On account of Lemma 5, in order to show that $\tilde{S}_p$ is maximal monotone it is enough to prove that $\tilde{S}_p+F+I$ is surjective, where I is the identity operator on $L^p(0, T; X)$. Below we shall show that this is indeed the case.

Let f be any element of $L^{p'}(0, T; X^*)$ and choose a sequence $\{f_n\}\subset D$ such that $f_n \xrightarrow{s} f$ in $L^{p'}(0, T; X^*)$. In view of the assumption (b) there exists, for each n, a $u_n\in D(S_p)$ such that $f_n\in u_n+Fu_n+S_p(u_n)$, or equivalently, $u_n(0)=u_n(T)$ and

$$\int_0^T (u_n'+u_n+Fu_n, u_n-v)_X dt \leqq \Psi(v)-\Psi(u_n) \tag{3.7}$$

for every $v\in D(\Psi)$. Observe now that by (b) there is at least one function $h_0 \in D(\Psi)\cap C([0, T]; H)$ such that $h_0(0)=h_0(T)$ and $h_0'\in L^{p'}(0, T; X^*)$. Taking h_0 in (3.7) as v, we obtain by integration by parts

$$\int_0^T (h_0'+u_n+Fu_n-f_n, u_n-h_0)_X dt \leqq \Psi(h_0)-\Psi(u_n).$$

The above inequality yields

$$\begin{aligned} &\Psi(u_n)+\int_0^T \|u_n\|_X^p dt+\int_0^T \|u_n\|_H^2 dt \\ &\leqq \Psi(h_0)+\int_0^T (\|h_0'\|_{X^*}+\|f_n\|_{X^*})(\|u_n\|_X+\|h_0\|_X)dt \\ &\quad +\int_0^T \|h_0\|_H\|u_n\|_H dt+\int_0^T \|u_n\|_X^{p-1}\|h_0\|_X dt. \end{aligned} \tag{3.8}$$

In view of the assumption (a) there are $f^*\in L^{p'}(0, T; X^*)$ and a number C such that

$$\Psi(v)\geqq \int_0^T (f^*, v)_X dt+C \qquad \text{for all} \quad v\in L^p(0, T; X).$$

Hence, by (3.8), we see that $\{u_n\}$ is bounded in $L^p(0, T; X)$ and $\{\Psi(u_n)\}$ is bounded. On the other hand, from Lemma 4 it follows that $\{u_n\}$ converges in H uniformly on $[0, T]$ to a function $u\in C([0, T]; H)$ with $u(0)=u(T)$. Thus $u\in L^p(0, T; X)$, $u_n \xrightarrow{w} u$ in $L^p(0, T; X)$ as $n\to\infty$, and

$$-\infty < \Psi(u) \leqq \liminf_{n\to\infty} \Psi(u_n) < +\infty$$

because of the lower semicontinuity of Ψ. We may assume, taking a subsequence if necessary, that $Fu_n \xrightarrow{w} g$ in $L^{p'}(0, T; X^*)$ as $n\to\infty$ for some $g \in L^{p'}(0, T; X^*)$.

Since $u \in D(\Psi)$, replacing v by u in (3.7) and using the monotonicity of the mapping $I+F$, we obtain

$$\limsup_{n\to\infty} \int_0^T \{(u_n', u_n - u)_X dt + \Psi(u_n) - \Psi(u)\} \leqq 0. \tag{3.9}$$

Moreover, since $u_n \xrightarrow{w} u$ in $L^p(0, T; X)$ and $Fu_n \xrightarrow{w} g$ in $L^{p'}(0, T; X^*)$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \int_0^T (Fu_n, u_n - u)_X dt \\ &\leqq \limsup_{n\to\infty} \int_0^T (Fu_n, u_n - v)_X dt + \int_0^T (g, v-u)_X dt \\ &\leqq \int_0^T (v', v-u)_X dt + \Psi(v) - \Psi(u) + \int_0^T (f-g-u, u-v)_X dt \end{aligned}$$

for every $v \in D(\Psi) \cap C([0, T]; H)$ such that $v(0) = v(T)$ and $v' \in L^{p'}(0, T; X^*)$. Take $v = u_n$ in the last expression of the above and let n tend to infinity. Then from (3.9) we find

$$\limsup_{n\to\infty} \int_0^T (Fu_n, u_n - u)_X dt \leqq 0.$$

This together with the uniform convexity of $L^p(0, T; X)$ implies that $u_n \xrightarrow{s} u$ in $L^p(0, T; X)$ and $Fu_n \xrightarrow{w} Fu$ in $L^{p'}(0, T; X^*)$. Thus $f-u-Fu \in \tilde{S}_p(u)$ by the definition of $\tilde{S}_p$. It follows that $\tilde{S}_p + F + I$ is surjective. This completes the proof of Theorem 2.

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