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(λ, μ)-*Absolutely Summing Operators*

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Introduction

Pietsch [9] introduced the concept of absolutely p-summing operators in normed spaces. This concept was extended in Ramanujan [10] to absolutely λ-summing operators by the aid of symmetric sequence spaces λ. On the other hand, Mityagin and Pelczyński [6] introduced the concept of (p, r)-absolutely summing operators in Banach spaces and this was recently extended in Miyazaki [7] to $(p, q; r)$-absolutely summing operators by using the sequence spaces $l_{p,q}$ and l_r. The object of this paper is to extend these two kinds of concepts to (λ, μ)-absolutely summing operators in normed spaces by making use of abstract sequence spaces λ and μ and to develop a theory of such operators.

In Section 1, we define the sequence spaces λ of type Λ and the sequence spaces μ of type M and define the (λ, μ)-absolutely summing operators. It is shown that $l_{p,q}$ is a space of type Λ and l_r is a space of type M. In Section 2, we state some basic properties of (λ, μ)-absolutely summing operators. We investigate in Section 3 some inclusion relations between the spaces of (λ_1, μ_1)- and (λ_2, μ_2)-absolutely summing operators. Section 4 is devoted to studying composition of two (λ, μ)-absolutely summing operators. Two spaces of (λ_1, μ_1)- and (λ_2, μ_2)-absolutely summing operators may happen to coincide, when their domain and range are particular normed spaces. These facts will be investigated in Section 5.

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§1. Notations and Definitions

For a sequence space λ the α-dual is denoted by $\lambda^{\times}$. If $\lambda^{\times\times}=\lambda$, then λ is said to be a perfect sequence or a Köthe space. We start with the sequence space c_0 of all scalar sequences converging to zero and the sequence space ω of all scalar sequences, which are given respectively an extended quasi-norm p and an extended norm q satisfying the following conditions:

(a) *If for any* $x=(x_1,\dots, x_n,\dots)\in c_0$ *and* $y=(y_1,\dots, y_n,\dots)\in\omega$ *we set* $x^i=(x_1, \dots, x_i, 0,\dots)$ *and* $y^i=(y_1,\dots, y_i, 0,\dots)$ *for* $i=1,2,\dots$, *then* $p(x^i)\to p(x)$ *and* $q(y^i)\to q(y)$.

(b) *p and q are both absolutely monotone.*

We shall then define the sequence space $\lambda\subset c_0$ (resp. $\mu\subset\omega$) to be the space consisting of all $x\in c_0$ (resp. $x\in\omega$) such that $p(x)<\infty$ (resp. $q(x)<\infty$).

Furthermore we assume that λ and μ satisfy the following conditions:

(c) *λ and μ are both the K-symmetric spaces. That is, if x_π is the sequence which is obtained as a rearrangement of the sequence x corresponding to a permutation π of the positive integers, then $p(x)=p(x_\pi)$ for each $x\in\lambda$ and each π and $q(y)=q(y_\pi)$ for each $y\in\mu$ and each π.*

(d) *μ is a Köthe space.*

(e) *The topology given by the norm q on μ is the Mackey topology of the dual pair $(\mu, \mu^\times)$ so that $\mu^\times=(\mu, q)'$.*

(f) *λ and μ have the norm preservation property. That is, if $x=(x_i)$ is such that $x_i=0$ for all $i\neq n$, then $p(x)=|x_n|$ and $q(x)=|x_n|$.*

We say the above λ and the space l_∞ to be spaces of type Λ and say the above μ to be a space of type M.

If μ is of type M, then we have $l_1\subseteq\mu\subseteq l_\infty$ and either $\mu\subseteq c_0$ or $\mu=l_\infty$.

We remark now that ϕ, ω, c and c_0 are not of type M and that any space of type M is also of type Λ.

In the following, we shall show that the Lorentz space $l_{p,q}$ $(1\leq p, q\leq\infty)$ is of type Λ.

Definition 1. *The Lorentz space $l_{p,q}$ is the collection of all sequences $(a_i)\in c_0$ such that $\|(a_i)\|_{l_{p,q}}<\infty$, where denoting by $(|a_i|^*)$ the non-increasing rearrangement of $(|a_i|)$ we put*

$$\|(a_i)\|_{l_{p,q}}=\begin{cases}\left(\sum_i i^{\frac{q}{p}-1}|a_i|^{*q}\right)^{\frac{1}{q}} & \text{if } 1\leq p<\infty,\quad 1\leq q<\infty,\\ \sup i^{\frac{1}{p}}|a_i|^* & \text{if } 1\leq p\leq\infty,\quad q=\infty.\end{cases}$$

Proposition 1. *The Lorentz space $l_{p,q}(1\leq p, q\leq\infty)$ is of type Λ.*

Proof. It suffices to show that $l_{p,q}$ satisfies the condition (a). Assume first that $1\leq p<\infty$ and $1\leq q<\infty$. If $a=(a_i)\in l_{p,q}$, we have $\sum_{i=1}^{\infty} i^{\frac{q}{p}-1}|a_i|^{*q}<\infty$. Here putting $(|a_i|^*)=(b_k)$, for any $\varepsilon>0$ we have a positive integer M such that $\sum_{i=M+1}^{\infty} i^{\frac{q}{p}-1}|b_i|^q<\varepsilon$. If we denote $b_i=a_{n_i}$ for $i=1,\dots, M$, there exists a positive integer N such that $\{a_1,\dots, a_N\}\supseteq\{a_{n_1},\dots, a_{n_M}\}$. Let $\{c_1,\dots, c_N\}$ be the non-increasing rearrangement of $\{a_1,\dots, a_N\}$. Then $c_i=b_i$ for $i=1,\dots, M$ and we have

$$\sum_{i=1}^{\infty} i^{\frac{q}{p}-1}|b_i|^q-\sum_{i=1}^{N} i^{\frac{q}{p}-1}|c_i|^q$$

$$= \sum_{i=1}^{\infty} i^{\frac{q}{p}-1} |b_i|^q - \left(\sum_{i=1}^{M} i^{\frac{q}{p}-1} |b_i|^q + \sum_{i=M+1}^{N} i^{\frac{q}{p}-1} |c_i|^q \right)$$

$$= \sum_{i=M+1}^{\infty} i^{\frac{q}{p}-1} |b_i|^q - \sum_{i=M+1}^{N} i^{\frac{q}{p}-1} |c_i|^q$$

$$= \sum_{i=M+1}^{N} i^{\frac{q}{p}-1} (|b_i|^q - |c_i|^q) + \sum_{i=N+1}^{\infty} i^{\frac{q}{p}-1} |b_i|^q < \varepsilon .$$

Therefore $\|a^i\|_{l_{p,q}}$ converges to $\|a\|_{l_{p,q}}$.

Next assume that $1 \leq p \leq \infty$ and $q = \infty$. If $a = (a_i) \in l_{p,q}$, we have $\sup_i i^{\frac{1}{p}} |a_i|^* = Q < \infty$. Hence if we put $(|a_i|^*) = (b_k)$, there exists a positive integer M such that $M^{\frac{1}{p}} |b_M| > Q - \varepsilon$. Hence taking N by the same way as in the above proof, we have $\sup_{1 \leq i \leq N} i^{\frac{1}{p}} |a_i|^* > Q - \varepsilon$. Hence $\|a^i\|_{l_{p,q}}$ converges to $\|a\|_{l_{p,q}}$.

Finally, in case of $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, if $\|a\|_{l_{p,q}} = \infty$, it is easy to show that $\|a^i\|_{l_{p,q}}$ tends to $\|a\|_{l_{p,q}}$ and the proof is complete.

Next we start with two normed linear spaces $(E, \| \ \|)$ and $(F, \| \ \|)$. Let μ be of type M. Then we shall denote by $\mu(E)$ the vector sequences $x = (x_n)$, $x_n \in E$, which are weakly contained in μ in the sense that for each $a \in E'$ the sequence $(<x_n, a>)$ of scalars is in μ.

Here suppose that $x = (x_n)$ belongs to $\mu(E)$. Then from a theorem of Pietsch [8] it follows that $\sup_{\|a\| \leq 1} q(|<x_n, a>|) < \infty$. We shall denote by ε_μ the functional defined on $\mu(E)$ by $\varepsilon_\mu(x) = \sup_{\|a\| \leq 1} q((|<x_n, a>|))$ which is also denoted by $\sup_{\|a\| \leq 1} \|(|<x_n, a>|)\|_\mu$. $\varepsilon_\mu(x)$ can easily be verified to be a norm. This gives $\mu(E)$ a natural topology.

Next let λ be of type Λ. Then we define the space $\lambda[F]$ as the space of all vector sequences $y = (y_n)$, $y_n \in F$, such that the sequence $(\|y_n\|) \in \lambda$. We denote by α_λ the functional defined on $\lambda[F]$ by $\alpha_\lambda(y) = p((\|y_n\|))$ which is also denoted by $\|(\|y_n\|)\|_\lambda$ or $\|(y_n)\|_{\lambda[F]}$. Thus $\lambda[F]$ is topologised in a natural way by the quasi-norm $\alpha_\lambda(y)$. We can easily show that $\mu(E) \supset \mu[E]$ for any μ of type M.

Definition 2. *Let E and F be normed linear spaces, let T be a linear mapping on E into F and let λ and μ be of type Λ and of type M respectively. Then the mapping T is said to be (λ, μ)-absolutely summing provided for each finite set of elements $x_1, \ldots, x_n$ in E the following inequality is satisfied:*

$$\|(Tx_i)\|_{\lambda[F]} \leq \rho \sup_{\|a\| \leq 1} \|(| < x_i, a > |)\|_\mu , \tag{1}$$

where ρ is constant.

Remark. $\|(Tx_i)\|_{\lambda[F]}$ appearing above is to be interpreted as the quasi-norm of the element $(Tx_i,\dots, Tx_n, 0,\dots)$ in the vector sequence space $\lambda[F]$ with a similar interpretation for $\|(|<x_i, a>|)\|_\mu$.

We denote by $\pi_{\lambda,\mu}(T)$ the least constant ρ satisfying (1) for any finite set $\{x_1,\dots, x_n\}$ in E and by $\pi_{\lambda,\mu}(E, F)$ the set of all (λ, μ)-absolutely summing operators. Then $\pi_{\lambda,\mu}(E, F)$ is a quasi-normed linear space with a quasi-norm $\pi_{\lambda,\mu}(T)$.

When $\lambda = l_{p,q}$ and $\mu = l_r$, the mappings T above are called $(p, q; r)$-absolutely summing operators and discussed extensively in Miyazaki [7].

§2. Elementary properties of (λ, μ)-absolutely summing operators

PROPOSITION 2. *Let $B(E,F)$ be the normed space of all bounded linear operators with the norm $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$, let λ be of type Λ and let μ be of type M. Then we have $\pi_{\lambda,\mu}(E, F) \subset B(E, F)$* and *$\|T\| \leq \pi_{\lambda,\mu}(T)$ for every $T \in \pi_{\lambda,\mu}(E, F)$.*

PROOF. By virtue of Definition 2, we have

$$\|(\|Tx\|, 0,\dots)\|_\lambda \leq \pi_{\lambda,\mu}(T) \sup_{\|a\|\leq 1} \|(| < x, a > |, 0,\dots)\|_\mu .$$

Therefore we have $\|Tx\| \leq \pi_{\lambda,\mu}(T)\|x\|$. Consequently we have

$$T \in B(E, F) \quad \text{and} \quad \|T\| \leq \pi_{\lambda\mu}(T).$$

Thus the proof is complete.

PROPOSITION 3. *Let λ be of type Λ, let μ be of type M and let l be a Banach sequence space satisfying $l \supset \phi$, $\|e_i\| = 1$* and *$l' \subset l^\times$. If there exists $\xi = (\xi_n)$ such that $\xi \in c_o$, $\xi \notin \lambda$ and $\xi \cdot l^\times \subset \mu$, then there exists a continuous linear mapping on l which is not (λ, μ)-absolutely summing.*

PROOF. The identity mapping T on l is linear and continuous. Define $(x^{(n)})$ in l by $x^{(n)} = \xi_n e_n$. Then if $a \in l' \subset l^\times$, we have $(<x^{(n)}, a>) = \xi a \in \xi \cdot l^\times \subset \mu$ and $(x^{(n)}) \in c_0(l)$. However $\|Tx^{(n)}\| = |\xi_n|$. Hence $(\|Tx^{(n)}\|) \notin \lambda$. Thus the proof is complete.

COROLLARY. *Assume that λ is of type Λ, $\lambda \subsetneqq c_0$ and μ is of type M. Then there exists a continuous linear mapping on c_0 which is not (λ, μ)-absolutely summing.*

PROOF. Since $l_1 \subseteq \mu \subseteq l_\infty$ and there exists a $\xi \in c_0$ which does not belong to λ, the condition of Proposition 3 is satisfied.

THEOREM 1. *Let λ be of type Λ and μ be of type M. Let us consider the* *following properties of* $T: E \to F$.

(i) *T is a* (λ, μ)*-absolutely summing operator.*

(ii) *If* $x=(x_i) \in \mu(E) \cap c_0(E)$, *then* $\hat{T}x=(Tx_i) \in \lambda[F]$.

(iii) *If* $x=(x_i) \in \mu(E)$, *then* $\hat{T}x=(Tx_i) \in \lambda[F]$.

Then

(1) (i) *and* (ii) *are equivalent.*

(2) *If* λ *is of type* M, (i), (ii) *and* (iii) *are equivalent.*

(3) *Let* λ *be of type* M. *Then even if* λ *and* μ *do not satisfy the condition* (f), (i) *and* (iii) *are equivalent.*

PROOF. (1) (i)$\Rightarrow$(ii): Let (i) be valid and let $x=(x_i) \in \mu(E) \cap c_0(E)$. For each fixed n, consider $x^n=(x_1,\dots, x_n, 0,\dots)$. Then we obtain

$$\|(\|Tx_1\|,\dots, \|Tx_n\|, 0,\dots)\|_\lambda \le \rho \sup_{\|a\|\le 1} \|(|<x_1, a>|,\dots, |<x_n, a>|, 0,\dots)\|_\mu$$

and since the norm on μ is absolutely nomotone, the above expression is $\le \rho\varepsilon_\mu(x)$. Since λ satisfies the condition (a), it follows that $\|(\|Tx_i\|)\|_\lambda < \infty$. By Proposition 2 $(\|Tx_i\|)$ belongs to c_0. Consequently $\hat{T}x \in \lambda[F]$. Thus (i)$\Rightarrow$(ii) is proved.

(ii)$\Rightarrow$(i): Let (ii) be valid and let (i) be not valid. Then for any positive integer j there exists a finite set $\{x_i^j\}_{1\le j\le n(j)}$ in E satisfying $\sup_{\|a\|\le 1} \|(|<x_i^j, a>|)\|_\mu \le 1$ and $\|(\|Tx_i^j\|)\|_\lambda > j2^j$. From our assumptions it follows that the sequence x of vectors

$$\frac{x_1^1}{2},\dots, \frac{x_{n(1)}^1}{2}, \frac{x_1^2}{2^2},\dots, \frac{x_{n(2)}^2}{2^2},\dots, \frac{x_1^j}{2^j},\dots, \frac{x_{n(j)}^j}{2^j},\dots$$

is in $\mu(E)$, and, since $\{x_i^j\}$ is bounded, x is contained in $c_0(E)$. Also since the quasinorm defining the topology of λ is absolutely monotone, it follows that $\hat{T}x \notin \lambda[F]$. This is a contradiction.

(2) (iii)$\Rightarrow$(ii) is clear. The proof of (ii)$\Rightarrow$(i) follows in the same way as in the proof of (i)$\Rightarrow$(ii) of (1) and the proof of (i)$\Rightarrow$(iii) follows in the same way as in the proof of (i)$\Rightarrow$(ii) of (1).

The analogous calculation of (1) shows the part (3) of the theorem. Thus our assertions are proved.

THEOREM 2. *Let* λ *and* μ *be of type* M. *Then the space* $\pi_{\lambda,\mu}(E, F)$ *is a normed linear space with the norm* $\pi_{\lambda,\mu}(T)$ *and if* F *is a Banach space,* $\pi_{\lambda,\mu}(E, F)$ *is complete.*

Proof. We omit the proof of $\pi_{\lambda,\mu}(T)$ being a norm and of $\pi_{\lambda,\mu}(E, F)$ being a normed linear space. Assuming that F is a Banach space, we shall prove that $\pi_{\lambda,\mu}(E, F)$ is complete. Let $\{T_n\}$ be a Cauchy sequence in $\pi_{\lambda,\mu}(E, F)$. Then for given $\varepsilon > 0$ the inequality $\|T_n - T_m\| \leq \pi_{\lambda,\mu}(T_n - T_m) < \varepsilon$ holds for $n, m > N$. Thus $\{T_n\}$ is a Cauchy sequence in the Banach space $B(E, F)$ and therefore there exists a $T \in B(E, F)$ such that $\lim_{n\to\infty} \|T_n - T\| = 0$. Since $\pi_{\lambda,\mu}(T_n - T_m) < \varepsilon$ for $n, m > N$, we get for $n, m > N$ and for each finite set $\{x_i\}_{1 \leq i \leq n}$ in E

$$\|(\|T_n x_i - T_m x_i\|)\|_\lambda \leq \varepsilon \sup_{\|a\| \leq 1} \|(|< x_i, a >|)\|_\mu .$$

Letting $m \to \infty$, we get

$$\|(\|T_n x_i - T x_i\|)\|_\lambda \leq \varepsilon \sup_{\|a\| \leq 1} \|(|< x_i, a >|)\|_\mu .$$

This implies $\pi_{\lambda,\mu}(T_n - T) \leq \varepsilon$ for any $n > N$. The proof is complete.

Proposition 4. *Let λ be of type Λ and μ be of type M.*

(i) *If $\mu \cap c_0 \not\subset \lambda$, then $\pi_{\lambda,\mu}(E, F) = \{0\}$;*

(ii) $\pi_{l_\infty,\mu}(E, F) = B(E, F)$.

Proof. (i) If possible, let $T(\neq 0) \in \pi_{\lambda,\mu}(E, F)$ and let $(a_n) \in \mu \cap c_0 \backslash \lambda$. Here a_i may be assumed to be positive for $i = 1, 2, \ldots$. Let x_0 be an element of E such that $\|x_0\| = 1$ and $\|Tx_0\| = V(\neq 0)$. Then we have $\left(\left\|T\frac{a_i}{V}x_0\right\|\right) = (a_i) \in \mu \cap c_0 \backslash \lambda$ but $\left(\left\|\frac{a_i}{V}x_0\right\|\right) = \left(\frac{a_i}{V}\right) \in \mu \cap c_0$. This contradicts $T \in \pi_{\lambda,\mu}(E, F)$, which proves (i).

(ii) Since μ satisfies the conditions (b) and (f), for each finite set of elements $x_1, \ldots, x_n$ in E the following inequality holds:

$$\sup_i \|Tx_i\| = \|Tx_{i_0}\| \leq \|T\| \|x_{i_0}\| \leq \|T\| \sup_{\|a\| \leq 1} \|(|< x_i, a >|)\|_\mu ,$$

where x_{i_0} is an element of $x_1, \ldots, x_n$. Thus our assertions are proved.

Theorem 3. *Let E, F and G be normed spaces, let λ be of type Λ and let μ be of type M.*

(i) *If $S \in B(E, F)$ and $T \in \pi_{\lambda,\mu}(F, G)$, then $TS \in \pi_{\lambda,\mu}(E, G)$ and $\pi_{\lambda,\mu}(TS) \leq \pi_{\lambda,\mu}(T)\|S\|$.*

(ii) *If $S \in \pi_{\lambda,\mu}(E, F)$ and $T \in B(F, G)$, then $TS \in \pi_{\lambda,\mu}(E, G)$ and $\pi_{\lambda,\mu}(TS) \leq \|T\| \pi_{\lambda,\mu}(S)$.*

Proof. (i) For each finite set of elements $x_1, \ldots, x_n$ in E, by our assumption the following inequality is valid:

$$\|(\|TSx_i\|)\|_\lambda \leq \pi_{\lambda,\mu}(T) \sup_{\|a\|\leq 1} \|(| < Sx_i, a > |)\|_\mu$$

$$\leq \pi_{\lambda,\mu}(T) \|S\| \sup_{\|a\|\leq 1} \left\|\left(\left| < x_i, \frac{S'a}{\|S\|} > \right| \right)\right\|_\mu$$

$$\leq \pi_{\lambda,\mu}(T)\|S\| \sup_{\|b\|\leq 1} \|(| < x_i, b > |)\|_\mu,$$

which proves (i).

The analogous calculation shows (ii) of the theorem. In fact, the following inequality holds:

$$\|(\|TSx_i\|)\|_\lambda \leq \|T\|\|(\|Sx_i\|)\|_\lambda \leq \|T\|\pi_{\lambda,\mu}(S) \sup_{\|a\|\leq 1} \|(| < x_i, a > |)\|_\mu .$$

Thus our assertions are proved.

COROLLARY. *Let λ be of type Λ and let μ be of type M. Then $\pi_{\lambda,\mu}(E, E)$ is a two sided ideal in $B(E, E)$ and for $S \in \pi_{\lambda,\mu}(E, E)$ and $T \in B(E, E)$, the following inequalities hold: $\pi_{\lambda,\mu}(ST) \leq \pi_{\lambda,\mu}(S)\|T\|$ and $\pi_{\lambda,\mu}(TS) \leq \|T\|\pi_{\lambda,\mu}(S)$.*

LEMMA 1. *Let λ be a space of type Λ. Then we have $\lambda \otimes E \subset \lambda[E]$.*

PROOF. Let $\hat{\phi}$ be the mapping on $\lambda \otimes E$ into $S(E)$, the linear space of all sequences with values in E, defined by $\hat{\phi}((c_i), x) = (c_i x) \in \lambda[E]$. Consequently by using the definition of tensor product, the linear mapping ϕ: $\sum_{i=1}^{n} (c_{ij}) \otimes x_i \to (\sum_{i=1}^{n} c_{ij} x_i)$ mapps $\lambda \otimes E$ into $\lambda[E]$ and ϕ is an algebraic isomorphism. Thus the proof is complete.

Now we denote by $\lambda \otimes_{\alpha_\lambda} F$ the quasi-normed space $\lambda \otimes F$ with the topology induced by the quasi-norm α_λ and also by $\mu \otimes_{\varepsilon_\mu} E$ the normed space $\mu \otimes E$ with the topology induced by the norm ε_μ.

PROPOSITION 5. *Let λ be of type Λ, let μ ($\neq l_\infty$) be of type M and let $\pi_{\lambda,\mu}(E, F) \neq 0$. Then the mapping T: $E \to F$ belongs to $\pi_{\lambda,\mu}(E, F)$ if and only if $I \otimes T$: $\mu \otimes_{\varepsilon_\mu} E \to \lambda \otimes_{\alpha_\lambda} F$ is continuous.*

PROOF. Assume that $I \otimes T$: $\mu \otimes_{\varepsilon_\mu} E \to \lambda \otimes_{\alpha_\lambda} F$ is continuous and T does not belong to $\pi_{\lambda,\mu}(E, F)$. Then for any positive integer j there exists a finite set $\{x_i^j\}_{1 \leq i \leq n(j)}$ in E satisfying $\alpha_\lambda((Tx_i^j)) > j\varepsilon_\mu((x_i^j))$. Since $\sum_{i=1}^{n} e_i \otimes x_i = \sum_{i=1}^{n} (0, \ldots, 0, x_i, 0, \ldots) = (x_i, \ldots, x_n, 0, \ldots)$, we have

$$\alpha_\lambda(I \otimes T(\sum_{i=1}^{n} e_i \otimes x_i^j))$$

$$= \alpha_\lambda(\sum_{i=1}^{n} e_i \otimes Tx_i^j) = \alpha_\lambda((Tx_i^j)) > j\varepsilon_\mu((x_i^j)) = j\varepsilon_\mu(\sum_{i=1}^{n} e_i \otimes x_i^j).$$

Consequently $I\otimes T$ is not continuous. This is a contradiction. Thus the sufficiency is proved. Conversely, assume that $T\in\pi_{\lambda,\mu}(E, F)$. Then $\hat{T}: \mu(E)\cap c_o(E) \to\lambda[F]$ is continuous. Therefore $I\otimes T: \mu\otimes_{\varepsilon_\mu}E\to\lambda\otimes_{\alpha_\lambda}F$ is continuous, for $\mu\otimes_{\varepsilon_\mu}E\subset\mu(E)\cap c_0(E)$ and $\hat{T}$ and $I\otimes T$ have the same values on $\mu\otimes E$. This completes the proof.

§3. Some inclusion relations between the spaces of (λ_1, μ_1)-and (λ_2, μ_2)-absolutely summing operators

Suppose that α and β are sequence spaces. We define $\alpha\cdot\beta=\{(x_ny_n): (x_n)\in\alpha, (y_n)\in\beta\}$. Here we denote by $D(\beta, \alpha)$ the set of diagonal matrices carrying β into α. We use the following results of Crofts [1].

Lemma 2. *$D(\beta, \alpha)\subset(\beta\cdot\alpha^\times)^\times$ and, if α is a Köthe space, $D(\beta, \alpha)=(\beta\cdot\alpha^\times)^\times$.*

Proposition 6. *Let λ_1 and λ_2 be of type Λ and let μ_1 and μ_2 be of type M. If $\mu_1\supset\mu_2$ and $\lambda_2\supset\lambda_1$, then $\pi_{\lambda_1,\mu_1}(E, F)\subset\pi_{\lambda_2,\mu_2}(E, F)$.*

Theorem 4. *Let λ_1 and λ_2 be of type Λ and let μ_1 and μ_2 be of type M. If there exists a sequence space $\nu\subset l_\infty$ satisfying the conditions $\nu\cdot\mu_2\subset\mu_1$ and $(\nu\cdot\lambda_1^\times)^\times\subset\lambda_2$, then we have $\pi_{\lambda_1,\mu_1}(E, F)\subset\pi_{\lambda_2,\mu_2}(E, F)$.*

Proof. Let T be (λ_1, μ_1)-absolutely summing on E into F and let $(x_n)\in\mu_2(E)\cap c_o(E)$. Then for each $\alpha=(\alpha_n)\in\nu$ and $a\in E'$ we have

$$(<\alpha_nx_n, a>) = \alpha(<x_n, a>)\in\nu\cdot\mu_2\subset\mu_1.$$

Since T is (λ_1, μ_1)-absolutely summing it follows that $|\alpha|(\|Tx_n\|)=(\|T(\alpha_nx_n)\|)\in\lambda_1$ and since λ_1 is solid, $\alpha(\|Tx_n\|)\in\lambda_1$ and therefore we have $(\|Tx_n\|)\in D(\nu\cdot\lambda_1)$. Hence by Lemma 2 $(\|Tx_n\|)\in(\nu\cdot\lambda_1^\times)^\times\subset\lambda_2$. Thus T is (λ_2, μ_2)-absolutely summing. This completes the proof.

Example. *Let $\lambda_1=l_1$, $\mu_1=l_1$, λ_2* be of type Λ and μ_2 be of type M such that $\mu_2\subset\lambda_2$. Then if we set $\nu=(\mu_1^\times\cdot\mu_2)^\times=\mu_2^\times$, we have $\nu\cdot\mu_2\subset l_1$ and $(\nu\cdot\lambda_1^\times)^\times=\mu_2\subset\lambda_2$ so that, by the above theorem, every absolutely summing mapping is (λ_2, μ_2)-absolutely summing.

§4. The composition of (λ, μ)-absolutely summing operators

Theorem 5. *Let E, F and G be normed spaces, let $1\le p$, $r_i\le\infty(i=1, 2)$*

be real numbers such that $\frac{1}{p}+\frac{1}{r_1}\leq\frac{1}{r_2}$ *and let* λ_1 *and* λ_2 *be sequence spaces of type* Λ *satisfying* $\lambda_2\supset\lambda_1\cdot l_p$. *Then for any* $T\in\pi_{l_p,l_p}(E,F)$ *and* $S\in\pi_{\lambda_1,l_{r_1}}(F,G)$ *the composition* ST *belongs to* $\pi_{\lambda_2,l_{r_2}}(E,G)$ *and satisfies* $\pi_{\lambda_2,l_{r_2}}(ST)\leq C\pi_{\lambda_1,l_{r_1}}(S)\pi_{l_p,l_p}(T)$ *where* C *is a constant.*

PROOF. By virtue of Proposition 6, it suffices to prove the assertion under the assumption $\frac{1}{r_2}=\frac{1}{p}+\frac{1}{r_1}$. Since T is absolutely p-summing operator, by Pietsch [9] there is a probability measure μ, that is, a regular positive Borel measure μ with total mass 1 on the weakly compact unit ball K' of E' such that $\|Tx\|\leq\pi_{l_p,l_p}(T)\left(\int_{K'}|<x,a>|^p d\mu(a)\right)^{\frac{1}{p}}$ for every $x\in E$. Let $\{x_i\}_{1\leq i\leq n}$ be an arbitrary finite set of elements in E. Put $x_i=x_i^0\xi_i$ where $\xi_i=\left(\int_{K'}|<x_i,a>|^{r_2}\right)^{\frac{1}{p}}$. Then, by our assumption, it follows that

$$\begin{aligned}\|(\|STx_i\|)\|_{\lambda_2} &\leq C\|(\|STx_i^0\|)\|_{\lambda_1}\cdot\|(\|\xi_i\|)\|_p\\ &\leq C\pi_{\lambda_1,l_{r_1}}(S)\sup_{\|b\|\leq 1}\|(|<Tx_i^0,b>|)\|_{l_{r_1}}\cdot\left(\sum_i\int_{K'}|<x_i,a>|^{r_2}\right)^{\frac{1}{p}},\end{aligned}$$

where C is a constant. The terms of the form $<Tx,b>$ can be written as

$$<Tx,b>=\int_{K'}<x,a>f(a)d\mu(a)\qquad\text{for each } x\in E$$

with an $f\in L_{p'}(K',\mu)$ satisfying the inequality

$$\left(\int_{K'}|f(a)|^{p'}d\mu(a)\right)^{\frac{1}{p'}}\leq\pi_{l_p,l_p}(T)\|b\|,\ \frac{1}{p}+\frac{1}{p'}=1. \tag{2}$$

In fact, let $E_p(K',\mu)$ be the subspace of $L_p(K',\mu)$ which is constituted by the rest classes $\hat{\phi}_x$ for $\phi_x(a)=<x,a>\in C(K')$ with $x\in E$. Then for each $b\in F'$ there exists a linear form β_b on $E_p(K',\mu)$ defined by $<\hat{\phi}_x,\beta_b>=<Tx,b>$ and it satisfies

$$|<\hat{\phi}_x,\beta_b>|\leq\|Tx\|\|b\|\leq\pi_{l_p,l_p}(T)\left(\int_{K'}|<x,a>|^p d\mu(a)\right)^{\frac{1}{p}}\|b\|.$$

Therefore there exists an $f\in L_{p'}(K',\mu)$, $\frac{1}{p}+\frac{1}{p'}=1$, such that

$$<Tx,b>=\int_{K'}<x,a>f(a)d\mu(a)\qquad\text{for each}\quad x\in E$$

and it satisfies (2). Hence by Hölder's inequality, we obtain

$$|<Tx, b>| \leq \int_{K'} |<x, a>||f(a)|d\mu(a)$$

$$= \int_{K'} | <x, a> |^{\frac{r_2}{p}}(| < x, a > |^{r_2}|f(a)|^{p'})^{\frac{1}{r_1}} |f(a)|^{\frac{p'}{r_2'}} d\mu(a)$$

$$\leq \left(\int_{K'} | < x, a > |^{r_2} d\mu(a)\right)^{\frac{1}{p}} \left(\int_{K'} | < x, a > |^{r_2} |f(a)|^{p'} d\mu(a)\right)^{\frac{1}{r_1}}$$

$$\times \left(\int_{K'} |f(a)|^{p'} d\mu(a)\right)^{\frac{1}{r_2'}}.$$

Replacing x by x_i^0 in the above inequality, we obtain

$$| < Tx_i^0, b > |^{r_1} \leq \left(\int_{K'} | < x_i, a > |^{r_2}|f(a)|^{p'} d\mu(a)\right)\left(\int_{K'} |f(a)|^{p'} d\mu(a)\right)^{\frac{r_1}{r'_2}}.$$

Finally, we get

$$\left(\sum_i | <Tx_i^0, b> |^{r_1}\right)^{\frac{1}{r_1}}$$

$$\leq \sup_{\|a\|\leq 1} \left(\sum_i | <x_i, a> |^{r_2}\right)^{\frac{1}{r_1}} \left(\int_{K'} |f(a)|^{p'} d\mu(a)\right)^{\frac{1}{p'}}$$

Consequently

$$\|(\|STx_i\|)\|_{\lambda_2} \leq C\pi_{\lambda_1,l_{r_1}}(S)\pi_{l_p,l_p}(T) \sup_{\|a\|\leq 1}\left(\sum_i | < x_i, a > |^{r_2}\right)^{\frac{1}{r_2}}$$

which completes the proof.

THEOREM 6. *Let E, F and G be normed spaces, $1 \leq p, r \leq \infty$, $\frac{1}{p}+\frac{1}{r} \leq 1$, and λ be of type Λ satisfying $l_p \cdot \lambda \subset l_1$. Then for any $T \in \pi_{l_p,l_p}(E, F)$ and any $S \in \pi_{\lambda,l_r}(F, G)$ the composition ST belongs to $\pi_{l_1,l_1}(E, G)$.*

PROOF. In case of $p=1$, this is clear by Theorem 3. We shall show this in case of $p>1$. Put $\frac{1}{p}+\frac{1}{p'}=1$. Then it satisfies $\lambda \subset l_{p'}$ and $l_r \supset l_{p'}$. By Proposition 6, $S \in \pi_{\lambda,l_r}(F, G) \subset \pi_{l_{p'},l_{p'}}(F, G)$. Hence applying Theorem 5 to S and T, we obtain $ST \in \pi_{l_1,l_1}(E, G)$. Thus the proof is complete.

§5. (λ, μ)-absolutely summing operators on special spaces E and F

LEMMA 3. *Let E be isomorphic to a subspace of $L_1(\mu)$ for a measure space (K, Σ, μ), let F be any normed space and let λ be of type Λ. Then $T \in B(E, F)$ belongs to $\pi_{\lambda,l_1}(E, F)$ if and only if for any $S \in B(l_\infty, E)$ the composition TS belongs to $\pi_{\lambda,l_1}(l_\infty, F)$.*

PROOF. By virtue of Theorem 3 it is clear that if $T\in\pi_{\lambda,l_1}(E,F)$ and $S\in B(l_\infty,E)$, then $TS\in\pi_{\lambda,l_1}(l_\infty,F)$. Conversely, we assume that $T\in B(E,F)$ satisfies the condition $TS\in\pi_{\lambda,l_1}(l_\infty,F)$ for any $S\in B(l_\infty,E)$ but $T\notin\pi_{\lambda,l_1}(E,F)$. Then there exists a sequence $\{x_i\}\subset E$ such that $\sum_i x_i$ converges unconditionally and

$$\|(\|Tx_i\|)\|_\lambda=\infty\,. \tag{3}$$

Here we define $S\in B(l_\infty,F)$ by $S((a_i))=\sum_i a_i x_i$ for each $(a_i)\in l_\infty$. On the other hand, from (3), there is a sequence $\{\eta_i\}\in c_o$ such that $\|(\eta_i\|Tx_i\|)\|_\lambda=\infty$, that is, $\|(\|TS(\eta_i e_i)\|)\|_\lambda=\infty$. Since $\sum_i|<\eta_i e_i, a>|<\infty$ for each $a\in l'_\infty$, that is, $(\eta_i e_i)\in l_1(l_\infty)\cap c_0(l_\infty)$, we have $TS\notin\pi_{\lambda,l_1}(l_\infty,F)$. This contradicts our assumption and the proof is complete.

THEOREM 7. *Let λ_1 and λ_2 be of type Λ.*

(i) *If $l_2\cdot\lambda_1^\times\supset\lambda_2^\times$ and λ_2 is a Köthe space, then we have $\pi_{\lambda_1,l_1}(E,F)\subset\pi_{\lambda_2,l_2}(E,F)$.*

(ii) *Let E and F be the same spaces as in Lemma 3. Then if $l_2\cdot\lambda_1^\times\subset\lambda_2^\times$ and λ_1 and λ_2 are Köthe spaces, we have $\pi_{\lambda_1,l_1}(E,F)\supset\pi_{\lambda_2,l_2}(E,F)$.*

PROOF. (i) Putting $\nu=(l_1^\times\cdot l_2)^\times=l_2$, we have $(l_2\cdot\lambda_1^\times)^\times\subset\lambda_2^{\times\times}=\lambda_2$ and $l_2\cdot l_2\subset l_1$. Therefore by Theorem 4 $\pi_{\lambda_1,l_1}(E,F)\subset\pi_{\lambda_2,l_2}(E,F)$.

(ii) Let $T\in\pi_{\lambda_2,l_2}(E,F)$. $S\in B(l_\infty,E)$ is always 2-absolutely summing. Since $(l_2\cdot\lambda_1^\times)\subset\lambda_2^\times$, it follows that $l_2\cdot\lambda_2\subset\lambda_1$. Therefore on account of Theorem 5, we have $TS\in\pi_{\lambda_1,l_1}(l_\infty,F)$. Hence by Lemma 3, we have $T\in\pi_{\lambda_1,l_1}(E,F)$, which completes the proof.

COROLLARY. *Assume that λ_1 and λ_2 is of type Λ, $l_2\cdot\lambda_1^\times=\lambda_2^\times$ and λ_1 and λ_2 are Köthe spaces. Let $1\le r\le 2$ and F be any normed space. Then we have $\pi_{\lambda_1,l_1}(l_r,F)=\pi_{\lambda_2,l_2}(l_r,F)$ and $\pi_{\lambda_1,l_1}(L_r(0,1),F)=\pi_{\lambda_2,l_2}(L_r(0,1),F)$.*

PROOF. This follows from Theorem 7 and the result [5] asserting that for $1\le r\le 2$ the spaces l_r and $L_r(0,1)$ are isomorphic to subspaces of $L_1(\mu)$.

THEOREM 8. *Let λ be of type Λ and let μ be of type M. If $l_2\cdot\mu^\times\supset\lambda^\times$, λ is a Köthe space and H is a Hilbert space, then we have $\pi_{\lambda,\mu}(H,H)=B(H,H)$.*

PFOOR. From [4] it is known that $\pi_{2,1}(H,H)=B(H,H)$. Therefore we may show that $\pi_{\lambda,\mu}(H,H)\cap\pi_{2,1}(H,H)$. But this follows from Theorem 4, for putting $\nu=(l_1^\times\cdot\mu)^\times=\mu^\times$ we have $\nu\cdot\mu\subset l_1$ and $(\mu^\times\cdot l_2)^\times\subset\lambda^{\times\times}=\lambda$. Hence we obtain $\pi_{\lambda,\mu}(H,H)\supset B(H,H)$. Thus the proof is complete.

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