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Oscillation and a Class of Odd Order Linear Differential Equations

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Introduction

Let q be a continuous function from $[0, \infty)$ to $(0, \infty)$. In studying oscillation for

$$u^{(m)}+qu = 0, \tag{1}$$

and related equations, many authors have recognized that the even and odd order cases have some fundamental differences. See, for example, A. G. Kartsatos [6], T. Kusano and H. Onose [10], G. Ladas, V. Lakshmikanthan, and J. S. Papadakis [11], Y. G. Sficas [15], Sficas and V. A. Staikos [16], and G. H. Ryder and D. V. V. Wend [14]. On the other hand, Ladas, Lakshmikantham, and Papadakis [11], Sficas and Staikos [16], and the present author [12] have observed that, for some purposes, the odd and even order cases coalesce if one replaces (1) by

$$u^{(m)}+(-1)^m qu = 0. \tag{2}$$

For example, the present author [12] has shown that

$$\int_0^\infty t^{m-1}q(t)dt < \infty$$

is a necessary and sufficient condition for the existence of a bounded nonoscillatory solution of (2), irrespective of the parity of m.

In the even order case, it is known that

$$\int_0^\infty t^{2n-2}q(t)dt = \infty \tag{3}$$

implies that every solution of

$$u^{(2n)}+qu = 0 \tag{4}$$

is oscillatory (see, for example, G. V. Anan'eva and V. I. Balaganskii [2], H. C. Howard [4], I. T. Kiguradze [8], V. A. Kondrat'ev [9], and C. A. Swanson [17, p. 175]). If (3) fails, the present author [13] has found two continuous functions ϕ and ψ from $[0, \infty)$ to $[0, \infty)$ such that if

$$w'' + \phi w = 0$$

is oscillatory then every solution of (4) is oscillatory, and such that if

$$w'' + \psi w = 0$$

is nonoscillatory then there exists a nonoscillatory solution of (4). Thinking of (4) as one case of (2), the purpose of the present work is to obtain analogies to these last results in the other case, i.e.,

$$u^{(2n+1)} - qu = 0 \tag{5}$$

where n is a positive integer.

Results

Before stating our results, we need to discuss some properties of nonoscillatory solutions of (5). First, there always is a nonoscillatory solution of (5). This is clear from the Volterra integral equation

$$u(t) = 1 + \frac{1}{(2n)!}\int_0^t (t-s)^{2n} q(s) u(s) ds\,, \tag{6}$$

the solution of which is everywhere positive and satisfies (5). If u is the solution of (6) then routine examination shows that $u^{(k)} > 0$ on $(0, \infty)$ for $k = 1, .., 2n+1$. We shall obtain herein results which ensure that under certain circumstances this is the only type of nonoscillatory solution which (5) may have.

Now suppose u is an eventually positive solution of (5). (Every nonoscillatory solution of (5) is either eventually positive or eventually negative, and since (5) is linear it suffices to consider the eventually positive case.) Since u is eventually positive, $u^{(2n+1)}$ is eventually positive. Since $u^{(2n+1)}$ is eventually positive, $u^{(2n)}$ is eventually one-signed. Since $u^{(2n)}$ is eventually one-signed, u^{2n-1} is eventually one-signed. Continuing this, we see that there is $c \geq 0$ such that none of $u, u', u'', \ldots, u^{(2n)}$ has any zeros in $[c, \infty)$.

LEMMA 1. *Let u be an eventually positive solution of* (5), *and find $c \geq 0$ such that none of $u, u', u'', \ldots, u^{(2n)}$ has any zeros in $[c, \infty)$. Then* (i) *and* (ii) *are equivalent.*

(i) $u^{(k)} > 0$ *on* $[c, \infty)$ *for* $k = 1, 2, \ldots, 2n$.

(ii) $u^{(2n)} > 0$ *on* $[c, \infty)$.

For expository convenience, we shall defer the proof of Lemma 1. In light of Lemma 1, and our earlier comments regarding (6), one sees that the relevant question, with regard to oscillation and nonoscillation, is: *Are there eventually positive solutions u of* (5) *with* $u^{(2n)}$ *eventually negative?*

THEOREM 1: *If*

$$\int_0^\infty t^{(2n-1)} q(t)dt = \infty , \tag{7}$$

then there is no eventually positive solution u of (5) *with* $u^{(2n)}$ *eventually negative.*

THEOREM 2: *If* (7) *fails, and the second order equation*

$$w''(t) + \left(\frac{1}{(2n-2)!}\int_t^\infty (s-t)^{2n-2} q(s)\,ds\right) w(t) = 0 \tag{8}$$

is oscillatory, then the conclusions of Theorem 1 *are true.*

THEOREM 3: *If the second order equation*

$$w''(t) + \frac{1}{(2n-1)!} t^{2n-1} q(t) w(t) = 0 \tag{9}$$

is nonoscillatory, then there exists an eventually positive solution u of (5) *with* $u^{(2n)}$ *eventually negative.*

Next we have a comparison theorem. Theorems 1, 2, and 3, taken together, create the impression that for "large" q the conclusions of Theorem 1 hold and for "small" q the conclusions of Theorem 3 hold. Theorem 4 reinforces this idea. Note that Theorem 4 is related to a recent even order result of A.G. Kartsatos [7].

THEOREM 4. *Let p be a continuous function from* $[0, \infty)$ *to* $(0, \infty)$ *such that* $p(t) \geq q(t)$ *whenever* $t \geq 0$, *and suppose the conclusions of* Theorem 1 *are true. Then*

$$v^{(2n+1)} - pv = 0 \tag{10}$$

has no eventually positive solution v with $v^{(2n)}$ *eventually negative.*

Since (5) is linear, if one wishes to think of nonoscillatory solutions instead of eventually positive solutions, one may ask: *Are there nonoscillatory solutions u of* (5) *with* $uu^{(2n)}$ *eventually negative?* When put this way, one sees that our work here is related to, but independent of, third order work of G. Villari [18], [19]. Since we know that (5) always has a nonoscillatory solution, Theorems 1 and 2 can be thought of a restricting the possible asymptotic behaviors of such solutions. Put another way, our results can be thought of as saying that in a certain sense there are not "very many" nonoscillatory solution of (5). On the other hand, we have not yet ensured the existence of oscillatory solutions. Furthermore, if Q is the solution space of (5), and if, whenever $1 \leq k \leq 2n+1$, z_k is the solution of

$$z_k(t) = \frac{t^{k-1}}{(k-1)!} + \int_0^t \frac{(t-s)^{2n}}{(2n)!} q(s) z_k(s) ds \tag{11}$$

on $[0, \infty)$, then each z_k is a nonoscillatory solution of (5) and $\{z_1, z_2, \dots, z_{2n+1}\}$ is a basis for Q. The following theorem clarifies this situation.

THEOREM 5. *Statements* (iii) *and* (iv) *are equivalent.*

(iii) *If u is an eventually positive solution of* (5) *then $u^{(2n)}$ is eventually positive.*

(iv) *There is a $2n$-dimensional subspace of Q each member of which is oscillatory.*

Furthermore, if (iii) *and* (iv) *are true and m is an integer in* $[0, 2n]$, *then there is a basis for Q consisting of m oscillatory members and* $2n+1-m$ *nonoscillatory members.*

If $n=1$ (the third order case), our theorem follows from results of S. Ahmad and A. C. Lazer [1] and G. D. Jones [5]. These authors have also shown that in the third order case the existence of a single nontrivial oscillatory solution implies that if u is a nonoscillatory solution then $uu^{(2n)}$ is eventually positive. The following fifth order example shows that this fails in general.

EXAMPLE. Suppose r is in $(0, 1)$. Now

$$(r+2)(r+1)r(r-1)(r-2) < r(r-1)(r-2)(r-3)(r-4).$$

Thus $\alpha > \gamma$ where

$$\alpha = \max\{r(r-1)(r-2)(r-3)(r-4) \colon 0 \leq r \leq 1\}$$

and

$$\gamma = \max\{r(r-1)(r-2)(r-3)(r-4) \colon 2 \leq r \leq 3\}$$

$$= \max\{r+2)(r+1)r(r-1)(r-2) \colon 0 \leq r \leq 1\}.$$

Suppose $\gamma < \beta < \alpha$, $n=2$, and q is given by $q(t) = \beta(t+1)^{-5}$. Since the polynomial equation

$$\rho(\rho-1)(\rho-2)(\rho-3)(\rho-4) - \beta = 0 \tag{12}$$

has two complex roots, we see that (5) has a nontrivial oscillatory solution. On the other hand, (12) has a real solution ρ in (0, 1); and u given by $u(t)=(t+1)^{\rho}$ is a positive solution of (5) with $u''''(t)=\rho(\rho-1)(\rho-2)(\rho-3)(t+1)^{\rho-4}<0$ whenever $t\geq 0$. The example is complete.

From Theorems 1, 2, 4, and 5, corollaries can be drawn giving conditions ensuring the existence of a $2n$-dimensional subspace of Q consisting solely of oscillatory solutions. We leave this to the reader.

PROOF OF LEMMA 1. It is clear that (i) implies (ii), so we shall show that the failure of (i) implies the failure of (ii). Suppose (i) fails. Let j be the largest integer such that $u^{(k)}>0$ on $[c, \infty)$ if $k\leq j$ (where we write $u=u^{(0)}$). By hypothesis, $j<2n+1$, and since $u^{(2n+1)}>0$ on $[c, \infty)$, we see $j\neq 2n$. Now $u^{(j+1)}<0$ on $[c, \infty)$, so $u^{(j)}$ is bounded. If $j+1\leq k\leq 2n$, and $u^{(k)}u^{(k+1)}>0$ on $[c, \infty)$, then $u^{(k)}$ is either positive and increasing or negative and decreasing. In either case, $u^{(k-1)}$ is unbounded. Clearly now, if $j+1\leq m\leq k$, then $u^{(m-1)}$ is unbounded, so $u^{(j)}$ is unbounded. But $u^{(j)}$ is bounded, so $u^{(k)}u^{(k+1)}<0$ on $[c, \infty)$ if $j+1\leq k \leq 2n$. Since $u^{(2n+1)}>0$, this says $u^{(2n)}<0$ on $[c, \infty)$. Although this completes the proof of the lemma, let us note that other observations can be made. In particular, it is clear that if $j+1\leq k\leq 2n$ then $u^{(k)}<0$ on $[c, \infty)$ if k is even and $u^{(k)}>0$ on $[c, \infty)$ if k is odd. Since $u^{(j+1)}<0$ on $[c, \infty)$, this says $j+1$ is even and j is odd. This last fact will be used without further comment in the remainder of our proofs.

LEMMA 2. *Let u be an eventually positive solution of* (5) *with $u^{(2n)}$ eventually negative. Let c and j be as in the Proof of* Lemma 1. *Then*

$$(13)\qquad (-1)^{k+1}u^{(k)}(t) = \frac{1}{(2n-k)!}\int_t^{\infty}(s-t)^{2n-k}q(s)u(s)ds$$

whenever $t\geq c$ and $j+1\leq k\leq 2n$, and

$$(14)\qquad u^{(j)}(t) \geq \frac{1}{(2n-j)!}\int_t^{\infty}(s-t)^{2n-j}q(s)u(s)ds$$

whenever $t\geq c$.

PROOF. Since $u^{(k)}u^{(k+1)}<0$ on $[c, \infty)$ if $j+1\leq k\leq 2n$, we see $u^{(k)}(\infty)=\lim_{t\to\infty}u^{(k)}(t)$ exists if $j\leq k\leq 2n$. Furthermore, if $j+1\leq k\leq 2n$, $u^{(k)}(\infty)=0$ since $u^{(k)}(\infty)$ and $u^{(k-1)}(\infty)$ both exist. Now, if $\tau\geq t\geq c$,

$$u^{(2n)}(\tau)-u^{(2n)}(t) = \int_t^{\tau}u^{(2n+1)}(s)ds$$

$$= \int_t^{\tau}q(s)u(s)ds,$$

so

$$-u^{(2n)}(t) = \int_t^\infty q(s)u(s)ds,$$

and (13) is true if $k=2n$. Suppose $j+2 \leq m \leq 2n$, and (13) is true for $k=m$. Now, if $\tau \leq t \leq c$,

$$(-1)^{m+1}u^{(m-1)}(\tau)-(-1)^{m+1}u^{(m-1)}(t)$$
$$= \frac{1}{(2n-m)!}\int_t^\tau\left(\int_s^\infty(\xi-s)^{2n-m}q(\xi)u(\xi)d\xi\right)ds,$$

so

$$(-1)^m u^{(m-1)}(t) = \frac{1}{(2n-m)!}\int_t^\infty\left(\int_s^\infty(\xi-s)^{2n-m}q(\xi)u(\xi)d\xi\right)ds$$
$$= \frac{1}{(2n-m+1)!}\int_t^\infty(s-t)^{2n-m+1}q(s)u(s)ds,$$

and (13) is true for $k=m-1$. By induction, the first part of the proof is complete. For (14), the same procedures suffice, but we have inequality because we have not shown $u^{(j)}(\infty)=0$, only $u^{(j)}(\infty)\geq 0$. This completes the proof.

PROOF OF THEOREM 1. We shall assume the existence of an eventually positive solution u of (5) with $u^{(2n)}$ eventually engative, and show that this violates (7). Let u be such a solution. Let c and j be as in the Proof of Lemma 1, and suppose $j>1$. Now $u^{(k)}>0$ on $[c, \infty)$ if $1\leq k\leq j$, so

$$u(t) \geq \frac{1}{(j-2)!}\int_c^t(t-s)^{j-2}u^{(j-1)}(s)ds \tag{15}$$

if $t\geq c$. This, and (14) say

$$u^{(j)}(c) \geq \frac{1}{(2n-j)!}\int_c^\infty(s-c)^{2n-j}q(s)\left(\frac{1}{(j-2)!}\int_c^s(s-\xi)^{j-2}u^{(j-1)}(\xi)d\xi\right)ds.$$

Since $u^{(j)}>0$, $u^{(j-1)}$ is increasing, so

$$u^{(j)}(c) \geq \frac{u^{(j-1)}(c)}{(2n-j)!(j-1)!}\int_c^\infty(s-c)^{2n-1}q(s)ds.$$

and we see

$$\int_c^\infty(s-\mathrm{c})^{2n-1}q(s)ds<\infty, \tag{16}$$

If $j=1$,

$$u'(c) \geq \frac{1}{(2n-1)!}\int_c^\infty (s-c)^{2n-j}q(s)u(s)ds$$
$$\geq \frac{u(c)}{(2n-1)!}\int_c^\infty (s-c)^{2n-j}q(s)ds\,,$$

and again (16) holds. But (16) implies the failure of (7), so the proof is complete.

Proof of Theorem 2. Again, let u be an eventually positive solution of (5) with $u^{(2n)}$ eventually negative. Let c and j be as in the Proof of Lemma 1, and suppose $j>1$. Now, from (13) and (15), if $t\geq c$,

$$-u^{(j+1)}(t) = \frac{1}{(2n-j-1)!}\int_t^\infty (s-t)^{2n-j-1}q(s)u(s)ds$$
$$\geq \frac{1}{(2n-j-1)!(j-2)!}\int_t^\infty (s-t)^{2n-j-1}q(s)\left(\int_c^s (s-\xi)^{j-2}u^{(j-1)}(\xi)d\xi\right)ds$$
$$\geq \frac{1}{(2n-j-1)!(j-2)!}\int_t^\infty (s-t)^{2n-j-1}q(s)\left(\int_t^s (s-\xi)^{j-2}u^{(j-1)}(\xi)d\xi\right)ds$$
$$\geq \frac{u^{(j-1)}(t)}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)ds\,,$$

so

$$u^{(j+1)}(t)/u^{(j-1)}(t) \leq -\frac{1}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)ds \tag{17}$$

if $t\geq c$. If $j=1$ then

$$-u''(t) = \frac{1}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)u(s)ds$$
$$\geq \frac{u(t)}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)ds\,,$$

whenever $t\geq c$, so (17) holds in either case. Let v be given on $[c,\infty)$ by $v(t) = u^{(j)}(t)/u^{(j-1)}(t)$, and note that $v(t)>0$ if $t\geq c$. Now

$$v'(t) = u^{(j+1)}(t)/u^{(j-1)}(t) - v(t)^2$$

if $t>c$, so (17) says

$$v'(t)+v(t)^2 \leq -\frac{1}{(2n-2)!}\int_t^\infty (s-t)^{2n-2}q(s)ds \tag{18}$$

if $t>c$. But a classical result of A. Wintner [20] (see also [17, Theorem 2.15, p. 63]) says that the existence of a positive solution of (18) on (c, ∞) implies nonoscillation for (8), and the proof is complete.

PROOF OF THEOREM 3. Suppose (9) is nonoscillatory, and let w be an eventually positive solution of (9). Find $c \geq 0$ such that $w(t)>0$ if $t \geq c$. Now $w'>0$ on $[c, \infty)$. If $\tau \geq t \geq c$,

$$w'(t) = w'(\tau)+\frac{1}{(2n-1)!}\int_t^\tau s^{2n-1}q(s)w(s)ds$$

$$\geq \frac{1}{(2n-1)!}\int_t^\tau s^{2n-1}q(s)w(s)ds\,,$$

so

$$w'(t) \geq \frac{1}{(2n-1)!}\int_t^\infty s^{2n-1}q(s)w(s)ds$$

$$\geq \frac{1}{(2n-1)!}\int_t^\infty (s-t)^{2n-1}q(s)w(s)ds\,.$$

Now standard iteration techniques say that there is a continuously differentiable function u from $[c, \infty)$ to $[w(c), \infty)$ such that $u(c)=w(c)$, such that $u(t) \leq w(t)$ whenever $t \geq c$, and such that

$$u'(t) = \frac{1}{(2n-1)!}\int_t^\infty (s-t)^{2n-1}q(s)u(s)ds \tag{19}$$

if $t \geq c$. Now $2n-1$ differentiations of (19) yield

$$u^{(2n)}(t) = -\int_t^\infty q(s)u(s)ds\,, \tag{20}$$

and then (5). Thus u solves (5) on $[c, \infty)$, and (20) says $u^{(2n)}<0$ on $[c, \infty)$. Clearly u can be extended to a solution of (5) on $[0, \infty)$, and this solution satisfies the requirements of Theorem 3, so the proof is complete.

PROOF OF THEOREM 4. We shall show that if there is an eventually positive solution v of (10) with $v^{(2n)}$ eventually negative then there is an eventually positive solution u of (5) with $u^{(2n)}$ eventually negative. Suppose v is an eventually positive solution of (10) with $v^{(2n)}$ eventually negative. Find $c \geq 0$ such that none of $v, v', v'', \ldots, v^{(2n)}$ has any zeros in $[c, \infty)$, and let j be the largest integer such that $v^{(k)}>0$ on $[c, \infty)$ if $k \leq j$. Suppose $j>1$. Let f be given on $[c, \infty)$ by

$$f(t) = v(c)+\sum_{k=1}^{j-1} v^{(k)}(c)(t-c)^k/k!\,.$$

Note that $f(t)>0$ if $t\geq c$ and

$$v(t)=f(t)+\frac{1}{(j-1)!}\int_c^t (t-s)^{j-1}v^{(j)}(s)ds \tag{21}$$

if $t\geq c$. Now (21) and the adaptation of (14) to v yield

$$\begin{aligned} v(t) &\geq f(t)+\frac{1}{(j-1)!}\int_c^t (t-s)^{j-1}\frac{1}{(2n-j)!}\left(\int_s^\infty (\xi-s)^{2n-j}p(\xi)v(\xi)d\xi\right)ds \\ &\geq f(t)+\frac{1}{(j-1)!}\int_c^t (t-s)^{j-1}\left(\frac{1}{(2n-j)!}\int_s^\infty (\xi)-s)^{2n-j}q(\xi)v(\xi)d\xi\right)ds \end{aligned}$$

if $t\geq c$. Now standard iteration arguments yield the existence of a continuous function u from $[c, \infty)$ to $[0, \infty)$ such that $u(t)\leq v(t)$ if $t\geq c$ and

$$u(t)=f(t)+\frac{1}{(j-1)!}\int_c^t (t-s)^{j-1}\left(\frac{1}{(2n-j)!}\int_s^\infty (\xi-s)^{2n-j}\, q(\xi)u(\xi)d\xi\right)ds \tag{22}$$

if $t\geq c$. Since $u\geq 0$ on $[c, \infty)$, (22) says $u\geq f$ on $[c, \infty)$, so u has no zeros in $[c, \infty)$. Now j differentiations of (22) yield

$$u^{(j)}(t)=\frac{1}{(2n-j)!}\int_t^\infty (s-t)^{2n-j}q(s)u(s)ds \tag{23}$$

if $t\geq c$, and $2n-j$ differentiations of (23) yield

$$u^{(2n)}(t)=-\int_t^\infty q(s)u(s)ds\,, \tag{24}$$

and then (5) if $t\geq c$. Now u can be extended to a solution of (5) on $[0, \infty)$, and this solution is eventually positive. Also, (24) says that $u^{(2n)}$ is eventually negative, so the proof is complete if $j>1$. If $j=1$, then

$$\begin{aligned} v'(t) &\geq \frac{1}{(2n-1)!}\int_t^\infty (s-t)^{2n-1}p(s)v(s)ds \\ &\geq \frac{1}{(2n-1)!}\int_t^\infty (s-t)^{2n-1}q(s)v(s)ds \end{aligned}$$

if $t\geq c$, so

$$\begin{aligned} v(t) &= v(c)+\int_c^t v'(s)ds \\ &\geq v(c)+\frac{1}{(2n-1)!}\int_c^t\left(\int_s^\infty (\xi-s)^{2n-1}q(\xi)v(\xi)\,d\xi\right)ds \end{aligned}$$

whenever $t\geq c$. Arguments virtually identical to those above can now be used to complete the proof if $j=1$, and we desist.

LEMMA 3. *Suppose* (iii) *is true, let* $\{w_m\}_{m=0}^{\infty}$ *be a* Q*-valued sequence, and suppose* $w_0^{(k)}(t)=\lim_{m\to\infty}w_m^{(k)}(t)$ *whenever* $t\geq 0$ *and* $k=0, 1,\dots, 2n+1$. *Suppose that* $\{\tau_m\}_{m=1}^{\infty}$ *is a* $[0, \infty)$*-valued sequence with* $\lim_{m\to\infty}\tau_m=\infty$ *and* $w_m(\tau_m)=0$ *whenever* $m\geq 1$. *Then* w_0 *is oscillatory.*

PROOF. Suppose w_0 is not oscillatory. We can, and do, assume w_0 is eventually positive. According to (iii) and Lemma 1, there is $c\geq 0$ such that $w_0^{(k)}(t)>0$ for $t\geq c$, $k=0, 1,\dots, 2n+1$. Clearly now there is an integer j with $w_j{}^{(k)}(c)>0$ for $k=0, 1,\dots, 2n+1$ and with $\tau_j>c$. Now

$$w_j(t) = w_j(c)+\textstyle\sum_{k=1}^{2n}\dfrac{(t-c)^k}{k!}w_j{}^{(k)}(c) + \displaystyle\int_c^t\frac{(t-s)^{2n}}{(2n)!}q(s)w_j(s)ds \tag{25}$$

whenever $t\geq c$. But since

$$w_j(c)+\textstyle\sum_{k=1}^{2n}\dfrac{(t-c)^k}{k!}w_j{}^{(k)}(c)>0$$

whenever $t\geq c$, standard iteration methods say that the solution of (25) is positive on $[c, \infty)$. But $w_j(\tau_j)=0$, so we have a contradiction and the proof is complete.

The technique in the proof of (iii)→(iv) in Theorem 5 is an adaptation to our present circumstance of a circle of ideas used by S. P. Hastings and Lazer [3], Ahmad and Lazer [1], and Jones [5].

PROOF OF THEOREM 5. Suppose (iv) is true and (iii) is false. Let u be an eventually positive solution of (5) with $u^{(2n)}$ eventually negative, and let M be a $2n$-dimensional subspace of Q, each member of which is oscillatory. Find $c\geq 0$ such that $u>0$ and $u^{(2n)}<0$ on $[c, \infty)$. Since u is not in M and M is $2n$-dimensional, every member of Q is of the form $au+y$, where y is in M. Find a such that $z_1=au+y$. Now $a\neq 0$, since z_1 is not in M. Also, $a>0$, for otherwise $z_1(t)<0$ whenever $t\geq c$ and $y(t)\leq 0$. It follows from the discussion preceeding Lemma 1 that if $y^{(2n)}$ is nonoscillatory then y is nonoscillatory, so $y^{(2n)}$ is oscillatory. Now $z_1^{(2n)}=au^{(2n)}+y^{(2n)}$, so $z_1^{(2n)}(t)<0$ whenever $t\geq c$ and $y^{(2n)}(t)\leq 0$, since $a>0$. But $z_1^{(2n)}(t)>0$ whenever $t>0$, so we have a contradiction, and the proof of (iv)→(iii) is complete.

Suppose (iii) is true. If k and j are positive integers, $1\leq k\leq 2n$, let $a(k,j)$ and $b(k,j)$ be real numbers such that

$$a(k,j)^2+b(k,j)^2 = 1 \tag{26}$$

and

$$a(k,j)z_k(j)+b(k,j)z_{2n+1}(j) = 0. \tag{27}$$

From (26), there is a subsequence $\{j_i\}_{i=1}^{\infty}$ of the positive integers such that

$$\alpha_k = \lim_{i\to\infty} a(k,j_i)$$

and

$$\beta_k = \lim_{i\to\infty} b(k,j_i)$$

exist for each k. For $k=1,\dots,2n$, let $y_k=\alpha_k z_k+\beta_k z_{2n+1}$. Let $M=span\{y_1,\dots, y_{2n}\}$. It is an immediate conclusion of (27) and Lemma 3 that each member of M is oscillatory. To verify $dim(M)=2n$ it suffices to show linear independence for $\{y_1,\dots, y_{2n}\}$. By (26), $\alpha_k^2+\beta_k^2=1$ for each k, so each y_k is nontrivial. Also, $\alpha_k\beta_k\neq 0$ for each k, for otherwise some y_k would be nonoscillatory. If $\{y_1,\dots, y_{2n}\}$ is linearly dependent then there is an integer j such that y_j is a linear combination of $y_1,\dots, y_{j-1}$. But the linear independence of $\{z_1,\dots, z_{2n+1}\}$ says this is impossible, so $\{y_1,\dots, y_{2n}\}$ is linearly independent, $dim(M)=2n$, and the proof of (iii)→(iv) is complete.

Finally, suppose (iii) and (iv) are true, and let m be an integer in $[0, 2n]$. If $m=0$, recall that $\{z_1,\dots, z_{2n+1}\}$ is a basis, and we are through. Suppose $m\geq 1$, and let $\{y_1,\dots, y_{2n}\}$ be as above. We claim that $\{y_1,\dots, y_m, z_{m+1},\dots, z_{2n+1}\}$ is a basis, and to show this it suffices to verify linear independence. Suppose $d_1,\dots, d_{2n+1}$ are numbers and

$$d_1y_1+\cdots+d_my_m+d_{m+1}z_{m+1}+\cdots+d_{2n+1}z_{2n+1} = 0,$$

$$d_1y_1+\cdots+d_my_m+d_{2n+1}z_{2n+1} = -d_{m+1}z_{m+1}-\cdots-d_{2n}z_{2n} \tag{28}$$

The left side of (28) is in $span\{z_1,\dots, z_m, z_{2n+1}\}$ and the right side of (28) is in $span\{z_{m+1},\dots, z_{2n}\}$, so

$$d_1y_1+\cdots+d_my_m+d_{2n+1}z_{2n+1} = 0 \tag{29}$$

and

$$d_{m+1}z_{m+1}+\cdots+d_{2n}z_{2n} = 0. \tag{30}$$

The linear independence of $\{z_1,\dots, z_{2n+1}\}$ and (30) yield

$$d_{m+1} = \cdots = d_{2n} = 0.$$

If $d_{2n+1}\neq 0$, then (29) says that z_{2n+1} is in M and is hence oscillatory. Thus $d_{2n+1}=0$. Now (29) says

$$d_1 y_1 + \cdots + d_m y_m = 0.$$

Since $\{y_1, \ldots, y_m\}$ is linearly independent, this says

$$d_1 = \cdots = d_m = 0,$$

and the proof is complete.

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