HIROSHIMA MATH. J.
5 (1975), 363-370

Potential Theoretic Properties for Accretive Operators

Bruce CALVERT

(Received December 24, 1974)
(Revised April 7, 1975)

Introduction. This note is intended to parallel some work of Kenmochi, Mizuta, the author, and others, who have studied potential theoretic properties for nonlinear monotone operators. It is related to work of Hunt, Yosida, Yamada, and Sato, who have dealt with linear accretive operators.

Given an accretive operator in a Banach lattice, we establish relationships between the modulus contraction, the domination principle, the majoration principle, the principle of the lower envelope, T-accretivity, and order-preserving resolvents. The results of this paper are presented as a series of definitions and propositions. Together these imply Theorem 1, which shows that all the conditions are equivalent under strong enough hypotheses.

It is hoped in future work to solve problems concerning the relationship between the accretive and monotone cases, operators acting in several spaces, the relation between reductions and variational inequalities, contractions onto the interval $[0, k]$, capacities, the principle of the convex envelope, the condenser principle, the balayage principle, and cones of potentials.

The author is very grateful to Professor Maeda for his helpful comments.

Potential Theoretic Properties. We recall [23] that a Banach lattice is a Banach space X over the real numbers $\boldsymbol{R}$, which is a lattice under the ordering $\leq$, satisfying the following. For x, y, z in X and $a \geq 0$ in $\boldsymbol{R}$, (1) $x \geq y$ implies $x+z \leq y+z$, (2) $x \leq y$ implies $ax \leq ay$, and (3) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. We write $\vee$ and $\wedge$ for supremum and infimum. We put $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$. We set $|x| = x^+ + x^-$. By $x \perp y$ we mean $|x| \wedge |y| = 0$.

We let $P(X)$ denote the power set of X; the subsets of X. Given $A: X \to P(X)$, we let $D(A)$ be the set of u in X for which Au is nonempty, and $R(A)$ is the union of the Au for u in X [11]. Given $A: X \to P(X)$, $B: X \to P(X)$, one defines $A+B$: $X \to P(X)$ by $(A+B)x = \{a+b: a \in Ax, b \in Bx\}$. For $\lambda \in \boldsymbol{R}$, $\lambda A: X \to P(X)$ is defined by $\lambda Ax = \{\lambda a: a \in Ax\}$. And $A^{-1}: X \to P(X)$ is defined by $x \in A^{-1}(a)$ if and only if $a \in Ax$.

Nonlinear accretive operators were introduced in [3, Definition 1]. Let X^* be the dual of X. We denote the pairing between X and X^* by parentheses.

Let $J: X \to P(X^*)$ be the duality map defined by $f \in Jx$ when $(x,f)=\|x\|^2=\|f\|^2$. Let $<,>_s: X \times X \to \boldsymbol{R}$ be defined by $<f,g>_s=\lim_{d\to 0+} d^{-1}(\|g+df\|^2-\|g\|^2)$. Given $A: X \to P(X)$, the following are equivalent [2], [11]:

(1) If $\lambda>0$, $x_1 \in Ax$, $y_1 \in Ay$, then $\|(x+\lambda x_1)-(y+\lambda y_1)\| \geq \|x-y\|$.

(2) If $x_1 \in Ax$, $y_1 \in Ay$, then $<x_1-y_1, x-y>_s \geq 0$.

(3) If $x_1 \in Ax$, $y_1 \in Ay$, then there is $f \in J(x-y)$ with $(x_1-y_1,f) \geq 0$.

If any and so all of these hold then A is called accretive.

Nonlinear T-accretive operators (where T stands for truncation) appeared in [2, 4, 5, 6, 7, 9, 13, 14, 15, 17, 20]. Let $J_K: X \to P(X^*)$ be defined by $f \in J_K x$ when $f \geq 0$, $(x,f)=\|x^+\|^2=\|f\|^2$. By [6, Remark], or [17, 1.1.4], $J_K(x) \subseteq J(x^+)$, and so $J_K(x)=\{f \in J(x^+): f \geq 0,\ (x^-,f)=0\}$. By [18] or [4, Prop 1.1] for $x \in X$, $\cap\{J_K(y): y^+=x^+\}$ is nonempty.

We let $\varphi_0(\ ,\ ): X \times X \to \boldsymbol{R}$ be defined by $\varphi_0(f,g)=\lim_{d\to 0+} d^{-1}(\|(g+df)^+\|^2 - \|g^+\|^2)$. Given $A: X \to P(X)$, the following are equivalent [6], [17]:

(1) If $\lambda>0$, $x_1 \in Ax$, $y_1 \in Ay$, then $\|((x+\lambda x_1)-(y+\lambda y_1))^+\| \geq \|(x-y)^+\|$.

(2) If $x_1 \in Ax$, $y_1 \in Ay$, then $\varphi_0(x_1-y_1, x-y) \geq 0$.

(3) If $x_1 \in Ax$, $y_1 \in Ay$, then there is $f \in J_K(x-y)$ with $(x_1-y_1,f) \geq 0$.

If any and so all of these hold then A is called T-accretive.

Given $A: X \to P(X)$, for $\lambda>0$ we write $J_\lambda=(I+\lambda A)^{-1}$ and $A_\lambda=\lambda^{-1}(I-J_\lambda)$ [11].

We say $A: X \to P(X)$ is m-accretive (m-T-accretive) when it is accretive (T-accretive) and $R(I+A)=X$ (giving $D(J_\lambda)=X$ for all $\lambda>0$). We use $\to$ to denote norm and $\rightharpoonup$ to denote weak convergence.

Definition. We say *the modulus contraction operates with respect to* $A: X \to P(X)$ if $u_1 \in Au$, $w_1 \in A(u+v^+)$ implies that for $f \in J_K(-v)$, $(w_1-u_1, f) \leq 0$.

PROPOSITION 1. *Let X be a Banach lattice. Given $A: X \to P(X)$, suppose that for $u_1 \in Au$ and $w_1 \in Aw$ and $f \in J(u-w)$, $(u_1-w_1,f) \geq 0$. Suppose $D(A)$ is a sublattice of X. Suppose the modulus contraction operates with respect to A. Then A is T-accretive.*

PROOF. Let $u_1 \in Au$, $w_1 \in Aw$. Let $f \in J_K(u-w)$. Then $f \in J(u-u \wedge w)$. Since $D(A)$ is a sublattice we may take $z_1 \in A(u \wedge w)$, giving $(u_1-z_1,f) \geq 0$. Since the modulus contraction operates, and $w=(u \wedge w)+(w-u)^+$, $(z_1-w_1,f) \geq 0$. Adding the two inequalities gives $(u_1-w_1,f) \geq 0$. q.e.d.

Definition. We say $A: X \to P(X)$ *satisfies the principle of the lower envelope* if $u_1 \in Au$, $v_1 \in Av$ and $u \wedge v \in D(A)$ implies there is $a_1 \in A(u \wedge v)$ with $a_1 \geq u_1 \wedge v_1$.

LEMMA 1. *If X is a Banach lattice, and $A: X \to X$ is hemicontinuous,* *i.e. continuous from strong to weak topologies, and T-accretive, then* $(Ax-Ay, f)\geq 0$ *for* $f\in J_K(x-y)$. (Cf. [16, Remark 4])

PROOF. For $\lambda>0$,

$$\begin{aligned}(A_\lambda x-A_\lambda y,f) &= \lambda^{-1}(x-y,f)-\lambda^{-1}(J_\lambda x-J_\lambda y,f)\\ &\geq \lambda^{-1}(x-y,f)-\lambda^{-1}((J_\lambda x-J_\lambda y)^+,f)\\ &\geq \lambda^{-1}(x-y,f)-\lambda^{-1}\|(J_\lambda x-J_\lambda y)^+\|\|f\|\\ &\geq \lambda^{-1}(x-y,f)-\lambda^{-1}\|(x-y)^+\|^2\\ &\geq 0.\end{aligned}$$

Since A is accretive in the equivalent norm $|x|_K=\|x^+\|+\|x^-\|$, $|J_\lambda x-x|_K \leq|\lambda Ax|_K$. Thus $J_\lambda x\to x$ as $\lambda\to 0$, and since A is hemicontinuous, $A_\lambda x\rightharpoonup Ax$. It follows that $(Ax-Ay,f)\geq 0$. q.e.d.

PROPOSITION 2. *Let X be a Banach lattice. Let $A: X\to P(X)$ be m-T-accretive. Suppose X^* is uniformly convex or $A: X\to X$ is continuous. Then A satisfies the principle of the lower envelope.*

PROOF. We recall from [9, Lemma 2] the following result. Let X be a Banach lattice with uniformly convex dual. Let A be m-T-accretive. Let $z\in D(A)$. Define $B_{\geq z}: X\to P(X)$ by

$$B_{\geq z}x = \begin{cases} \{f\leq 0: f\perp(x-z)\} & \text{for } x\geq z\\ \emptyset \quad \text{otherwise.} & \end{cases}$$

Then $A+B_{\geq z}$ is m-T-accretive.

Now suppose $A: X\to X$ is continuous and X is a general Banach lattice. By [9, Lemma 1] $B_{\geq z}$ is m-T-accretive. Since $B_{\geq z}$ and A are T-accretive they are accretive in the norm $|x|_K=\|x^+\|+\|x^-\|$. We recall from [1] that the sum of an m-accretive operator B and a continuous accretive operator $A: X\to X$ is m-accretive. Consequently, whether X^* is uniformly convex or $A: X\to X$ is continuous we have $R(\varepsilon I+A+B_{\geq u\wedge v})=X$ for $\varepsilon>0$. Let $\varepsilon x+a+f=(u_1+\varepsilon u)\wedge(v_1+\varepsilon v)$ with $a\in Ax$ and $f\in B_{\geq u\wedge v}x$. Take g in $\cap\{J_K(h): h^+=(x-u)^+\}$. Then $(x-u)^+\perp f$ gives $(f, g)=0$, i.e.

$$(1)\qquad (\varepsilon x+a-(u_1+\varepsilon u)\wedge(v_1+\varepsilon v), g) = 0.$$

From $(u_1+\varepsilon u)\wedge(v_1+\varepsilon v)\leq u_1+\varepsilon u$ we have

$$(2)\qquad ((u_1+\varepsilon u)\wedge(v_1+\varepsilon v)-(u_1+\varepsilon u), g)\leq 0.$$

If X^* is uniformly convex, J_K is single valued and so

$$(u_1 - a, g) \leq 0. \tag{3}$$

If A is continuous then (3) holds by Lemma 1. Adding (1) (2) and (3) gives $\varepsilon\|(x-u)^+\|^2 \leq 0$ and so $x \leq u$. Similarly $x \leq v$. Consequently $x = u \wedge v$. Taking $a_1 = a$ gives $\varepsilon(u \wedge v) + a_1 \geq (u_1 + \varepsilon u) \wedge (v_1 + \varepsilon v)$. Letting $\varepsilon \to 0$ we obtain $a_1 \geq u_1 \wedge v_1$. q.e.d.

Definition. We say $A: X \to P(X)$ *satisfies the domination principle* if $u_1 \in Au$, $v_1 \in Av$, $u_1 \wedge v_1 \geq f$ and $(u_1 - f) \perp (u-v)^+$ implies $u \leq v$.

We say that A is strictly accretive if for $u_1 \in Au$, $v_1 \in Av$, $u \neq v$, for all $f \in J(u-v)$ we have $(u_1 - v_1, f) > 0$.

PROPOSITION 3. *Let X be a Banach lattice. Let $A: X \to P(X)$ be strictly accretive. Let $D(A)$ be a sublattice of X. Suppose A satisfies the principle of the lower envelope. Then A satisfies the domination principle.*

PROOF. Suppose $u_1 \in Au$, $v_1 \in Av$, $u_1 \wedge v_1 \geq f$, and $(u_1 - f) \perp (u-v)^+$. Take $a_1 \in A(u \wedge v)$ with $a_1 \geq u_1 \wedge v_1$. Take $g \in \cap\{J_K(h): h^+ = (u-v)^+\}$. Then $a_1 \geq f$ gives $(f - a_1, g) \leq 0$. Since $(u_1 - f) \perp (u-v)^+$ we have $(u_1 - f, g) = 0$. Adding, we obtain $(u_1 - a_1, g) \leq 0$. By strict accretivity, $u = u \wedge v$, and $u \leq v$. q.e.d.

We say $U: X \to P(X)$ is order preserving if $x_1 \in Ux$, $y_1 \in Uy$, $x \leq y$ implies $x_1 \leq y_1$. We say $U: D(U) \subseteq X \to X$ is T-nonexpansive if $x, y \in D(U)$ implies $\|(Ux - Uy)^+\| \leq \|(x-y)^+\|$ [4].

PROPOSITION 4. *Let X be a Banach lattice. Let A satisfy the domination principle. Then $(I + \lambda A)^{-1}$ is order preserving for $\lambda > 0$.*

PROOF. Let $\lambda > 0$, $f \leq g$, $u_1 \in Au$, $v_1 \in Av$, $u + \lambda u_1 = f$, $v + \lambda v_1 = g$. Then

$$u_1 \wedge v_1 \geq \lambda^{-1}(f - (u \vee v)).$$

Also, since $u_1 - \lambda^{-1}(f - (u \vee v)) = \lambda^{-1}(u-v)^-$,

$$u_1 - \lambda^{-1}(f - u \vee v) \perp (u-v)^+.$$

By the domination principle, $u \leq v$. q.e.d.

PROPOSITION 5. *Let X be a Banach lattice. Let A be accretive and let $(I + \lambda A)^{-1}$ be order preserving for $\lambda \in (0, \varepsilon)$. Then the modulus contraction operates with respect to A_λ, $\lambda \in (0, \varepsilon)$. If A is m-accretive and X^* is uniformly convex and A is single valued, or $A: X \to X$ is continuous, then the modulus* *contraction operates with respect to A.*

PROOF. Given $\lambda\in(0,\varepsilon)$, u, $u+v^+\in D(A_\lambda)$, $f\in J_K(-v)$, since $(v^+,f)=0$ and $f\geq 0$,

$$(A_\lambda(u+v^+)-A_\lambda u,f)=\lambda^{-1}((u+v^+)-(I+\lambda A)^{-1}(u+v^+)-u+(I+\lambda A)^{-1}u,f)$$
$$\leq 0.$$

If A is continuous, or if A is single valued and m-accretive and X^* is uniformly convex, then $A_\lambda u\rightharpoonup Au$ and $A_\lambda(u+v^+)\rightharpoonup A(u+v^+)$ as $\lambda\to 0$ for $u,u+v^+\in D(A)$. This gives $(A(u+v^+)-Au,f)\leq 0$. q.e.d.

PROPOSITION 6. *Let $A: X\to P(X)$ be given in the Banach lattice X. If A_λ is T-accretive for $0<\lambda<\varepsilon$, then A is T-accretive.*

PROOF. Given $\delta>0$, let $u_1\in Au$, $v_1\in Av$, $u+\delta u_1=f$, and $v+\delta v_1=g$. For $\lambda\in(0,\delta)\cap(0,\varepsilon)$, put $u+\lambda u_1=x_\lambda$, $v+\lambda v_1=y_\lambda$. Then $x_\lambda+(\delta-\lambda)A_\lambda x_\lambda\ni f$ and $y_\lambda+(\delta-\lambda)A_\lambda y_\lambda\ni g$. Since A_λ is T-accretive, $\|(x_\lambda-y_\lambda)^+\|\leq\|(f-g)^+\|$. Letting $\lambda\to 0$, we have $\|(u-v)^+\|\leq\|(f-g)^+\|$. q.e.d.

Definition. We say $A: X\to P(X)$ *satisfies the majoration principle* if for $f\in X$, $u_1\in Au$, $f\leq y_1\in Ay$, $\varepsilon>0$, $\varepsilon(u_1-f)\leq(y-u)^+$ implies $u\leq y$.

REMARK. The next result follows [21], [22], [24]. I have not obtained the result for $R(A)$ dense as in [21]. Note that one may approximate f in the closure of $R(A)$ for m-accretive A by $(x_\lambda)_{\lambda>0}$ where $x_\lambda-A((f-x_\lambda)/\lambda)\ni 0$. Note the condition $J_\lambda(y-u+z+\lambda f)\leq(y-u+z)$ holds if $Au\ni u_1\geq f$ as in [22], [24] where $f=0$, or A is linear as in [21], [22], [24].

PROPOSITION 7. *Let X be a Banach lattice. Given $A: X\to P(X)$ we suppose $R(A)=X$ and $(\lambda I+A)^{-1}$ is order preserving and single valued for $\lambda\geq 0$. Suppose that if $f\leq y_1\in Ay$, $u_1\in Au$, $(u_1-f)^++f\in Az$ and $\lambda>0$, then*

$$(I+\lambda A)^{-1}(y-u+z+\lambda f)\leq(y-u+z).$$

Then A satisfies the majoration principle.

PROOF. Let $f\in X$, $u_1\in Au$, $f\leq y_1\in Ay$, $\varepsilon>0$, $\varepsilon(u_1-f)\leq(y-u)^+$. Define $A_f: X\to P(X)$ by $A_f(p)=A(p)-f$. Note that $(\lambda I+A_f)^{-1}$ is order preserving and single valued for $\lambda\geq 0$. Let $(u_1-f)^++f\in Az$ and write $x=y-u+z$. Then $(I+\varepsilon A_f)^{-1}x=(I+\varepsilon A)^{-1}(x+\varepsilon f)\leq x$. We have $(I+\varepsilon A_f)^{-1}(z+\varepsilon(u_1-f)^+)=z\leq z+\varepsilon(u_1-f)^+$. Let $v=(z+\varepsilon(u_1-f)^+)\wedge x$. Since $(I+\varepsilon A_f)^{-1}$ is order preserving, $(I+\varepsilon A_f)^{-1}v\leq v$. This implies $(A_f)_\varepsilon v=\varepsilon^{-1}(v-(I+\varepsilon A_f)^{-1}v)\geq 0$. Now $z=(I+\varepsilon A_f)^{-1}(z+\varepsilon(u_1-f)^+)\geq(I+\varepsilon A_f)^{-1}v=v-\varepsilon(A_f)_\varepsilon v$, giving $\varepsilon(A_f)_\varepsilon v\geq v-z$. Hence $\varepsilon(A_f)_\varepsilon v\geq(v-z)^+$. Now $v=(\varepsilon(u_1-f)^+\wedge(x-z))+z$, giving $(v-z)^+=\varepsilon(u_1-f)^+$

$\wedge (x-z)^+ = \varepsilon(u_1-f)^+ \wedge (y-u)^+ = \varepsilon(u_1-f)^+$ by our original supposition. It follows that $\varepsilon(A_f)_\varepsilon v \geq \varepsilon(u_1-f)^+$. Since A_f^{-1} preserves order, $(I+\varepsilon A_f)^{-1} v \geq z$. Hence, $x \geq v \geq (I+\varepsilon A_f)^{-1} v \geq z$. But $x \geq z$ means $y \geq u$. q.e.d.

We say $A: X \to P(X)$ is strictly T-accretive if for $u_1 \in Au$, $v_1 \in Av$, $(u-v)^+ \neq 0$, we have $(u_1-v_1, f) > 0$ for all $f \in J_K(u-v)$.

PROPOSITION 8. *Let X be a Banach lattice. Let A be strictly T-accretive. Then A satisfies the majoration principle.*

PROOF. Let $f \in X$, $u_1 \in Au$, $f \leq y_1 \in Ay$, $\varepsilon > 0$, $\varepsilon(u_1-f) \leq (y-u)^+$. We want to show $u \leq y$, or $(u-y)^+ = 0$. Take g in $\cap \{J_K(h), h^+ = (u-y)^+\}$. Since $\varepsilon(u_1-f) \leq (u-y)^-$, $(u_1-f)^+ \perp (u-y)^+$. It follows that

$$\begin{aligned}(u_1-y_1, g) &\leq (u_1-f, g)\\ &\leq ((u_1-f)^+, g)\\ &= 0.\end{aligned}$$

Since A is strictly T-accretive, $(u-y)^+ = 0$.

PROPOSITION 9. *Let X be a Banach lattice. Suppose A satisfies the majoration principle. Then $(I+\varepsilon A)^{-1}$ is order preserving for $\varepsilon > 0$.*

PROOF. Let $u + \varepsilon u_1 \leq y + \varepsilon y_1$, with $\varepsilon > 0$, $u_1 \in Au$, and $y_1 \in Ay$. Put $f = y_1$ in the definition of the majoration principle. We have $\varepsilon(u_1-y_1) \leq y-u \leq (y-u)^+$. By the majoration principle, $u \leq y$. q.e.d.

PROPOSITION 10. *Let X be a Banach lattice. Let A be m-accretive. Suppose $\varepsilon > 0$ and $A + \varepsilon I$ is T-accretive. Then A is T-accretive.*

PROOF. Let $\lambda \in (0, \varepsilon^{-1})$, $x_1 \in Ax$, $y_1 \in Ay$, and suppose $x + \lambda x_1 = f \geq g = y + \lambda y_1$. Take h in $J_K(y-x)$ with $((y_1+\varepsilon y)-(x_1+\varepsilon x), h) \geq 0$. This implies $(1-\lambda\varepsilon)(y-x, h) \leq (g-f, h) \leq 0$. Since $0 < \lambda\varepsilon < 1$, we have $\|(y-x)^+\|^2 = 0$. Thus, $(I+\lambda A)^{-1}$ is order preserving.

By Proposition 5, the modulus contraction operates with respect to A_λ. Since A is m-accretive, by [16, Remark 4] (cf. Lemma 1), A_λ satisfies the conditions of Proposition 1. By Proposition 1, A_λ is T-accretive, and by Proposition 6, A is T-accretive. q.e.d.

THEOREM 1. *Let X be a Banach lattice, and let $A - \varepsilon I$ be m-accretive, $\varepsilon > 0$. Suppose either $A: X \to X$ is continuous or A is single valued and X^* is uniformly convex and $D(A)$ is a sublattice of X. Then the following are equivalent:*

(1) *The modulus contraction operates with respect to A.*
(2) *A is T-accretive.*
(3) *A satisfies the principle of the lower envelope.*
(4) *A satisfies the domination principle.*
(5) *For $\lambda>0$, $(I+\lambda A)^{-1}$ is order preserving.*
(6) *A satisfies the majoration principle.*

PROOF. We note J and J_K are single valued if X^* is uniformly convex.

(1) implies (2) by Proposition 1. Note that if A is continuous then for $f\in J(u-w)$, $(Au-Aw,f)\geq 0$ [16, Remark 4].

(2) implies (3) by Proposition 2.

(3) implies (4) by Proposition 3. Note that if $A-\varepsilon I$ is accretive, $\varepsilon>0$, then A is strictly accretive; in case X^* is uniformly convex since J is single valued and if A is continuous by $((A-\varepsilon I)u-(A-\varepsilon I)w,f)\geq 0$ for all $f\in J(u-w)$.

(4) implies (5) by Proposition 4.

(5) implies (6) by Proposition 5.

(2) implies (1) by Propositions 8 and 10. In fact, by Proposition 10, $A-\varepsilon I$ is T-accretive. Note that if $A: X\to X$ is continuous and $A-\varepsilon I$ is T-accretive then A is strictly T-accretive, since $((A-\varepsilon I)u-(A-\varepsilon I)w,f)\geq 0$ for all f in $J_K(u-w)$ by Lemma 1. If X^* is uniformly convex and $A-\varepsilon I$ is T-accretive then A is strictly T-accretive since J_K is singlevalued.

(6) implies (5) by Proposition 9. q.e.d.

REMARK. Without assuming A is continuous or X^* is uniformly convex and A is single valued, we have A_ε satisfies (3) for $\varepsilon>0$ implies (2). For by Theorem 1, A_ε is T-accretive, and by Proposition 6, A is T-accretive.

Bibliography

[1] V. Barbu, Continuous perturbations of nonlinear m-accretive operators in Banach spaces, Boll. U. M. I. (4) **6** (1972), 270–278.

[2] P. Bénilan, Equations d'évolution dans un espace de Banach quelconque et applications. Thèse, Orsay, 1972.

[3] F. Browder, Fixed point theorems for nonlinear semi-contractive mappings in Banach spaces, Arch. Rat. Mech. Anal. **11** (1966), 259–269.

[4] B. Calvert, Nonlinear evolution equations in Banach lattices, Bull. Amer. Math. Soc. **76** (1970), 845–850.

[5] B. Calvert, Nonlinear equations of evolution, Pacific J. Math. **39** (1971), 293–350.

[6] B. Calvert, Semigroups in an ordered Banach space, J. Math. Soc. Japan **23** (1971), 311–319.

[7] B. Calvert, On T-accretive operators, Annali di Mat. Pura Appl. **94** (1972), 291–314.

[8] B. Calvert, Potential theoretic properties for monotone operators, Boll. U. M. I. (4) **5** (1972), 473–489.

[9] B. Calvert, Functions excessive with respect to a nonlinear resolvent, J. Math. Anal.

Appl. **50** (1975), 303–313.

[10] G. A. Hunt, Markov processes and potentials II, Ill. J. Math. **1** (1957), 316–369.

[11] T. Kato, Accretive operators and nonlinear evolution equations in Banach spaces, Nonlinear Functional Analysis, Proc. Symp. Pure Math., Vol. 13 Part I, F. Browder Ed., Amer. Math. Soc (1970), 138–161.

[12] N. Kenmochi and Y. Mizuta, Potential theoretic properties of the gradient of a convex function on a functional space, to appear in Nagoya Math. J.

[13] N. Kenmochi and Y. Mizuta, The gradient of a convex function on a regular functional space and its potential theoretic properties, Hiroshima Math. J. **4** (1974), 743–763.

[14] Y. Konishi, Nonlinear semi-groups in Banach lattices, Proc. Japan. Acad. **47** (1971), 24–28.

[15] Y. Konishi, Some examples of nonlinear semi-groups in Banach lattices, J. Fac. Sci. Univ. Tokyo, Sect. IA **18** (1972), 537–543.

[16] R. H. Martin, A global existence theorem for autonomous differential equations in a Banach space, Proc. Amer. Math. Soc. **26** (1970), 307–314.

[17] C. Picard, Opérateurs T-accrétifs, ø-accretifs et génération de semi-groupes non linéaires, Thèse, Orsay, 1972.

[18] R. Phillips, Semi-groups of positive contraction operators, Czech. Math. J. **12** (1962), 294–313.

[19] K. Sato, On dispersive operators in Banach lattices, Pacific J. Math. **33** (1970), 429–443.

[20] K. Sato, A note on nonlinear dispersive operators, J. Fac. Sci. Univ. Tokyo, Sect. IA **18** (1971–72), 465–473.

[21] K. Sato, Positive pseudo-resolvents in Banach lattices, to appear in J. Fac. Sci. Univ. Tokyo.

[22] A. Yamada, On the correspondence between potential operators and semi-groups associated with Markov process. Z. Wahrscheinlichkeitstheorie verw. Geb. **15** (1970), 230–238.

[23] K. Yosida, Functional Analysis, Academic Press, Berlin-Göttingen-Heidelberg, 1965.

[24] K. Yosida, Positive pseudo-resolvents and potentials, Proc. Japan Acad. **41** (1965), 1–5.

[25] K. Yosida, Positive resolvents and potentials, Z. Wahrscheinlichkeitstheorie verw. Geb. **8** (1967), 210–218.

Department of Mathematics,
University of Auckland,
New Zealand