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Locally Ascendantly Coalescent Classes of Lie Algebras

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Introduction

In the recent study of infinite-dimensional Lie algebras the concepts of coalescence, ascendant coalescence and local coalescence have been investigated considerably in detail [1–12]. However, very little is known concerning local ascendant coalescence. The only fact known about it is that the class $\mathfrak{N}$ of nilpotent Lie algebras over a field of characteristic 0 is locally ascendantly coalescent [5, 6]. Therefore it is desirable for us to know more about this concept. The purpose of this paper is to investigate the properties of locally ascendantly coalescent classes and to show that several known classes and others are locally ascendantly coalescent.

We shall show that if a class $\mathfrak{X}$ is a Q-closed and locally ascendantly coalescent (resp. ascendantly coalescent) subclass of $(\mathrm{E}\mathfrak{A})_{(\omega)}$ then the class $\mathfrak{X}_{(\omega)}$ is locally ascendantly coalescent (resp. ascendantly coalescent). We shall also show that, for the classes $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathfrak{X} \leq \mathfrak{Y} \leq \acute{\mathrm{M}}\mathfrak{X}$, $\mathfrak{X}$ is locally ascendantly coalescent if and only if so is $\mathfrak{Y}$. The applications of these results to the special case where $\mathfrak{X}$ is the class $\mathfrak{N}$ of nilpotent Lie algebras over a field of characteristic 0 yield that the following classes are locally ascendantly coalescent: $\mathfrak{N}_{(\omega)}$, the class $\mathfrak{D}$ (resp. $\mathfrak{D}'$) of Lie algebras L such that every subalgebra of L is a subideal (resp. an ascendant subalgebra), the class $\mathfrak{Ft}$ of Fitting algebras, the class $\mathfrak{B}$ of Baer algebras, the class $\mathfrak{Gr}$ of Gruenberg algebras, and the class $\mathfrak{Z}$ of hypercentral algebras.

1.

Throughout this paper, we shall be concerned with Lie algebras over an arbitrary field Φ which are not necessarily finite-dimensional, and we denote by $\mathfrak{X}$ an arbitrary class of Lie algebras over Φ, unless otherwise specified.

$\mathfrak{X}$ is ascendantly coalescent provided the join of two ascendant $\mathfrak{X}$-subalgebras of any Lie algebra L is always an ascendant $\mathfrak{X}$-subalgebra of L. $\mathfrak{X}$ is locally ascendantly coalescent [5] provided whenever H and K are ascendant $\mathfrak{X}$-subalgebras of a Lie algebra L, for every finite subset F of the join $\langle H, K\rangle$ there exists an ascendant $\mathfrak{X}$-subalgebra X of L such that $F \subseteq X \leq \langle H, K\rangle$. Any ascendantly coalescent class of Lie algebras is obviously locally ascendantly coalescent.

We denote by $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$, $\mathfrak{F}$ and $\mathfrak{G}$ respectively the classes of nilpotent, solvable, finite-dimensional and finitely generated Lie algebras. It is well known that, when Φ is of characteristic 0, the classes $\mathfrak{F}$, $\mathfrak{N} \cap \mathfrak{F}$ and $\mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ are ascendantly coalescent and the class $\mathfrak{N}$ is locally ascendantly coalescent [6].

For any class $\mathfrak{X}$, $\acute{\mathrm{M}}\mathfrak{X}$ is the class of Lie algebras L such that every finite subset of L lies inside an ascendant $\mathfrak{X}$-subalgebra of L [10], and $\acute{\mathrm{N}}\mathfrak{X}$ is the class of Lie algebras which are generated by their ascendant $\mathfrak{X}$-subalgebras [8]. $\mathfrak{X}$ is Q-closed (resp. S-closed) provided every quotient algebra (resp. subalgebra) of an $\mathfrak{X}$-algebra is always an $\mathfrak{X}$-algebra. For any class $\mathfrak{X}$ of Lie algebras, $\mathfrak{X}_{(\omega)}$ (resp. $\mathfrak{X}_{\omega}$) denotes the class of Lie algebras L such that $L/L^{(\omega)} \in \mathfrak{X}$ (resp. $L/L^{\omega} \in \mathfrak{X}$) [9], where

$$L^{(\omega)} = \bigcap_{i=0}^{\infty} L^{(i)}, \qquad L^{\omega} = \bigcap_{i=1}^{\infty} L^{i}.$$

We shall give the following two lemmas for later use. The first lemma is a slight modification of the statement (1) before Theorem 3.2.5 in [5].

LEMMA 1.1. *If H is an ascendant $\mathfrak{G}$-subalgebra of a Lie algebra L, then $H^{(\omega)}$ and H^{ω} are ascendantly stable in L and especially characteristic ideals of L.*

PROOF. Let L be an ascendant subalgebra of a Lie algebra K. Then H is an ascendant subalgebra of K, whence $H \lhd^{\sigma} K$ for some ordinal σ. Let x be any element of K and let M be the finite set of generators of H. Denote by Λ the set of all ordinals $\lambda \leq \sigma$ for which

$$[x, {}_{n}M] \subseteq H_{\lambda}$$

for some positive integer n. Then Λ is non-empty and therefore Λ contains the first ordinal μ. It is easily seen that $\mu = 0$. Hence

$$[x, {}_{n}M] \subseteq H \qquad \text{for some } n.$$

It follows that

$$[x, {}_{n}H] \subseteq H.$$

Hence

$$[x, H^{n+m-1}] \subseteq [x, {}_{n+m-1}H] \subseteq H^{m}$$

for $m = 1, 2, 3, \ldots$, and therefore

$$[x, H^{\omega}] \subseteq H^{\omega}.$$

Similarly

$$[x, H^{(n)}] \subseteq [x, H^{2^n}] \subseteq [x, {}_{2^n}H] \subseteq H^{(1)}.$$

By induction on m we have

$$[x, H^{(n+m-1)}] \subseteq H^{(m)}$$

for $m=1, 2, 3,\dots$. Therefore

$$[x, H^{(\omega)}] \subseteq H^{(\omega)}.$$

Hence H^{ω} and $H^{(\omega)}$ are ideals of K. Thus H^{ω} and $H^{(\omega)}$ are ascendantly stable in L.

LEMMA 1.2. *Every perfect ascendant subalgebra of a Lie algebra L is an ideal of L.*

This is Proposition 1.3.5 in [5].

2.

We begin with the following

PROPOSITION 2.1. (a) *If $\mathfrak{X}$ is locally ascendantly coalescent, then $\mathfrak{X} \cap \mathfrak{G}$ is ascendantly coalescent.*

(b) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are* s*-closed and locally ascendantly coalescent, then so is $\mathfrak{X} \cap \mathfrak{Y}$.*

(c) *If $\mathfrak{X}$ is $\acute{\text{M}}$-closed and locally ascendantly coalescent and $\mathfrak{Y}$ is ascendantly coalescent, then $\mathfrak{X} \cap \mathfrak{Y}$ is ascendantly coalescent.*

PROOF. (a) The proof is immediate.

(b) Let H and K be ascendant $\mathfrak{X} \cap \mathfrak{Y}$-subalgebras of a Lie algebra L. Put $J=\langle H, K\rangle$ and let F be any finite subset of J. Then there exist subalgebras X and Y of J containing F such that

$$X \in \mathfrak{X}, \qquad X \text{ asc } L,$$

$$Y \in \mathfrak{Y}, \qquad Y \text{ asc } L.$$

Since $\mathfrak{X}$ and $\mathfrak{Y}$ are s-closed, it follows that $X \cap Y$ is an ascendant $\mathfrak{X} \cap \mathfrak{Y}$-subalgebra of L. Hence $\mathfrak{X} \cap \mathfrak{Y}$ is locally ascendantly coalescent.

(c) Let H and K be ascendant $\mathfrak{X} \cap \mathfrak{Y}$-subalgebras of a Lie algebra L. Put $J=\langle H, K\rangle$. Then J is an ascendant $\mathfrak{Y}$-subalgebra of L. For any finite subset F of J, there exists an ascendant $\mathfrak{X}$-subalgebra X of L such that

$$F \subseteq X \leq J.$$

It follows that X is an ascendant $\mathfrak{X}$-subalgebra of J. Hence $J \in \acute{\mathrm{M}}\mathfrak{X} = \mathfrak{X}$. Therefore $\mathfrak{X} \cap \mathfrak{Y}$ is ascendantly coalescent.

We generalize Theorem 2.2 in [11] in the following

THEOREM 2.2. *If $\mathfrak{X}$ is* Q*-closed and locally ascendantly coalescent, then $\mathfrak{X}_{(\omega)} \cap \mathfrak{G}$ and $\mathfrak{X}_{\omega} \cap \mathfrak{G}$ are ascendantly coalescent.*

PROOF. Let H and K be ascendant $\mathfrak{X}_{\omega} \cap \mathfrak{G}$-subalgebras of a Lie algebra L. Put $J = \langle H, K \rangle$. Then $J \in \mathfrak{G}$. Put $I = H^{\omega} + K^{\omega}$. Then I is an ideal of L by Lemma 1.1. Hence

$$H+I/I \text{ asc } L/I, \qquad K+I/I \text{ asc } L/I.$$

Since $\mathfrak{X}$ is Q-closed,

$$H+I/I \in \mathfrak{X} \cap \mathfrak{G}, \qquad K+I/I \in \mathfrak{X} \cap \mathfrak{G}.$$

Since $\mathfrak{X}$ is locally ascendantly coalescent, $\mathfrak{X} \cap \mathfrak{G}$ is ascendantly coalescent. Hence

$$J/I \in \mathfrak{X}, \qquad J/I \text{ asc } L/I.$$

Now

$$J^{\omega} \supset H^{\omega} + K^{\omega} = I.$$

By Q-closedness of $\mathfrak{X}$ it follows that

$$J/J^{\omega} \in \mathfrak{X}.$$

Therefore J is an ascendant $\mathfrak{X}_{\omega} \cap \mathfrak{G}$-subalgebra of L. Thus $\mathfrak{X}_{\omega} \cap \mathfrak{G}$ is ascendantly coalescent.

Ascendant coalescence of $\mathfrak{X}_{(\omega)} \cap \mathfrak{G}$ is similarly proved.

Over any field Φ of characteristic 0 the following classes are ascendantly coalescent [5, 11]:

$$\mathfrak{N} \cap \mathfrak{F}, \quad \mathfrak{N}_{(\omega)} \cap \mathfrak{F}, \quad \mathrm{E}\mathfrak{A} \cap \mathfrak{F}, \quad \mathfrak{F}$$

$$\text{Min}, \quad \text{Min-asc}, \quad \text{Min-si}, \quad \text{Min-}\lhd^{\alpha}\ (\alpha > 1),$$

$$\text{Min-}\lhd \cap \mathfrak{G}, \quad \text{Min-}\lhd \cap \text{Max-}\lhd.$$

We assert that the class $\mathrm{E}\mathfrak{A} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G}$ is ascendantly coalescent. In fact, let H and K be ascendant $\mathrm{E}\mathfrak{A} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G}$-subalgebras of a Lie algebra L. Put $J = \langle H, K \rangle$. Since $\mathfrak{N}_{\omega} \cap \mathfrak{G}$ is ascendantly coalescent by Theorem 2.2, J is an ascendant $\mathfrak{N}_{\omega} \cap \mathfrak{G}$-subalgebra of L. Put $I = H^{\omega} + K^{\omega}$. Then I is a solvable ideal of L as the sum of two solvable ideals. Hence $H+I/I$ and $K+I/I$ are ascendant $\mathfrak{N} \cap \mathfrak{G}$-subalgebras of L/I. Now the class $\mathfrak{N} \cap \mathfrak{G}$ is equal to $\mathfrak{N} \cap \mathfrak{F}$ and ascendantly coalescent, and therefore $J/I \in \mathfrak{N} \cap \mathfrak{G}$. It follows that J is solvable. Therefore $\mathrm{E}\mathfrak{A} \cap \mathfrak{N}_\omega \cap \mathfrak{G}$ is ascendantly coalescent, as was asserted.

By Theorem 2.2 it now follows that

$$\mathrm{E}\mathfrak{A}_{(\omega)} \cap \mathfrak{N}_\omega \cap \mathfrak{G} = (\mathrm{E}\mathfrak{A} \cap \mathfrak{N}_\omega \cap \mathfrak{G})_{(\omega)} \cap \mathfrak{G}$$

is ascendantly coalescent.

Now by the facts shown above and by applying Theorem 2.2 for the classes $\mathfrak{F}$, $\mathfrak{N}$, Min-$\lhd \cap \mathfrak{G}$ and Min-$\lhd \cap$ Max-$\lhd$ we have the following

COROLLARY 2.3. *Over any field Φ of characteristic* 0 *the following classes are ascendantly coalescent*:

$$\mathfrak{F}_{(\omega)} \cap \mathfrak{G}, \quad \mathfrak{F}_\omega \cap \mathfrak{G},$$

$$\mathfrak{N}_{(\omega)} \cap \mathfrak{G}, \quad \mathfrak{N}_\omega \cap \mathfrak{G},$$

$$\mathrm{E}\mathfrak{A} \cap \mathfrak{N}_\omega \cap \mathfrak{G}, \quad \mathrm{E}\mathfrak{A}_{(\omega)} \cap \mathfrak{N}_\omega \cap \mathfrak{G},$$

$$(\text{Min-}\lhd)_{(\omega)} \cap \mathfrak{G}, \quad (\text{Min-}\lhd)_\omega \cap \mathfrak{G},$$

$$(\text{Min-}\lhd \cap \text{Max-}\lhd)_{(\omega)} \cap \mathfrak{G}, \quad (\text{Min-}\lhd \cap \text{Max-}\lhd)_\omega \cap \mathfrak{G}.$$

For the classes related with minimal conditions we have the following relations:

$$(\text{Min})_{(\omega)} = (\text{Min-asc})_{(\omega)} = (\text{Min-si})_{(\omega)} = (\text{Min-}\lhd^\alpha)_{(\omega)} = \mathfrak{F}_{(\omega)} \quad (\alpha > 1),$$

$$(\text{Min})_\omega = (\text{Min-asc})_\omega = (\text{Min-si})_\omega = (\text{Min-}\lhd^\alpha)_\omega = \mathfrak{F}_\omega \quad (\alpha > 1),$$

$$\mathfrak{F}_{(\omega)} \cap \mathfrak{G} \le (\text{Min-}\lhd \cap \text{Max-}\lhd)_{(\omega)} \cap \mathfrak{G} \le (\text{Min-}\lhd)_{(\omega)} \cap \mathfrak{G}$$

$$\le \mathrm{E}\mathfrak{A}_{(\omega)} \cap \mathfrak{N}_\omega \cap \mathfrak{G},$$

$$\mathfrak{F}_\omega \cap \mathfrak{G} \le (\text{Min-}\lhd \cap \text{Max-}\lhd)_\omega \cap \mathfrak{G} \le (\text{Min-}\lhd)_\omega \cap \mathfrak{G} \le \mathfrak{N}_\omega \cap \mathfrak{G}.$$

By Proposition 2.1 over any field of characteristic 0 the intersection of any locally ascendantly coalescent class and any class in the above Corollary 2.3 is ascendantly coalescent.

REMARK. In the case where the basic field Φ is of arbitrary characteristic, we have the following statement:

If either (a) $\mathfrak{X}$ is Q-closed and locally ascendantly coalescent and $\mathfrak{X} \cap \mathfrak{N}$ is locally ascendantly coalescent, or (b) $\mathfrak{X}$ is $\{\mathrm{I}, \mathrm{Q}, \mathrm{E}\}$-closed and $\mathfrak{X} \cap \mathfrak{N}$ is locally ascendantly coalescent, then the classes

$$\mathfrak{X} \cap \mathfrak{N}_{(\omega)} \cap \mathfrak{G}, \quad \mathfrak{X} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G}, \quad \mathfrak{X} \cap \mathrm{E}\mathfrak{A} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G}$$

are all ascendantly coalescent.

3.

We first observe that $(\mathrm{E}\mathfrak{A})_{(\omega)}$ is the class of Lie algebras L such that $L^{(\omega)} = L^{(n)}$ for some positive integer n. For subclasses of the class $(\mathrm{E}\mathfrak{A})_{(\omega)}$ we can show the following statements which are analogous to Theorem 4.1 in [9].

THEOREM 3.1. *Let $\mathfrak{X}$ be a* Q*-closed subclass of $(\mathrm{E}\mathfrak{A})_{(\omega)}$.*
(a) *If $\mathfrak{X}$ is locally ascendantly coalescent, then so is $\mathfrak{X}_{(\omega)}$.*
(b) *If $\mathfrak{X}$ is ascendantly coalescent, then so is $\mathfrak{X}_{(\omega)}$.*

PROOF. (a) Let H and K be ascendant $\mathfrak{X}_{(\omega)}$-subalgebras of a Lie algebra L. Since $\mathfrak{X} \leq (\mathrm{E}\mathfrak{A})_{(\omega)}$,

$$\mathfrak{X}_{(\omega)} \leq (\mathrm{E}\mathfrak{A})_{(\omega)(\omega)} = (\mathrm{E}\mathfrak{A})_{(\omega)}.$$

Hence $H/H^{(\omega)}$ and $K/K^{(\omega)}$ are solvable and therefore

$$H^{(\omega)} = H^{(n)}, \qquad K^{(\omega)} = K^{(n)}$$

for some integer n. This shows that $H^{(\omega)}$ and $K^{(\omega)}$ are perfect. Hence these are ideals of L by Lemma 1.2. If we put $I = H^{(\omega)} + K^{(\omega)}$, then I is an ideal of L. Therefore

$$H+I/I \in \mathfrak{X}, \quad H+I/I \ \text{asc} \ L/I,$$

$$K+I/I \in \mathfrak{X}, \quad K+I/I \ \text{asc} \ L/I.$$

Put $J = \langle H, K \rangle$ and let F be any finite subset of J. If we denote by $\bar{F}$ the image of F under the natural homomorphism of L onto L/I, $\bar{F}$ is a finite subset of J/I. Since $\mathfrak{X}$ is locally ascendantly coalescent, there exists a subalgebra X of L such that

$$\bar{F} \subseteq X/I \leq J/I,$$

$$X/I \in \mathfrak{X}, \quad X/I \ \text{asc} \ L/I.$$

Therefore

$$F \subseteq X \leq J, \qquad X \ \text{asc} \ L.$$

Since $H^{(\omega)}$ and $K^{(\omega)}$ are perfect,

$$H^{(\omega)} \le X^{(\omega)}, \qquad K^{(\omega)} \le X^{(\omega)}.$$

It follows that $X/X^{(\omega)} \in \mathfrak{X}$ and hence $X \in \mathfrak{X}_{(\omega)}$. Thus $\mathfrak{X}_{(\omega)}$ is locally ascendantly coalescent.

(b) Let H and K be ascendant $\mathfrak{X}_{(\omega)}$-subalgebras of a Lie algebra L. Put $I = H^{(\omega)} + K^{(\omega)}$. Then by using the fact that $H^{(\omega)}$ and $K^{(\omega)}$ are ideals of L by Lemma 1.2 we deduce that $H+I/I$ and $K+I/I$ are ascendant $\mathfrak{X}$-subalgebras of L/I. Put $J = \langle H, K \rangle$. Since $\mathfrak{X}$ is ascendantly coalescent,

$$J/I \in \mathfrak{X}, \qquad J/I \text{ asc } L/I.$$

Clearly $I \subset J^{(\omega)}$. These, combined with Q-closedness of $\mathfrak{X}$, imply

$$J/J^{(\omega)} \in \mathfrak{X}, \qquad J \text{ asc } L.$$

Thus $\mathfrak{X}_{(\omega)}$ is ascendantly coalescent.

The proof is complete.

COROLLARY 3.2. *Let the basic field Φ be of characteristic* 0. *Then*

(a) $\mathfrak{N}_{(\omega)}$ *is locally ascendantly coalescent.*

(b) *The classes*

$$(\mathfrak{N} \cap \mathfrak{F})_{(\omega)}, \quad \mathfrak{F}_{(\omega)}, \quad (\text{Min-}\lhd \cap \mathfrak{G})_{(\omega)},$$

$$(\text{Min-}\lhd \cap \text{Max-}\lhd \cap \mathfrak{G})_{(\omega)} \quad \textit{and} \quad (\text{E}\mathfrak{A} \cap \mathfrak{G})_{(\omega)} \cap \mathfrak{N}_{\omega}$$

are ascendantly coalescent.

PROOF. Since $\mathfrak{N}$ is locally ascendantly coalescent, by Theorem 3.1 $\mathfrak{N}_{(\omega)}$ is locally ascendantly coalescent. Since $\mathfrak{N} \cap \mathfrak{F}$ and $\mathfrak{F}$ are ascendantly coalescent, $(\mathfrak{N} \cap \mathfrak{F})_{(\omega)}$ and $\mathfrak{F}_{(\omega)}$ are ascendantly coalescent by Theorem 3.1. The other classes are ascendantly coalescent by Corollary 2.3 and Theorem 3.1, since

$$(\text{Min-}\lhd \cap \mathfrak{G})_{(\omega)} = ((\text{Min-}\lhd)_{(\omega)} \cap \mathfrak{G})_{(\omega)},$$

$$(\text{E}\mathfrak{A} \cap \mathfrak{G})_{(\omega)} \cap \mathfrak{N}_{\omega} = (\text{E}\mathfrak{A}_{(\omega)} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G})_{(\omega)},$$

$$(\text{Min-}\lhd \cap \text{Max-}\lhd \cap \mathfrak{G})_{(\omega)} = ((\text{Min-}\lhd \cap \text{Max-}\lhd)_{(\omega)} \cap \mathfrak{G})_{(\omega)}.$$

REMARK. As for the class $\text{E}\mathfrak{A} \cap \mathfrak{F}$ we have

$$(\text{E}\mathfrak{A} \cap \mathfrak{F})_{(\omega)} = \mathfrak{F}_{(\omega)}.$$

4.

We may ask whether the closure operations $\acute{\text{M}}$ and $\acute{\text{N}}$ preserve the local ascendant coalescence of classes of Lie algebras. In this section we shall give several answers to this question. We begin with

PROPOSITION 4.1. *If $\mathfrak{X}$ is locally ascendantly coalescent, then*

$$\acute{\mathrm{M}}\mathfrak{X} = \acute{\mathrm{N}}\mathfrak{X} \leq \mathrm{L}\mathfrak{X}.$$

PROOF. Let L be an $\acute{\mathrm{N}}\mathfrak{X}$-algebra. Then L is generated by ascendant $\mathfrak{X}$-subalgebras of L. If F is a finite subset of L, it follows that F is contained in a subalgebra H generated by a finite number of ascendant $\mathfrak{X}$-subalgebras of L. By the assumption that $\mathfrak{X}$ is locally ascendantly coalescent, we can find an ascendant $\mathfrak{X}$-subalgebra X of L such that

$$F \subseteq X \leq H.$$

Therefore L is an $\acute{\mathrm{M}}\mathfrak{X}$-algebra. Thus $\acute{\mathrm{M}}\mathfrak{X} \leq \acute{\mathrm{M}}\mathfrak{X}$. The statement is now immediate since

$$\acute{\mathrm{M}}\mathfrak{X} \leq \acute{\mathrm{N}}\mathfrak{X}, \qquad \acute{\mathrm{M}}\mathfrak{X} \leq \mathrm{L}\mathfrak{X}.$$

In the following theorem we shall show, in a more general form, the fact that if $\mathfrak{X}$ is locally ascendantly coalescent then so is $\acute{\mathrm{M}}\mathfrak{X}$.

THEOREM 4.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes such that $\mathfrak{X} \leq \mathfrak{Y} \leq \acute{\mathrm{M}}\mathfrak{X}$. Then $\mathfrak{X}$ is locally ascendantly coalescent if and only if $\mathfrak{Y}$ is locally ascendantly coalescent.*

PROOF. Assume that $\mathfrak{X}$ is locally ascendantly coalescent. Let H and K be ascendant $\mathfrak{Y}$-subalgebras of a Lie algebra L. Put $J = \langle H, K \rangle$. If F is a finite subset of J, then there exist finite sets $A \subseteq H$ and $B \subseteq K$ such that

$$F \subseteq \langle A, B \rangle \leq J.$$

Since H is an $\acute{\mathrm{M}}\mathfrak{X}$-subalgebra, there exists an ascendant $\mathfrak{X}$-subalgebra M of H containing A. Similarly there exists an ascendant $\mathfrak{X}$-subalgebra N of K containing B. Then

$$F \subseteq \langle M, N \rangle.$$

Now M and N are ascendant $\mathfrak{X}$-subalgebras of L and $\mathfrak{X}$ is locally ascendantly coalescent. Therefore there exists an ascendant $\mathfrak{X}$-subalgebra X of L such that

$$F \subseteq X \leq \langle M, N \rangle.$$

Clearly X belongs to $\mathfrak{Y}$ and

$$F \subseteq X \leq J.$$

Therefore $\mathfrak{Y}$ is locally ascendantly coalescent.

Conversely, assume that $\mathfrak{Y}$ is locally ascendantly coalescent. Let H and K be ascendant $\mathfrak{X}$-subalgebras of L. Then H and K are ascendant $\mathfrak{Y}$-subalgebras of L. Put $J=\langle H, K\rangle$ and let F be any finite subset of J. Then there exists an ascendant $\mathfrak{Y}$-subalgebra Y of L such that

$$F \subseteq Y \leq J.$$

Since $\mathfrak{Y} \leq \acute{\mathrm{M}}\mathfrak{X}$, there exists an ascendant $\mathfrak{X}$-subalgebra X of Y such that

$$F \subseteq X \leq Y.$$

Clearly X is an ascendant $\mathfrak{X}$-subalgebra of L and

$$F \subseteq X \leq J.$$

Therefore $\mathfrak{X}$ is locally ascendantly coalescent.

The proof is complete.

Over any field Φ of characteristic 0 there are several classes related to the class $\mathfrak{N}$ as follows.

$\mathfrak{D}$: The class of Lie algebras L such that every subalgebra of L is a subideal of L.

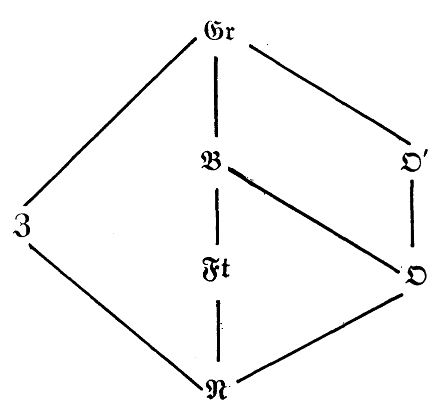
$\mathfrak{Ft}$: The class of Fitting algebras, that is, the class of Lie algebras which are generated by their nilpotent ideals.

$\mathfrak{B}$: The class of Baer algebras, that is, the class of Lie algebras which are generated by their nilpotent subideals.

$\mathfrak{Gr}$: The class of Gruenberg algebras, that is, the class of Lie algebras which are generated by their nilpotent ascendant subalgebras.

$\mathfrak{Z}$: The class of hypercentral Lie algebras.

We now define the new class $\mathfrak{D}'$ as follows: A Lie algebra L belongs to $\mathfrak{D}'$ if and only if every subalgebra of L is an ascendant subalgebra.



This diagram is obtained by adding $\mathfrak{D}'$ to Fig. 2 and Fig. 3 in [5] and shows the interrelation among the above classes. The inclusions $\mathfrak{D}<\mathfrak{D}'<\mathfrak{Gr}$ are strict, as we shall show it by examples in Section 5.

COROLLARY 4.3. *Let the basic field Φ be of characteristic* 0. *Then the following classes are locally ascendantly coalescent*:

$$\mathfrak{D},\ \mathfrak{D}',\ \mathfrak{Ft},\ \mathfrak{B},\ \mathfrak{Gr},\ \mathfrak{Z},$$

$$\mathrm{M}\mathfrak{D}',\ \mathrm{M}\mathfrak{Z},\ \mathrm{M}(\mathfrak{N}_{(\omega)}),\ \acute{\mathrm{M}}(\mathfrak{N}_{(\omega)}).$$

PROOF. By definition $\mathfrak{Gr}=\acute{\mathrm{N}}\mathfrak{N}$. Since $\mathfrak{N}$ is locally ascendantly coalescent, we have

$$\mathfrak{Gr}=\acute{\mathrm{M}}\mathfrak{N}$$

by Proposition 4.1. By Corollary 3.2 $\mathfrak{N}_{(\omega)}$ is locally ascendantly coalescent. Therefore, taking account of the above diagram, we can use Theorem 4.2 to establish the statement.

We can show that, if $\mathfrak{X}\leq\mathfrak{Y}\leq\acute{\mathrm{N}}\mathfrak{X}$, in the definition of local ascendant coalescence of $\mathfrak{Y}$ "for any two ascendant $\mathfrak{Y}$-subalgebras" may be replaced by "for any finite number of ascendant $\mathfrak{X}$-subalgebras". Namely, we have the following

THEOREM 4.4. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes such that $\mathfrak{X}\leq\mathfrak{Y}\leq\acute{\mathrm{N}}\mathfrak{X}$. Then the following statements are equivalent.*

(a) *$\mathfrak{Y}$ is locally ascendantly coalescent.*

(b) *For any finite number of ascendant $\mathfrak{X}$-subalgebras $H_1,\dots, H_n$ of a Lie algebra L and for any finite subset F of $\langle H_1,\dots, H_n\rangle$, there exists an ascendant $\mathfrak{Y}$-subalgebra Y of L such that*

$$F\subseteq Y\leq\langle H_1,\dots, H_n\rangle.$$

This can be proved in the same way as Theorem 3.1 in [12]. We have only to replace there N, local coalescence and subideals respectively by $\acute{\mathrm{N}}$, local ascendant coalescence and ascendant subalgebras.

5.

In this section we shall show by examples that the class $\mathfrak{D}'$, which has been introduced in Section 4, is different from both $\mathfrak{D}$ and $\mathfrak{Gr}$.

First let $L=A+\Phi y$ be the Lie algebra over a field Φ of characteristic 0 defined in [6] as follows: A is an abelian subalgebra with basis e_0, e_1, e_2,... and

$$[e_i, y] = \begin{cases} e_{i-1} & \text{for} \quad i > 0 \\ 0 & \text{for} \quad i = 0. \end{cases}$$

Then L belongs to the class $\mathfrak{D}'$. In fact, let H be any subalgebra of L. If $H \leq A$, then

$$H \lhd A \lhd L.$$

If $H \nleq A$, put

$$H_i = H + \Phi e_0 + \Phi e_1 + \cdots + \Phi e_{i-1} \qquad \text{for} \quad i = 1, 2, \ldots,$$

$$H_\omega = \bigcup_{i=1}^{\infty} H_i.$$

Then

$$H = H_0 \lhd H_1 \lhd H_2 \lhd \ldots, \qquad H_\omega = L.$$

Therefore $H \operatorname{asc} L$. Thus $L \in \mathfrak{D}'$. However L does not belong to $\mathfrak{D}$. In fact, assume that $L \in \mathfrak{D}$. Then Φy is a subideal of L. Hence

$$\Phi y \lhd^n L$$

for some positive integer n. It follows that

$$e_0 = [e_n, {}_n y] \in \Phi y,$$

which is a contradiction. Therefore $L \notin \mathfrak{D}$. Thus $\mathfrak{D} < \mathfrak{D}'$.

Secondly, we consider the Roseblade-Stonehewer algebra L in [5]. As a vector space L is the sum of an abelian ideal V and a subalgebra J generated by two abelian subideals of L. It is known that $J = I_L(J)$. Hence J is not an ascendant subalgebra of L. Therefore $L \notin \mathfrak{D}'$. However $L \in \mathfrak{Gr}$, since L is generated by three abelian subideals. Therefore $\mathfrak{D}' < \mathfrak{Gr}$.

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