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Forced Oscillations in General Ordinary Differential Equations with Deviating Arguments

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1. Introduction

Recently quite a few authors have spent considerable effort in finding conditions to ensure that nonoscillatory solutions of both ordinary and their companion retarded differential equations approach zero asymptotically. For these criteria, the reader is referred to [3, 5, 6, 8, 9] and references cited in them. However the literature is very scanty about similar results in regard to oscillatory solutions of these equations. Our purpose here is to find conditions to ensure that the oscillatory solutions of the general n-th order equation

$$(1) \qquad (r(t)y'(t))^{(n-1)}+a(t)y_{\tau}(t)=f(t),\ y_{\tau}(t)\equiv y(t-\tau(t))$$

approach to zero as $t\to\infty$.

We now give definitions and assumptions that hold in the rest of this paper:

(i) $\tau(t)$, $r(t)$, $a(t)$, $f(t)$ are real, continuous and defined on the whole real line R.

(ii) $r(t)$ and $\tau(t)$ are positive on R. $\tau(t)$ is bounded above by $K_0>0$.

We call a function $h(t)\in C[0,\infty)$ oscillatory if it has arbitrarily large zeros. Otherwise $h(t)$ is called nonoscillatory on the half line $[0,\infty)$.

In what follows only continuous and extendable solutions of equations (1) and (2) will be considered. The term "solution" applies only to such solutions in this manuscript.

2. Main results

LEMMA 1. *Suppose $p_1>p_2>p_3>p_4>\cdots>p_{n-2}$ are respectively the zeros of*

$$(r(t)y'(t))',\ (r(t)y'(t))'',\dots,\ (r(t)y'(t))^{(n-3)},\ (r(t)y'(t))^{(n-2)},$$

where $y(t)$ is a solution of equation (1). *Further suppose that $t_1<p_{n-2}$ and $t_2>p_1$ are zeros of $y(t)$. Suppose*

$$M = \max|y(t)|, \qquad t \in [t_1, t_2].$$

If $|y_\tau(t)| \le M$ *in* $[t_1, t_2]$, *then*

(3) $$4 \le \left(\int_{t_1}^{t_2} 1/r(t)dt\right)\left(\int_{t_1}^{t_2} \frac{(t-t_1)^{n-2}}{(n-2)!}|a(t)|dt + \frac{1}{M}\int_{t_1}^{t_2} \frac{(t-t_1)^{n-2}}{(n-2)!}|f(t)|dt\right).$$

PROOF. On repeated integration from equation (1) we have

(4) $$\pm(r(t)y'(t))' + \int_t^{p_1}\int_{s_2}^{p_2}\int_{s_3}^{p_3}\cdots\int_s^{p_{n-2}} a(s)y_\tau(s)dsds_{n-2}\cdots ds_2$$
$$= \int_{t_1}^{p_1}\int_{s_2}^{p_2}\cdots\int_s^{p_{n-2}} f(s)ds\cdots ds_2.$$

Since $p_1 > p_2 > p_3 > \cdots > p_{n-2}$ we get from (4)

$$|(r(t)y'(t))'| \le \int_t^{p_1}\int_{s_2}^{p_1}\cdots\int_{s_{n-2}}^{p_1} |a(s)|\,|y_\tau(s)|dsds_{n-2}\cdots ds_2$$
$$+ \int_t^{p_1}\int_{s_2}^{p_1}\cdots\int_{s_{n-2}}^{p_1} |f(s)|dsds_{n-2}\cdots ds_2,$$

which gives

(5) $$|(r(t)y'(t))'| \le \int_t^{p_1} \frac{(s-t)^{n-3}}{(n-3)!}|a(s)|\,|y_\tau(s)|ds + \int_t^{p_1} \frac{(s-t)^{n-3}}{(n-3)!}|f(s)|ds.$$

Let

(6) $$M = |y(t_0)|, \qquad t_0 \in [t_1, t_2].$$

Now

$$\pm M = y(t_0) = \int_{t_1}^{t_0} y'(t)dt,$$

which yields

(7) $$M \le \int_{t_1}^{t_0} |y'(t)|dt.$$

Similarly

$$\pm M = -\int_{t_0}^{t_2} y'(t)dt$$

gives

(8) $$M \le \int_{t_0}^{t_2} |y'(t)|dt.$$

Adding (7) and (8) we have

$$2M \leq \int_{t_1}^{t_2} |y'(t)| dt$$

$$= \int_{t_1}^{t_2} \frac{1}{\sqrt{r}} \cdot [r(t)\,|y'(t)|]^{1/2}\,|y'(t)|^{1/2} dt.$$

By Schwarz's inequality we have

$$4M^2 \leq \int_{t_1}^{t_2} \frac{1}{r(t)}\, dt \cdot \int_{t_1}^{t_2} (r(t)y'(t))y'(t)dt. \tag{9}$$

Integrating the second integral by parts we have

$$\frac{4M^2}{\int_{t_1}^{t_2} 1/r(t)dt} \leq -\int_{t_1}^{t_2} y(t)(r(t)y'(t))'dt \tag{10}$$

since $y(t_1)=y(t_2)=0$. From (10) we get

$$\frac{4M^2}{\int_{t_1}^{t_2} 1/r(t)dt} \leq \int_{t_1}^{t_2} |y(t)|\,|r(t)y'(t))'|\,dt. \tag{11}$$

From (6) and (11) we have

$$\frac{4M^2}{\int_{t_1}^{t_2} 1/r(t)dt} \leq M\int_{t_1}^{t_2} |r(t)y'(t))'|\,dt$$

$$\frac{4M}{\int_{t_1}^{t_2} 1/r(t)dt} \leq \int_{t_1}^{t_2} |(r(t)y'(t))'|\,dt. \tag{12}$$

From (5) and (12) we get

$$\frac{4M}{\int_{t_2}^{t_1} 1/r(t)dt} \leq \int_{t_1}^{t_2}\int_{s}^{p_1} \frac{(x-s)^{n-3}}{(n-3)!} |y_\tau(x)|\,|a(x)|\,dxds \tag{13}$$

$$+\int_{t_1}^{t_2}\int_{s}^{p_1} \frac{(x-s)^{n-3}}{(n-3)!}\,|f(x)|\,dxds.$$

Dividing by M and noting that $t_2 > p_1$ we have from (13)

$$\frac{4}{\int_{t_1}^{t_2} 1/r(t)dt} \leq \int_{t_1}^{t_2}\int_{s}^{t_2} \frac{(x-s)^{n-3}}{(n-3)!}\,|a(x)|\,dxds \tag{14}$$

$$+\frac{1}{M}\int_{t_1}^{t_2}\int_{t_s}^{t_2} \frac{(x-s)^{n-3}}{(n-3)!}\,|f(x)|\,dxds.$$

From (14) we have

$$4 \le \int_{t_1}^{t_2} \frac{1}{r(t)} dt \left[\int_{t_1}^{t_2} \frac{(s-t_1)^{n-2}}{(n-2)!} |a(s)|\, ds + \frac{1}{M} \int_{t_1}^{t_2} \frac{(s-t_1)^{n-2}}{(n-2)!} |f(s)|\, ds \right]$$

and the proof is complete.

THEOREM 1. *Let $y(t)$ be an oscillatory solution of equation* (1). *Suppose further that*

(15) $$\int^{\infty} t^{n-2} |f(t)| dt < \infty$$

(16) $$\int^{\infty} t^{n-2} |a(t)| dt < \infty$$

(17) $$\int^{\infty} \frac{1}{r(t)} dt < \infty .$$

Then $y(t) \to 0$ as $t \to \infty$.

PROOF. Suppose to the contrary that $y(t) \nrightarrow 0$ as $t \to \infty$. Then

(18) $$\liminf_{t \to \infty} |y(t)| = 0$$

and

(19) $$\limsup_{t \to \infty} |y(t)| > 2d$$

for some $d > 0$. Due to oscillatory nature of $y(t)$, $(r(t)y'(t))^{(n-2)}$ must be oscillatory. In fact if $(r(t)y'(t))^{(n-2)}$ is nonoscillatory, then $r(t)y'(t)$ assumes one sign eventually. Since $r(t) > 0$, $y'(t)$ becomes nonoscillatory which in turn forces $y(t)$ to be nonoscillatory, a contradiction. Hence $(r(t)y'(t))^{(n-2)}$ is oscillatory. Similarly $(r(t)y'(t))^{(n-3)}, (r(t)y'(t))^{(n-4)}, \ldots, (r(t)y(t))'$ are all oscillatory. Let T be large enough so that

$$T < t_1 + K_0 < q < p_{n-2} < p_{n-3} < \cdots < p_3 < p_2 < p_1 < t_2$$

are points where

(20) $$y(t_1) = 0, \quad y(q) = 0, \quad y(t_2) = 0$$

(21) $$(r(p_i)y'(p_i))^{(i)} = 0, \qquad i = 1, 2, 3, \ldots, n-2.$$

Let

$$M_0 = \max |y(t)|, \qquad t \in [t_1, t_2].$$

We shall show that $y(t)$ is bounded. Suppose not. Let $q_1 > t_2$ be the first point such that $y(q_1) > M_0$. Let $t_3 > q_1$ be the smallest zero of $y(t)$. Let

(22) $$L_1 = \max|y(t)|, \qquad t \in [t_1, t_3].$$

Let

(23) $$L_2 = \max|y(t)|, \qquad t \in [t_2, t_3].$$

Then $L_1 \geq L_2$. Let

(24) $$L_1 = y(t_q), \qquad t_q \in [t_1, t_3].$$

Since by construction $L_1 > M_0$ we must have a point t_q such that

(25) $$t_3 > t_q \geq q_1 > t_2.$$

Hence

(26) $$L_2 \geq L_1$$

From (24) and (25)

$$L_1 = L_2 = L_0.$$

Thus $\max|y(t)|$ in $[t_1, t_3]$ is achieved at a point $t_{q_1} \geq t_q$ and

$$t_{q_1} \in [t_2, t_3].$$

Thus

(27) $$L_0 = \max|y(t)|, \quad t \in [t_1, t_3] \text{ and achieved in } [t_2, t_3].$$

Now for $t \in [q, t_3]$

$$t - \tau(t) \geq t - K_0$$

and by construction

$$t_1 < t - K_0 < t_3 \qquad \text{for} \quad t \in [q, t_3].$$

Hence

(28) $$\max|y_\tau(t)| \leq L_0, \qquad t \in [q, t_3],$$

(29) $$\max|y(t)| \leq L_0, \qquad t \in [q, t_3],$$

and

(30) $$y(t_{q_1}) = L_0.$$

Replacing M by L_0, t_2 by t_3 and t_1 by q we get from conclusion (3) of Lemma 1

$$(31) \qquad \frac{4}{\int_q^{t_3} 1/r(t)dt} \leq \int_q^{t_3} \frac{(t-t_1)^{n-2}}{(n-2)!} |a(t)| dt + \frac{1}{L_0} \int_q^{t_3} \frac{(t-t_1)^{n-2}}{(n-2)!} |f(t)| dt.$$

Since in (31), the right hand side can be made as small as we please and the left hand side can be made as large as we please in view of (15), (16) and (17), and choices of q, t_3, this is a contradiction and hence $y(t)$ is bounded.

In fact looking at the proof more carefully we have shown that

$$(32) \qquad |y(t)| \leq M_0, \qquad t \in [t_2, \infty)$$

and hence

$$(33) \qquad |y_\tau(t)| \leq M_0, \qquad t \in [t_2+K_0, \infty).$$

Let now

$$(34) \qquad T < t_2+K < t_3 < e_{n-2} < e_{n-1} < \cdots < e_3 < e_2 < e_1 < T_0$$

be such that

$$(35) \qquad y(t_2) = y(t_3) = 0$$

and

$$(36) \qquad (r(e_i)y'(e_i))^{(i)} = 0$$

$i=1, 2,\ldots, n-2$; T_0 is such that

$$\left.\begin{array}{l} \max |y(t)| > d \\ \max |y_\tau(t)| > d \end{array}\right] \quad t \in [t_3, T_0].$$

Let $t_4 > T_0$ be such that $y(t_4)=0$. Let

$$M_1 = \max |y(t)|, \qquad t \in [t_3, t_4].$$

Then

$$(37) \qquad d < M_1 \leq M_0.$$

Now in the proof of Lemma 1 we recourse to inequality (13). Replacing t_1 by t_3, t_2 by t_4, M by M_1 and 'p's by 'e's we have

$$(38) \qquad \frac{4M_1}{\int_{t_3}^{t_4} 1/r(t)dt} \leq \int_{t_3}^{t_4}\int_s^{e_1} \frac{(x-s)^{n-3}}{(n-3)!} |y_\tau(x)| |a(x)| dxds.$$

From (37) and (38) and the fact that $e_1 < t_4$ we have

$$\frac{4d}{\int_{t_3}^{t_4} 1/r(t)dt} \leq M_0 \int_{t_3}^{t_4}\int_{s}^{t_4} \frac{(x-s)^{n-3}}{(n-3)!} |a(x)| dxds \tag{39}$$

$$+ \int_{t_3}^{t_4}\int_{s}^{t_4} \frac{(x-s)^{n-3}}{(n-3)!} |f(x)| dxds$$

$$= M_0 \int_{t_3}^{t_4} \frac{(s-t_3)^{n-2}}{(n-2)!} |a(s)| ds + \int_{t_3}^{t_4} \frac{(s-t_3)^{n-2}}{(n-2)!} |f(s)| ds.$$

Since right hand side of (39) can be made arbitrarily small and left hand side arbitrarily large by proper choice of t_3 and t_4, a contradiction is obtained. This completes the proof.

EXAMPLE 1. Consider the equation

$$(e^t y'(t))'' + e^{-t-2\pi} \sin t y(t-\pi) = 4e^{-t} \cos t + 2e^{-t} \sin t - e^{-3t} \sin^2 t. \tag{40}$$

All conditions of Theorem 1 are satisfied. Hence all oscillatory solutions of equation (40) approach to zero as $t \to \infty$. One such solution is

$$y(t) = e^{-2t} \sin t.$$

REMARK. It is not possible to violate condition (17) on $r(t)$ if (15) and (16) hold. The following example indicates this fact.

EXAMPLE 2. The equation

$$y'''(t) + e^{-t} y(t) = \frac{3 \sin(\ln t)}{t^3} + \frac{\cos(\ln t)}{t^3} + e^{-t} \sin(\ln t), \tag{41}$$

$$t > 0, \qquad \tau(t) \equiv 0$$

has

$$y(t) = \sin(\ln t)$$

as an oscillatory solution not approaching zero. Only the condition on $r(t)$ is violated.

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