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A New Family in the Stable Homotopy Groups of Spheres II

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Statement of Results

This paper is a continuation of [5] with the same title. We shall use all notations defined in Part I [5].

In Theorem A, we constructed nonzero elements $\rho_{t,r}$, $t \geqq 1, 1 \leqq r \leqq p-1$, of order p in G_*, the stable homotopy group of spheres. Here p denotes always a fixed prime integer with $p \geqq 5$. The following result is a sequel to Theorem A. We put $q = 2(p-1)$.

THEOREM **AII.** *There exist nonzero elements*

$$\rho_{t,0} \in G_{(tp+t-1)pq-2}, \qquad t \geqq 2,$$

of order p such that

$$\rho_{t,1} \in \langle \rho_{t,0}, p, \alpha_1 \rangle .$$

REMARK. For $t=1$, there is no element $\rho_{1,0}$ with $\rho_{1,1} \in \langle \rho_{1,0}, p, \alpha_1 \rangle$ This fact is equivalent to the nontriviality of the differential on $E_2^{2p-1, p^2 q}$, proved by H. Toda [8], in the Adams spectral sequence. The first element $\rho_{2,0}$ coincides with the element ρ_0 constructed in [2].

We recall the stable homotopy rings $\mathscr{A}_*(M)$ and $\mathscr{A}_*(X(r))$, $r \geqq 1$, of M $= S^1 \cup_p e^2$ and $X(r) = S^2 M \cup_{\alpha^r} CS^{rq+2} M$; α being the generator of $\mathscr{A}_q(M) = Z_p$ (see Definition 1.1). We consider elements in $\mathscr{A}_*(M)$ and $\mathscr{A}_*(X(p))$ corresponding to the ones in Theorem A II, and obtain the following two results as sequels to Theorems B and C.

THEOREM **BII.** *There exist nonzero elements*

$$\rho_0(t) \in \mathscr{A}_{(tp+t-1)pq-1}(M), \qquad t \geqq 2,$$

such that $\rho_0(t)\alpha = \rho(t)$, $\rho_0(t)\alpha^{p-1} = \alpha^{p-1}\rho_0(t) = \beta_{(tp)}$, $\rho_0(t)\alpha^p = \alpha^p \rho_0(t) = 0$ *and* $\pi_* i^* \rho_0(t) = \rho_{t,0}$

Here $\rho(t)$ and $\beta_{(t)}$ are the elements in $\mathscr{A}_*(M)$ introduced in Theorem B and [9], respectively, and we denote the cofiberings for M and $X(r)$ by $S^1 \xrightarrow{i} M \xrightarrow{\pi} S^2$ and $S^2M \xrightarrow{j_r} X(r) \xrightarrow{k_r} S^{rq+3}M$. Recall also the natural map A: $S^qX(r-1) \to X(r)$.

THEOREM CII. *There exist nonzero elements*

$$R(p)^{(t)} \in \mathscr{A}_{t(p+1)pq}(X(p)), \qquad t \geqq 2,$$

such that $R(p)^{(t)}A = AR(p-1)^t$ *and* $k_{p*}j_p^*R(p)^{(t)} = \rho_0(t)$, *where* $R(p-1)^t$ *is the t-times composition of the element* $R(p-1)$ *constructed in Theorem* C.

A sequel to Theorem D is given as follows:

THEOREM DII. *The element* $R(p)^{(t)}$ *induces the multiplication by an element congruent to* $[V]^{pt}$ *modulo* $[CP(p-1)]^{p-1}$ *on the complex bordism theory, where* $[V] \in \Omega_*^U$ *is the class of the Milnor manifold for the prime p with* dim $V = 2(p^2-1)$. *Hence the mapping cone of* $R(p)^{(t)}$ *realizes a cyclic* Ω_*^U*-module*

$$\Omega_*^U/(p, [CP(p-1)]^p, [V]^{pt} + [N_t])$$

for some $[N_t] \in ([CP(p-1)]^{p-1})$ *with* $\dim N_t = 2t(p^2-1)$.

The rings $\mathscr{A}_*(M)$ and $\mathscr{A}_*(X(r))$ form differential algebras over Z_p with differentials D and θ defined in [1] and [9], respectively.

PROPOSITION E. *For even t, the elements* $\rho_0(t)$ *and* $R(p)^{(t)}$ *can be chosen so that* $D(\rho_0(t)) = 0$ *and* $\theta(R(p)^{(t)}) = 0$.

In §8, we shall compute $\mathscr{A}_*(M)$ completely in degree $<(2p^2+p)q-4$ (Theorem 8.10) and partially in higher degree (Proposition 8.11), by the same techniques as [4]. In §9, Theorems AII–DII and Proposition E will be proved. From Proposition E, we shall, in § 10, slightly generalize Theorems A′–D′ for even t. The results are Theorems A′II–D′II.

§8. Some results on $\mathscr{A}_*(M)$

We recall the structure of the ring $\mathscr{A}_*(M)$, $M = S^1 \cup_p e^2$, from [4], (cf. [1]). This is a differential graded algebra over Z_p with differential D of degree $+1$ [4; (1.6)]. The subalgebra

$$K_* = \sum_k K_k = \operatorname{Ker} D$$

is commutative [4; (1.11)], and there are direct sum decompositions

(8.1) $$\mathscr{A}_k(M) = K_k + \delta^* K_{k+1} = K_k + \delta_* K_{k+1},$$

where the right and the left translations δ^* and δ_* by the element $\delta = i\pi \in \mathscr{A}_{-1}(M)$ are monomorphic on K_* [1; Th. A (c)]. For any $\gamma \in G_k$, the smash product $\gamma \wedge 1_M$ lies in K_k [4; Lemma 3.1], and the subgroup $G_k \wedge 1_M$ of K_k is naturally isomorphic to $G_k \otimes Z_p$[4; Lemma 3.3]. For any $\gamma \in G_{k-1}$ of order p, there is an element $[\gamma] \in K_k$ such that $\pi_* i^*[\gamma] = \gamma$ [4; Lemma 3.2], where $S^1 \xrightarrow{i} M \xrightarrow{\pi} S^2$ is the cofibering for M. This element $[\gamma]$ is determined up to the subgroup $G_k \wedge 1_M$ The subgroup $[G_{k-1} * Z_p]$ of K_k, which consists of those elements $[\gamma]$ for $\gamma \in G_{k-1} * Z_p$, is isomorphic to $G_{k-1} * Z_p$ by the homomorphism $\pi_* i^*$ [4; Lemma 3.3], and there is a direct sum decomposition

$$(8.2) \qquad K_k = [G_{k-1} * Z_p] + G_k \wedge 1_M \ (\approx G_{k-1} * Z_p + G_k ® Z_p).$$

The equalities (8.1) and (8.2) provide that $\mathscr{A}_k(M)$is computed from G_{k-1}, G_k and G_{k+1} [4; Th. 3.5]. If $\gamma \in G_{k-1} * Z_p$,then there is a relation [4; (3.5)]

$$(8.3) \qquad \gamma \wedge 1_M = [\gamma]\delta - (-1)^k \delta[\gamma].$$

Let $A_*(\alpha, \delta)$ be the subalgebra generated by the element $\alpha \in \mathscr{A}_q(M)$, $q = 2(p-1)$, and δ. Then a Z_p-basis for $A_k = A_k(\alpha, \delta)$ is given as follows [4; Th. 4.1]:

$$(8.4) \qquad A_{rq} = Z_p\{\alpha^r\}^{*)}, \quad A_{rq-1} = Z_p\{\alpha^r\delta, \alpha^{r-1}\delta\alpha\},$$

$$A_{rq-2} = Z_p\{\alpha^{r-1}\delta\alpha\delta\} \qquad \textit{for} \quad r \geqq 1;$$

$$A_0 = Z_p\{1_M\}, \quad A_{-1} = Z_p\{\delta\}, \quad A_k = 0 \qquad \textit{for other} \quad k,$$

and hence,

(8.5) *for* fe^O *and* $r \geqq 1$, α^r_* *and* α^{r*}: $A_k(\alpha, \delta) \rightarrow A_{k+rq}(\alpha, \delta)$ are *isomorphic.*

We have also [4; pp. 648-651]

$$(8.6) \qquad [\alpha_r] = \alpha^r, \quad \alpha_r \wedge 1_M = r(\alpha^r\delta - \alpha^{r-1}\delta\alpha),$$

$$\alpha'_{rp} \wedge 1_M = r(\alpha^{rp}\delta - \alpha^{rp-1}\delta\alpha), \quad \alpha''_{rp^2} \wedge 1_M = r(\alpha^{rp^2}\delta - \alpha^{rp^2-1}\delta\alpha)$$

In [4; Th. 0.1] we computed the algebra $\mathscr{A}_*(M)$ up to degree $(p^2+3p+1)q-6$ from our results on G_* [3; Th. A]. We have recently determined G_* in higher degrees ([6; Th. C], [2; Th. 4.1]), and so we can easily continue to compute $\mathscr{A}_*(M)$.

LEMMA 8.7. *There exists an element* $\kappa_{(s)} \in K_{k(s)}$, $1 \leqq s \leqq p-3$, *such that* $\pi_* i^* \kappa_{(s)} = \kappa_s$, *the generator of the p-component of* $G_{k(s)-1}$ [6], *where* fc(s) = $(p^2+(s+2)p+s+1)q-4$. *For* $1 \leqq s \leqq p-4$, $\kappa_{(s)}$ is *unique and satisfies* $\alpha\kappa_{(s)} = \beta_{(1)}\kappa_{(s)} = 0$.

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*) $Z_p\{d_1, \ldots, d_n\}$stands for the Z_p-module with basis $d_1, \ldots, d_n$.

PROOF. Since $K_{k(s)}=Z_p$, generated by $[\kappa_s]$, and $K_{k(s)+q}=K_{k(s)+pq-1}=0$ $(s \leqq p-4)$, we have the result by setting $\kappa_{(s)}=[\kappa_s]$. *q. e. d.*

We defined the elements $\beta_{(s)}=[\beta_s](s\geqq 1, s\not\equiv 0 \bmod p)$, $\varepsilon=[\varepsilon']$ and $\lambda=[\lambda_1]$ in K_* ([4; §5], [6; §22]), where β_s, ε' and λ_1 are the generators of G_* ([3; Th. A], [6; Th. C]). For the generators λ' and μ of G_*, we also define the following two elements

(8.8) $\bar{\lambda} = [\quad {}_{(2p^2+1)q-4},$

(8.9) $\bar{\mu} = \mu \wedge 1_M \in K_{(2p^2+p-1)q-5}.$

Now let y be any element in the p-component of G_k, $(p^2+3p+1)q-7\leqq k \leqq(2p^2+p)q-4$. For y of order greater than p, i.e., $y=\alpha'_{rp}$, α''_{rp^2} or μ, the element $y \wedge 1_M$ is given by (8.6) or (8.9). For y of order p, it suffices by (8.3) to determine $[\gamma]$. Furthermore, by [4; Prop. 3.8], it suffices to do for indecomposable y, i.e., $\gamma=\alpha_r$, β_s, κ_s, λ', λ_i, for which we have $[y]=\alpha^r$, $\beta_{(s)}$, $\kappa_{(s)}$, I, $\lambda\alpha^{i-1}$ by (8.6), [4; (5.9)], Lemma 8.7, (8.8), [6; Th. 22.2], respectively. Thus, from Theorem C of [6], we have obtained the following result.

THEOREM 8.10. *The following elements give a Z_p-basis for $\mathscr{A}_k(M)$, $(p^2+3p+1)q-6\leqq k\leqq(2p^2+p)q-5$, ($a$, fee{0, 1}, $0\leqq r<p$ and $s\geqq 1$ unless otherwise stated):*

$\alpha^t\delta^a$, $\alpha^{t-1}\delta\alpha\delta^a$ *for* $p^2+3p+1 \leqq t \leqq 2p^2+p-1$;

$\delta^a(\beta_{(1)}\delta)^{t-1}\beta_{(1)}\delta^b$ *for* $p+4 \leqq t \leqq 2p+1$;

$\delta^a(\beta_{(1)}\delta)^r\beta_{(s)}\delta^b$ *for* $4 \leqq s \leqq 2p-1, s \neq p, p+3 \leqq r+s \leqq 2p-1$

and for $(r, s) = (p-1, p+1)$;

$\delta^a\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)}\delta^b$ *for* $4 \leqq s \leqq p-1$, *for* $r=0, p+1 \leqq s \leqq 2p-2$

and for $r=1$, $s=2p-2$;

$\delta^a(\alpha\delta)^c(\beta_{(1)}\delta)^r\beta_{(2)}\delta\beta_{(p)}\delta^b$ *for* $c\in\{0, 1\}, 2 \leqq r \leqq p-1$

except for $b=c=1$, $r=p-1$;

$\delta^a(\beta_{(1)}\delta)^r\bar{\varepsilon}\delta^b$ *for* $4\leqq r\leqq p-1$;

$\delta^a(\beta_{(1)}\delta)^r\kappa_{(s)}\delta^b$ *for* $1 \leqq s \leqq p-3$, $r+s \leqq p-2$

$\delta^a\bar{\lambda}\delta^b$; $\delta^a\lambda\alpha^i\delta^b$ *for* $0 \leqq i \leqq p-4$; $\delta\lambda\alpha^i\delta\alpha\delta^b$ *for* $0 \leqq i \leqq p-5$; $\bar{\mu}\delta^a$.

We can also determine completely the ring structure in the cited range, but we omit the details. For example, the relation [5; Prop. 7.3. (iii)] implies that Toda's relation $(\beta_{(1)}\delta)^p\beta_{(s)}=0$ ($s\geqq 2, s\not\equiv -1 \bmod p$) [9; Cor. 5.7] also holds for $s\equiv -1 \bmod p$. In [7; Cor. 2] and [6; Th. 22.4], several new relations have been obtained. The relation $(\beta_{(1)}\delta)^p\bar{\varepsilon}=0$ clearly holds. The following corresponds to the result [6; Cor. 21.5] in G_*:

$$\alpha\delta\beta_{(2p-1)} = z((\beta_{(1)}\delta)^{p-1}\beta_{(p+1)} + (\delta\beta_{(1)})^{p-1}\delta\beta_{(p+1)}), \quad z \not\equiv 0 \bmod p.$$

We can obtain relations among α, $\beta_{(s)}$, $\kappa_{(t)}$ and ones among α, $\bar{\lambda}$, λ similar to (ii)-(iv) and (vi)-(vii) of [4; Th. 0.1], respectively, and also obtain analogues to (ix)-(xi), and so on.

We have computed $\mathscr{A}_*(M)$ up to degree corresponding to [6; Th. C]. We can make further computations corresponding to the recent result [2; Th. 4.1] on G_*, but can not determine the ring structure because [2; Th. 4.1] does not give some products in G_*.

In Part I, we gave the elements p(2) and $\sigma(2)$ in K_* such that $\rho(2)\alpha^{p-2} = \beta_{(2p)}$, $\rho(2)\alpha^{p-1}=0$, $\sigma(2)\alpha^{p-3}=\beta_{(1)}\beta_{(2p-1)}$ and $\sigma(2)\alpha^{p-2}=0$. We also introduced in [2; Lemma 5.3] a unique element $\rho\in K_*$ with $\rho\alpha=\rho(2)$.

PROPOSITION 8.11. (i) *The group* $\mathscr{A}_k(M)$, $k=(2p^2+p+i)q-\varepsilon$, $0\leqq i\leqq p$, $\varepsilon=0, 1, 2$, is *the direct sum of* $A_k(\alpha, \delta)$ *in* (8.4) *and the following subgroup* $A_{i,\varepsilon}$:

$$\begin{aligned}
A_{i,2} = {} & Z_p\{\rho\delta, \delta\rho, [\mathrm{v}], [\gamma]\} && \textit{for } i=0,\\
& Z_p\{\sigma(2), \rho\alpha\delta, \delta\rho\alpha, (\beta_{(1)}\delta)^{p-2}\kappa_{(1)}\} && \textit{for } i=1,\\
& Z_p\{\sigma(2)\alpha, \rho\alpha^2\delta, \delta\rho\alpha^2, \delta(\beta_{(1)}\delta)^{p-3}\kappa_{(2)}\delta\} && \textit{for } i=2,\\
& Z_p\{\sigma(2)\alpha^{i-1}, \rho\alpha^i\delta, \delta\rho\alpha^i\} && \textit{for } 3\leqq i\leqq p-2,\\
& Z_p\{\rho\alpha^{p-1}\delta, \delta\rho\alpha^{p-1}\} && \textit{for } i=p-1,\\
& Z_p\{\rho\alpha^{p-1}\delta\alpha, \beta_{(1)}\delta\eta\delta, \delta\beta_{(1)}\delta\eta\}(+Z_p\{(\delta\beta_{(1)})^2\delta\bar{\lambda}\delta\} \textit{ if } p=5) && \textit{for } i=p;\\
A_{i,1} = {} & Z_p\{\rho, \delta\eta\delta\} && \textit{for } i=0,\\
& Z_p\{\rho\alpha^i\} && \textit{for } i=1 \textit{ and for } 3\leqq i\leqq p-1,\\
& Z_p\{\rho\alpha^2, (\beta_{(1)}\delta)^{p-3}\kappa_{(2)}\delta, (\delta\beta_{(1)})^{p-3}\delta\kappa_{(2)}\}\\
& \qquad (+Z_p\{(\delta\beta_{(1)})^{2p+2}\delta\} \textit{ if } p=5) && \textit{for } i=2,\\
& Z_p\{\beta_{(1)}\delta\eta\}(+Z_p\{(\beta_{(1)}\delta)^2\bar{\lambda}\delta, (\delta\beta_{(1)})^2\delta\bar{\lambda}\} \textit{ if } p=5) && \textit{for } i=p;\\
A_{i,0} = {} & Z_p\{\eta\delta, \delta\eta\}(+Z_p\{\delta\beta_{(1)}\delta\bar{\lambda}\delta\} \textit{ if } p=5) && \textit{for } i=0,\\
& Z_p\{(\beta_{(1)}\delta)^{p-3}\kappa_{(2)}\}(+Z_p\{(\beta_{(1)}\delta)^{2p+2}, (\delta\beta_{(1)})^{2p+2}\} \textit{ if } p=5) && \textit{for } i=2,
\end{aligned}$$

$0(+Z_p\{\delta\beta_{(1)}\bar{\mu}\delta\}$ *if* $p=5)$ *for* $i=3$,

$0(+Z_p\{(\beta_{(1)}\delta)^2\bar{\lambda}\}$ *if* $p=5)$ *for* $i=p$,

0 *for* $i=1$ *and for* $4\leqq i\leqq p-1$.

In the above, *we put* $\eta=(\beta_{(1)}\delta)^{p-2}\beta_{(p+2)}$

(ii) *The element* $i\rho\alpha^{i-1}\delta\alpha\in A_{i,2}, i\geqq 1$, *is equal to* $ia\sigma(2)\alpha^{i-1}+(i-1)\rho\alpha^i\delta-\delta\rho\alpha^i$ *(modulo* $(\beta_{(1)}\delta)^{p-2}\kappa_{(1)}$ *if* $i=1$), ***where the*** *coefficient* $a\in Z_p$ *is independent of* i. ***In particular,***

$$2\rho\alpha^{p-3}\delta\alpha=a\beta_{(1)}\beta_{(2p-1)}+3\rho\alpha^{p-2}\delta+\delta\rho\alpha^{p-2},$$

$$\rho\alpha^{p-2}\delta\alpha=2\rho\alpha^{p-1}\delta+\delta\rho\alpha^{p-1}.$$

Also the following equality holds:

$$\delta\rho\alpha^{p-1}\delta\alpha=\rho\alpha^{p-1}\delta\alpha\delta.$$

PROOF. From discussions similar to those in Theorem 8.10, it is easy to see (i) with $\sigma(2)\alpha^{i-1}$ replaced by $[\rho_i']$. The relations $\sigma(2)\alpha^{i-1}\not\equiv 0 \bmod Z_p\{\rho\alpha^i\delta+\delta\rho\alpha^i\}(+Z_p\{(\beta_{(1)}\delta)^{p-2}\kappa_{(1)}\}$ if $i=1$), which provides to replace $[\rho_i']$ by $\sigma(2)\alpha^{i-1}$, and (ii) follow from the discussions in [2; §5] and [4; §§5–6]. *q.e.d.*

COROLLARY 8.12. ***The*** *elements* $\rho_j'\in G_{(2p^2+p+j)q-3}$, $1\leqq j\leqq p-2$, *given in* [2; Th. 4.1] ***can be taken,*** *up to nonzero* ***coefficients,*** *such that* $\rho_j=\pi\sigma(2)\alpha^{j-1}i$. *For these* ρ_j, ***there are*** *relations* $\rho_j\alpha_k=ka\rho_{j+k}'$ *for* $j\geqq 0$, fe^1, $j+k\geqq 2$, *where we interpret* $\rho_k'=0$ ***for*** $k\geqq p-1$.

§ 9. Proof of Theorems AII–DII

In this section, we shall prove Theorems AII-DII and Proposition E. We first prove Theorems **AII, BII** and **DII** assuming **CII.**

PROOF of **CII⇒DII.** Consider the induced homomorphism

$$R(p)_*^{(t)}: S^{n(t)}\tilde{\Omega}_*^U(X(p))\longrightarrow\tilde{\Omega}_*^U(X(p)),$$

where $n(t)=t(p^2+p)q$ and flg(Jf(p)) $=\Omega_*^U/(p,[P]^p)\ \xi(p)$, $P=CP(p-1)$, $\deg\xi(p)=3$ (Proposition 3.2). Since A and $R(p-1)$ induce the multiplications by [P] and by an element congruent to $[V]^p$ modulo $[P]^{p-2}$, we see from the commutativity $AR(p-1)^t=R(p)^{(t)}A$ that $R(p)_*^{(t)}$ is the multiplication by an $[M]$ such that $[M][P]=[V]^{pt}[P]+[N][P]^{t(p-2)+1}$ for some $[N]$. Hence we have $[M]=[V]^{pt} \bmod [P]^{p-1}$ in $\Omega_*^U/(p,[P]^p)$. *q.e.d.*

In the same way as Definition 4.8, we define elements in *rf*(M)* and G_*

from $R(p)^{(t)}$.

DEFINITION 9.1. *Let* $t \geqq 2$.

$$\rho_0(t) = k_p R(p)^{(t)} j_p \in \mathscr{A}_{(tp+t-1)pq-1}(M),$$

$$\rho_{t,0} = \pi\rho_0(t)i = \pi k_p R(p)^{(t)} j_p i \in G_{(tp+t-1)pq-2}.$$

Then the following relation is easily seen from the commutativity $AR(p-1)^t = R(p)^{(t)}A$.

(9.2) $$\rho_0(t)\alpha = \rho(t), \quad \rho_{t,1} \in \langle \rho_{t,0}, P, \alpha_1 \rangle,$$

where $\rho(t) = k_{p-1}R(p-1)^t j_{p-1}$ *and* $\rho_{t,1} = \pi\rho(t)i$ *(Definition* 4.8).

PROOF of DII⇒AII, BII. This is similar to the proofs of Theorems A and B [5; p. 105]. It suffices by (9.2) to show $\rho_{t,0} \neq 0$.

Let h be the MU-Hurewicz homomorphism. By DII, $h(R(p)^{(t)}j_p i) = [V]^{p^t}\xi(p) \bmod [P]^{p-1}\xi(p)$, which is not contained in the image of $l_{p*}h = hl_{p*}$ (l_p is the inclusion $Y(p) \subset X(p)$, see (3.6)), by Proposition 3.9. Hence $R(p)^{(t)}j_p i \notin \operatorname{Im} l_{p*}$, which is equivalent to $\rho_{t,0} = (\pi k_p)_*(R(p)^{(t)}j_p i) \neq 0$. *q.e.d.*

Next we prove Theorem CII. To prove CII, we prepare some lemmas.

LEMMA 9.3. *The kernel of*

$$k_{p-1*}j_1^*\colon \{X(1), X(p-1)\}_{(2p^2+2p-1)q-1} \longrightarrow \mathscr{A}_{(2p^2+p)q-2}(M)$$

is equal to $Z_p\{j_{p-1}k_1R(1)^2\}$ ($+Z_p\{j_{p-1}\xi k_1\}$ *if* $p=5$), *where* $\xi = (\beta_{(1)}\delta)^2\lambda$.

PROOF. By Proposition 8.11 and (8.5), we have the following results:

(1) $\mathscr{A}_{(2p^2+2p)q}(M)/\operatorname{Im}\alpha^* = 0$ ($+Z_p\{\xi\}$ if $p=5$),

(2) $\mathscr{A}_{(2p^2+2p-1)q-1}(M) \cap \operatorname{Ker}\alpha^* = Z_p\{\beta_{(2p)} = k_1R(1)^2j_1\}$,

(3) $\mathscr{A}_{(2p^2+2p-1)q}(M)/\operatorname{Im}\alpha^* = 0$,

(4) $\mathscr{A}_{(2p^2+2p-2)q-1}(M) \cap \operatorname{Ker}\alpha^* = 0$,

(5) $\mathscr{A}_{(2p^2+p+1)q-1}(M)/\operatorname{Im}\alpha^* = 0$.

We compute $\{X(1), M\}_k$ for some k by applying the above results to the exact sequence (1.3)*. From (1) and (2), we have

(6) $\{X(1), M\}_{(2p^2+2p-1)q-3} = Z_p\{k_1R(1)^2\}$ ($+Z_p\{\xi k_1\}$ if $P=5$).

From (3) and (4), we have $\{X(1), M\}_{(2p^2+2p-2)q-3} = 0$, and hence, by (1.3),

(7) $j_{1*}\colon \{X(1), M\}_{k-2} \to \mathscr{A}_k(X(1))$ *and* $j_{p-1*}\colon \{X(1), M\}_{k-2} \to \{X(1), X(p-1)\}_k$ *are monomorphic for* $k = (2p^2+2p-1)q-1$.

We also see from (5) that j_1^*: $\{X(1),M\}_{(2p^2+p)q-4} \to \mathscr{A}_{(2p^2+p)q-2}(M)$ is monomorphic. Therefore

$$\operatorname{Ker} k_{p-1*}j_1^* = \operatorname{Ker} k_{p-1*} = \operatorname{Im} j_{p-1*},$$

which is equal to the desired result by (6) and (7). *q. e. d.*

LEMMA 9.4. *$\mathscr{A}_{(2p^2+2p-1)q-1}(X(1)) = Z_p\{\beta^{2p}\delta_1 = \delta_1\beta^{2p}, \beta^{2p-1}\delta_1\beta\}$ $(+Z_p\{j_1 \xi k_1\}$ if $p=5)$, where $\beta^p = R(1)$, $\delta_1 = j_1k_1$[9] and $\xi = (\beta_{(1)}\delta)^2\lambda$.*

PROOF. By Proposition 8.11 and (8.5), $\mathscr{A}_{(2p^2+2p-1)q-1}(M)/\operatorname{Im}\alpha^* = 0$ and $\mathscr{A}_{(2p^2+2p-2)q-2}(M) \sqcap \operatorname{Ker}\alpha^* = Z_p$, generated by $-\beta_{(1)}\beta_{(2p-1)} = \beta_{(2p-1)}\beta_{(1)} = k_1\beta^{2p-1}\delta_1\beta j_1$ Hence $\{X(1), M\}_{(2p^2+2p-2)q-4} = Z_p\{k_1\beta^{2p-1}\delta_1\beta\}$. From this and (6)-(7) in the proof of Lemma 9.3, the lemma easily follows. *q. e. d.*

PROOF of Theorem **CII**. Consider the elements $R(p-1)^2 j_{p-1}k_1$ and $R(p-1)j_{p-1}k_1R(1)$ in $\{X(1), X(p-1)\}_{(2p^2+2p-1)q-1}$. Since fcJ^O and $\varepsilon\beta_{(p)} = \varepsilon^2\alpha^{p-2} = 0$by [6; (22.2)],these elements lie in Ker fc$_{p-1*}j_1^*$. Hence, by Lemma 9.3, we can put

$$R(p-1)^2 j_{p-1}k_1 = xj_{p-1}k_1R(1)^2 + yj_{p-1}\xi k_1,$$

$$R(p-1)j_{p-1}k_1R(1) = x'j_{p-1}k_1R(1)^2 + y'j_{p-1}\xi k_1,$$

for some x, y, x', $y' \in Z_p$ $(y = y' = 0$ if $p \geqq 7)$. Consider the B_*^{p-2}-images of these equalities. Then by Theorem C (c) and (1.4), we get

$$R(1)^2 j_1k_1 = xj_1k_1R(1)^2 + yj_1\xi k_1,$$

$$R(1)j_1k_1R(1) = x'j_1k_1R(1)^2 + y'j_1\xi k_1.$$

Since $\delta_1 = j_1k_1$ commutes with $R(1) = \beta^p$ [9], it follows from Lemma 9.4 that $x = x' = 1$ and $y = y' = 0$, i.e., $R(p-1)^2 j_{p-1}k_1 = j_{p-1}k_1R(1)^2$ and $R(p-1)j_{p-1}k_1R(1) = j_{p-1}k_1R(1)^2$. We obtain therefore

$$(*) \qquad R(p-1)^t j_{p-1}k_1 = j_{p-1}k_1R(1)^t \quad \text{for } t \geqq 2.$$

By Lemma 1.5, $\Delta = j_{p-1}k_1$ is contained in the following sequence of cofiberings

$$X(1) \xrightarrow{\Delta} S^{q+1}X(p-1) \xrightarrow{A} SX(p) \xrightarrow{B^{p-1}} SX(1).$$

By Lemma 2.5 (i), (*) yields the existence of an element $R(p)^{(t)} \in \mathscr{A}_*(X(p))$, $t \geqq 2$, such that $R(p)^{(t)}A = AR(p-1)^t$ and

$$(9.5) \qquad B^{p-1}R(p)^{(t)} = R(1)^t B^{p-1}.$$

Consider the element $\rho_0(2)=k_pR(p)^{(2)}j_p$. This satisfies $\rho_0(2)\alpha=\rho(2)=\rho\alpha$ by (9.2) and [2; Lemma 5.3]. By Proposition 8.11 and (8.5), we have

$$\rho_0(2) = \rho + x\delta\eta\delta \qquad \text{for some} \quad x \in Z_p.$$

Since $\mathscr{A}_k(M)=A_k(\alpha, \delta)$ for $k=(2p^2+2p-1)q$ by Proposition 8.11, the Toda bracket $\langle \alpha^{p-2}, \delta\eta\delta, \alpha\rangle$ contains zero. So there is an element $S\in\{X(1), X(p-1)\}_{(2p^2+2p-2)q}$ such that $k_{p-2}Sj_1=\delta\eta\delta$. Then, replacing $R(p)^{(2)}$ by $R(p)^{(2)}-xA^2SB^{p-1}$, we obtain $k_pR(p)^{(2)}j_p=\rho$. For this $R(p)^{(2)}$, the relations $R(p)^{(2)}A=AR(p-1)^2$ and (9.5) also hold, because $B^{p-1}A=0$: $S^qX(p-1)\rightarrow X(p)\rightarrow X(1)$.

q.e.d.

Finally we consider the element $\theta(R(p)^{(2)})$*).

LEMMA 9.6. *The following composition is monomorphic*:
$\theta j_{p*}: \mathscr{A}_{(2p^2+2p)q+1}(M) \longrightarrow \{M, X(p)\}_{(2p^2+2p)q+3} \longrightarrow \{M, X(p)\}_{(2p^2+2p)q+4}$.

PROOF. Put $l=(2p^2+2p)q+1$. From the results on G_{l+1}, G_l and G_{l-1} [2; Th. 4.1], we have $\mathscr{A}_l(M)=0$ if $p\geqq 7$, and $=i_*\pi^*G_{l+1}=Z_p$ if $p=5$. If $p\geqq 7$, the lemma holds obviously, and so we consider the case $p=5$.

By Proposition 1.13, $\theta(j_p)=0$ and so, by Proposition 1.9 (i), $\theta j_{p*}=j_{p*}\theta$, where θ in the right side coincides with $-D$ by Proposition 1.12. We notice that

(9.7) D *is monomorphic on the subgroup* $i_*\pi^*G_{k+1}$ *of* $\mathscr{A}_k(M)$.

For, $i_*\pi^*G_{k+1}=\delta^*(G_k\wedge 1_M)=\delta_*(G_k\wedge 1_M)$, and δ^* and δ_* are right inverses of D.

In particular, D on $\mathscr{A}_l(M)$ is monomorphic. Since $\mathscr{A}_{l-pq+1}(M)=Z_p$ generated by $\beta_{(1)}\delta\lambda$ and since $\alpha_*^p(\beta_{(1)}\delta\lambda)=0$, j_{p*} on $\mathscr{A}_{l+1}(M)$ is also monomorphic. Thus, $\theta j_{p*}=-j_{p*}D$ is monomorphic. *q.e.d.*

LEMMA 9.8. *The following composition is monomorphic:*
$k_p^*\theta$: $\{M, X(p)\}_{(2p^2+3p)q+4} \longrightarrow \{M, X(p)\}_{(2p^2+3p)q+5} \longrightarrow \mathscr{A}_{(2p^2+2p)q+2}(X(p))$.

PROOF. Put $l=(2p^2+3p)q+2$. If $p\geqq 7$, then $\mathscr{A}_l(M)=\mathscr{A}_{l-pq-1}(M)=0$ and hence $\{M, X(p)\}_{l+2}=0$. So we consider the case $p=5$.

Since $\mathscr{A}_{l-2q}(M)=0$, $\alpha_*^p=0$: $\mathscr{A}_{l-pq}(M)\rightarrow\mathscr{A}_l(M)$. Since $\mathscr{A}_{l-2q+1}(M)=Z_p$ generated by $\delta\beta_{(2)}\delta\beta_{(2p-1)}\delta=i\beta_2\beta_{2p-1}\pi$ and since $\alpha^2\delta\beta_{(2)}=0$ [4; §5], we have also $\alpha_*^p=0$: $\mathscr{A}_{l-pq+1}(M)\rightarrow\mathscr{A}_{l+1}(M)$. Hence, we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{A}_l(M) & \longrightarrow & \{M, X(p)\}_{l+2} & \longrightarrow & \mathscr{A}_{l-pq-1}(M) \\
& \downarrow{\scriptstyle -D} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle D} \\
0 \longrightarrow & \mathscr{A}_{l+1}(M) & \longrightarrow & \{M, X(p)\}_{l+3} & \longrightarrow & \mathscr{A}_{l-pq}(M).
\end{array}$$

*) For the definition and properties of θ, see § 1,

From the results on G_* [2; Th. 4.1], $\mathscr{A}_k(M) = i_*\pi^*G_{k+1}$ for $k=l$, $l-pq-1$. By (9.7), D's in the above diagram are monomorphic, and hence θ is also monomorphic.

We have $\mathscr{A}_{l-pq-3q}(M)=0$ and $\mathscr{A}_{l-3q+1}(M) = Z_p\{(\beta_{(1)}\delta)^2\beta_{(2p-1)}\}$ for $p=5$. Hence $\alpha^{3*}=0$: $\{M, X(p)\}_{l-3q+3} \to \{MX(p)\}_{l+3}$, and k_*^* is monomorphic. Therefore the lemma follows. *q.e.d.*

LEMMA 9.9. $\mathscr{A}_{(2p^2+2p)q+1}(X(p)) \cap \operatorname{Ker}\theta \cap \operatorname{Ker} k_{p*}j = 0$.

PROOF. Let ξ be any element in the left side. Then $\xi j_p = 0$ by Lemma 9.6. Write $\xi=\eta k_p$. Then $\theta(\eta)k_p=0$ and $\eta=0$ by Lemma 9.8. Therefore $\xi=0$. *q.e.d.*

PROOF of Proposition E. It is easily seen that the element $\xi=\theta(R(p)^{(2)})$ satisfies $\theta(\xi)=0$ and $k_p\xi j_p=0$. Then $\theta(R(p)^{(2)})=0$ by Lemma 9.9. By setting $R(p)^{(2t)}=(R(p)^{(2)})^t$, we obtain $\theta(R(p)^{(2t)})=0$ and $D(\rho_0(2t))=0$. *q.e.d.*

§ 10. Generalization of Theorems A′–D′

We considered in § 6 a generalization of the elements in Theorems A and B for $t\equiv 0 \bmod p$ and obtained Theorems A'-D'. In the same way, we shall generalize the elements in Theorems AII and BII for $t\equiv 0 \bmod 2p$.

The following result corresponds to Lemma 6.1.

LEMMA 10.1. *Let* $\Delta=j_pk_p$. *Then*

$$\lambda_{X(p)}(\rho\delta) = R(p)^{(2)}\Delta - \Delta R(p)^{(2)}.$$

PROOF. Since $\langle\alpha^p, \delta\eta\delta, \alpha^p\rangle=0$, $\eta=(\beta_{(1)}\delta)^{p-2}\beta_{(p+2)}$, there is an element $S'\in\mathscr{A}_{(2p^2+2p)q}(X(p))$ with $k_pS'j_p=\delta\eta\delta$ (We can take $S'=A^2SB^{p-1}$ for the element S in the proof of Theorem CII). Then, by routine calculations, the following results are verified:

$$(10.2) \qquad \mathscr{A}_{(2p^2+p)q-1}(X(p)) = Z_p\{R(p)^{(2)}\Delta\Delta R(p)^{(2)}, S'\Delta, \Delta S'\}$$

$$(+Z_p\{j_p\xi k_p\} \text{ if } p=5), \quad \text{where } \xi = (\beta_{(1)}\delta)^2\lambda.$$

$$(10.3) \qquad k_p^*j_{p*}\mathscr{A}_{(2p^2+p)q-1}(M) = Z_p\{\Delta R(p)^{(2)}\Delta S'\Delta\}.$$

Put $d=\lambda_{X(p)}(\rho\delta)-R(p)^{(2)}\Delta+\Delta R(p)^{(2)}$. Then, $dA=Ad=0$ in the same way as in Lemma 6.1. Hence, $d=0$ for $p\geqq 7$ and $d=xj_p\xi k_p$, $x\in Z_p$, for $p=5$. It follows easily from Theorem CII and (9.5) that $B^{p-1}\lambda_{X(p)}(\rho\delta)A^{p-1}=0$, $B^{p-1}(R(p)^{(2)}\Delta-\Delta R(p)^{(2)})A^{p-1}=R(1)^2\delta_1-\delta_1R(1)^2=0$ and $B^{p-1}j_p\xi k_pA^{p-1}=j_1\xi k_1$, which is nonzero by Lemma 9.4, Hence $x=0$ and $d=0$ for $p=5$. *q.e.d.*

From Proposition 1.12, we have

(10.4) $R(p)^{(2)}\Delta-\Delta R(p)^{(2)}$ *commutes with any element in* $\mathscr{A}_*(X(p))\cap \mathrm{Ker}\,\theta$.

By Proposition E, we obtain

THEOREM 10.5. *For* $R=R(p)^{(2)}$ *and* $A=j_pk_p$,

$$R^2\Delta-2R\Delta R+\Delta R^2=0 \qquad in\ \mathscr{A}_*(X(p)).$$

Then all the relations in Corollary 6.4 are also verified for these R and A. In particular, $R^p\Delta=\Delta R^p$ holds. From this relation together with $R(r)^{2p}A=AR(r-1)^{2p}$, $2\leqq r\leqq p-1$, and $R^pA=AR(p-1)^{2p}$, we can construct the following elements $R'(r)^{(2)}\in\mathscr{A}_*(X(r))$, $p\leqq r\leqq 2p$, in the same manner as in the proof of Theorem C′.

THEOREM C′II. *There exist nonzero elements*

$$R'(r)^{(2)}\in\mathscr{A}_{2(p^3+p^2)q}(X(r)), \qquad p\leqq r\leqq 2p,$$

satisfying the following relations:

(i) $R'(p)^{(2)}=(R(p)^{(2)})^p$,

(ii) $AR'(r-1)^{(2)}=R'(r)^{(2)}A$ *for* $p+1\leqq r\leqq 2p$,

(iii) $B^pR'(r)^{(2)}=R(r-p)^{2p}B^p$ *for* $p+1\leqq r\leqq 2p$,

where $R(p)^{2p}=(R(p)^{(2)})^p$.

REMARK. The squares of the elements $R'(r), p\leqq r\leqq 2p-2$, constructed in Theorem C′ also satisfy (ii), but may possibly differ from the above elements $R'(r)^{(2)}$.

The following results are also obtained by the same techniques.

THEOREM ATI. *The elements*
$\rho'_{2tp,r}=\pi k_{2p-r-1}(R'(2p-r-1)^{(2)})^tj_{2p-r-1}i\in G_{(tp^3+tp^2-2p+r+1)q-2}$, $-1\leqq r\leqq p-1$, $t\geqq 1$, *are nonzero and satisfy*

$$\rho'_{2tp,r}\in\langle\rho'_{2tp,r-1}, P, \alpha_1\rangle \qquad for \quad 0\leqq r\leqq p\ (\rho'_{2tp,p}=\rho_{2tp,0})$$

THEOREM B′II. *The element* $\rho_0(2tp)$ *in Theorem EII is strictly divisible by* α^p, *and hence* $\beta_{(2tp^2)}$ *is strictly divisible by* α^{2p-1}.

Here we say that ξ *is strictly divisible by* η if $\xi=\eta\zeta=\zeta\eta$ for some ζ.

THEOREM D′II. *The complex bordism module of the mapping cone of* $R'(r)^{(2)}, p\leqq r\leqq 2p$, *is isomorphic to*

$$\Omega_*^U/(p, [CP(p-1)]^r, [V]^{2p^2}+[N_r]) \quad for\ some \quad [N_r]\in([CP(p-1)]^{r-1}).$$

Let $BP_*(\ \ )$ be the Brown-Peterson homology theory for the prime p ($\geqq 5$). $BP_*=BP_*(S^0)$ is a polynomial ring on generators v_i of degree $2(p^i-1)$ over the integers localized at p. An ideal I of BP_* is called realizable if there is a CW-complex (or- spectrum) X with $BP_*(X)=BP_*/I$.

We consider the ideal $I_{r,s}=(p, v_1^r, v_2^s)$, where $r\geqq 1$, $s=ap^f, a\geqq 1$, $a\not\equiv 0 \bmod p$, $f\geqq 0$. R. S. Zahler proved [10] that $I_{r,s}$ is not realizable if $r>p^f$, and we consider the converse conclusion. In general, the converse is negative. In fact, $I_{p,p}$ is not realizable. Since the element β realizes the multiplication by v_2 on $BP_*(\ )$, $I_{1,s}$ is realized by the mapping cone of β^s. We see therefore that $I_{r,s}$, $s\not\equiv 0 \bmod p$, is realizable if and only if r=1. Similarly we see the realizability of $I_{r,s}$ for the following four cases:

$$1\leqq r\leqq p-1, f\geqq 1; \qquad p\leqq r\leq 2p-2, f\geqq 2;$$

$$r=p, f\geqq 1,\ s\geqq 2p; \qquad r=2p-1, 2p, f\geqq 2,\ a\equiv 0 \bmod 2;$$

by Theorems D, D', DII, D'II, respectively. In particular, for $f=1$, realizable $I_{r,s}$ are exhausted by the above.

PROPOSITION 10.6. *The ideal* (p, v_1^r, v_2^{tp}), $r\geqq 1$, $t\geqq 1$, $t\not\equiv 0 \bmod p$, *is realizable if and only if* $r\leqq p$ *and* $(r, t)\neq(p, 1)$.

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