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On the Commutativity of Torsion and Injective Hull

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Introduction

Throughout this note A denotes a commutative ring with a unit and all modules are unitary A-modules. For any module M, if L is a submodule of M and S is a subset of M, then we put $(L: S)=\{x \in A; xS \subseteq L\}$, in particular $0(S)=(0: S)$. For any filter F of ideals of A, we have an operation upon the lattice of submodules of any A-module M, as follows. If L is a submodule of M, we define $C(L, M)=\{x \in M; (L: x) \in F\}$. Especially we rewrite $C(0, M)=T(M)$; $C(M, E(M))=D(M)$, where $E(M)$ is an injective hull of M. Our main purpose is to answer the question: With the above notations, let F' be another filter and T', D' be the associated operators relative to F'. Can we have the equalities
(1) $D'(T(M))=T(D'(M))$, (2) $D'(M/T(M))=D'(M)/D'(T(M))$ and
(3) $D(\mathrm{Hom}(N, M))=\mathrm{Hom}(N, D(M))$?

The above equalities have been obtained, in [8], in a special case using the local property.

§ 1. Notation and Preliminaries

Let F be a filter of ideals of A. When L is a submodule of an A-module M, we put $C(L, M)=\{x \in M; (L: x) \in F\}$. Especially we rewrite $C(0, M)=T(M)$, which is called the F-torsion of M; $C(M, E(M))=D(M)$; $C(\mathfrak{a}, A)=c(\mathfrak{a})$. It is easy to see that, for any submodule N of M, $C(L, M) \cap N=C(L \cap N, N)$ and $C(L, M)/L=T(M/L)$. We denote the class of A-modules M such that $T(M)=M$ by $\mathscr{T}$ and the class of A-modules M such that $T(M)=0$ by $\mathscr{F}$. The following facts are easy and well-known:

(1) The class $\mathscr{T}$ is closed under submodule, image and direct sum (such class will be called a weak torsion class). Hence a module M belongs to $\mathscr{T}$ if and only if $Ax \in \mathscr{T}$ for any element x in M.

(2) T is a left exact subfunctor. Namely, the functor T satisfies the properties: (i) $T(M) \subseteq M$, (ii) if L is a submodule of M, then $T(L)=T(M) \cap L$, and (iii) for any homomorphism $f: M \to N$, $f(T(M)) \subset T(N)$ (such functor is called a left exact preradical).

(3) The operator c satisfies the properties: (i) $\mathfrak{a} \subseteq c(\mathfrak{a})$, (ii) $c(\mathfrak{a} \cap \mathfrak{b})=c(\mathfrak{a}) \cap c(\mathfrak{b})$ and (iii) $(c(\mathfrak{a}): x)=c(\mathfrak{a}: x)$, for any ideals $\mathfrak{a}$, $\mathfrak{b}$ and any element x in A

(such operator c will be called a modular operation).

(4) The class $\mathscr{F}$ is the right annihilator of $\mathscr{T}$, i.e., an A-module M belongs to $\mathscr{F}$ if and only if $\mathrm{Hom}_A(N, M)=0$ for any module N in $\mathscr{T}$ (cf. [2], [6]). Hence $\mathscr{F}$ is closed under submodule, group extension and direct product. Further $\mathscr{F}$ is closed under essential extension. And an A-module M belongs to $\mathscr{F}$ if and only if $Ax \in \mathscr{F}$ for any element x in M.

(5) (Relations among F, T, $\mathscr{T}$ and c) For any ideal $\mathfrak{a}$ of A, the following statements are equivalent: (a) $\mathfrak{a} \in F$, (b) $A/\mathfrak{a} \in \mathscr{T}$, (c) $T(A/\mathfrak{a})=A/\mathfrak{a}$ and (d) $c(\mathfrak{a})=A$. Let us note that, for any ideal $\mathfrak{a}$, $c(\mathfrak{a})=\mathfrak{a}$ if and only if $A/\mathfrak{a} \in \mathscr{F}$. Further note that $c(\mathfrak{a})$ is the union of ideals $(\mathfrak{a}: \mathfrak{b})$, where $\mathfrak{b}$ runs through F.

The above notations will be fixed throughout this note.

PROPOSITION 1. *The following conditions for a filter F are equivalent*:
(a) *For any ideal $\mathfrak{a}$, $c(\mathfrak{a})=c^2(\mathfrak{a})$.*
(b) *For any ideals $\mathfrak{a}$, $\mathfrak{b}$, if $\mathfrak{b}/\mathfrak{a} \in \mathscr{T}$, $\mathfrak{b} \in F$, then $\mathfrak{a} \in F$.*
(c) *For any ideal $\mathfrak{a}$, if $c(\mathfrak{a}) \in F$ then $\mathfrak{a} \in F$.*
(d) *For any module M, $M/T(M) \in \mathscr{F}$.*
(e) *$\mathscr{T}$ is the left annihilator of $\mathscr{F}$.*
(f) *$\mathscr{T}$ is closed under group extension.*
(g) *For any submodule L of a module M with $L \in \mathscr{T}$, $C(L, M)=T(M)$.*

PROOF. (a)$\Rightarrow$(b)$\Rightarrow$(c) follow from the fact that $\mathfrak{b}/\mathfrak{a} \in \mathscr{T}$ if and only if $\mathfrak{b} \subseteq c(\mathfrak{a})$. (c)$\Rightarrow$(a): If $x \in c^2(\mathfrak{a})$, then $(c(\mathfrak{a}): x)=c(\mathfrak{a}: x) \in F$, hence $(\mathfrak{a}: x) \in F$ by (c). (c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(g) are rather obvious. (g)$\Rightarrow$(a) follows from the equalities: $c^2(\mathfrak{a})/\mathfrak{a} = C(c(\mathfrak{a})/\mathfrak{a}, A/\mathfrak{a})=c(\mathfrak{a})/\mathfrak{a}$.

DEFINITION 1. A filter satisfying the above condition is said to be *idempotent* (cf. [1], [3] and [6]). The associated operator c is called a modular closure operator. The associated functor T is called a left exact radical or torsion radical (cf. [4], [7], [9]). And the class $\mathscr{T}$ will be called a torsion class (cf. [2], [9]).

PROPOSITION 2. *The following conditions for an A-module M are equivalent*:
(a) *$M=D(M)$, i.e. $E(M)/M \in \mathscr{F}$.*
(b) *If $0 \to L \to N \to K \to 0$ be an exact sequence of modules with $K \in \mathscr{T}$, then any homomorphism $L \to M$ can be extended to a homomorphism $N \to M$.*
(c) $\mathrm{Ext}_A^1(L, M)=0$ *for any* $L \in \mathscr{T}$.
(d) $\mathrm{Ext}_A^1(A/\mathfrak{a}, M)=0$ *for any* $\mathfrak{a} \in F$.
(e) *Any exact sequence $0 \to M \to N \to K \to 0$ with $K \in \mathscr{T}$ is split.*
(f) *Let $0 \to M \to N \to K \to 0$ be an exact sequence with $K \in \mathscr{T}$. Then for any element x in K there exists an inverse image y of x in N such that $0(y)=0(x)$.*

PROOF. (b)$\Rightarrow$(e)$\Rightarrow$(f)$\Rightarrow$(a) and the equivalence of (b), (c) and (d) are obvious

(e. g. see [9]). (a)$\Rightarrow$(b): Under the assumption in (b), we can construct a commutative diagram of modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & N & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \end{array}$$

Since $h=0$, $g(N)\subseteq M$, which completes the proof.

Definition 2. An A-module M is said to be *F-injective* or *F-divisible* if M satisfies the conditions above. The class of F-injective modules will be denoted by $\mathscr{D}$.

Corollary 1. *The class $\mathscr{D}$ is closed under group extension and direct product.*

Corollary 2. *If M is F-injective, then for any module N containing M, an exact sequence $0\to M\to C(M, N)\to T(N/M)\to 0$ is split. And $C(M, N)$ $=M+T(N)$, furthermore $0\to T(M)\to T(N)\to T(N/M)\to 0$ is exact.*

Definition 3. The intersection $\mathscr{D}\cap\mathscr{F}$ will be denoted by $\mathscr{F}_d$, whose member will be said to be *F-closed*.

Corollary 3. *Let M be an F-closed module and L its submodule. Then L is F-closed if and only if $M/L\in\mathscr{F}$.*

Proof. Let $\mathfrak{a}$ be an ideal in F. Since $\mathrm{Hom}_A(A/\mathfrak{a}, M)=\mathrm{Ext}_A^1(A/\mathfrak{a}, M)=0$, $\mathrm{Hom}_A(A/\mathfrak{a}, M/L)\simeq\mathrm{Ext}_A^1(A/\mathfrak{a}, L)$.

Remark 1. If F is an idempotent filter, then $D(M)$ is F-injective for any module M. $D(M)$ is the only submodule D of $E(M)$ so that $D/M\in\mathscr{T}$ and $E(M)/D$ $\in\mathscr{F}$. Consequently we can say that $D(M)$ is an *F-injective hull* of an A-module M.

Notice. For each filter F the class $\mathscr{T}$ is a Serre subcategory if and only if F is idempotent. See [10] for the terminology. Further we can say that F is idempotent if and only if $\mathscr{T}$ is a localizing subcategory. Recently an idempotent filter is called a Gabriel topology by Bo Stenström.

§2. Splitting filters

Theorem 1. *The following conditions for a filter F are equivalent:*

(a) *For any module M, if $M\notin\mathscr{T}$, then there exists a non-zero submodule L of M with $L\in\mathscr{F}$.*

(b) *$\mathscr{T}$ is closed under essential extension.*

(c) *For any module M, $E(T(M))=T(E(M))$.*

(d) *If an A-module M is injective, then so is $T(M)$.*

(e) *For any ideal $\mathfrak{a}$ of A, there exists $\mathfrak{b}$ in F such that $\mathfrak{a}=c(\mathfrak{a})\cap\mathfrak{b}$.*

(f) *For any ideal $\mathfrak{a}$ with $\mathfrak{a}\notin F$, there exists $a\in A-c(\mathfrak{a})$ such that $(\mathfrak{a}: a)=c(\mathfrak{a}: a)$.*

PROOF. This can be obtained by a modification of the proof of [8, Theorem 2]. So we shall omit the proof.

NOTICE. A part of Theorem 1 has already been known (see [3] and [9]). Recently S. Itoh, in [5], has shown the equivalence of (a)–(d) in Theorem 1 when $\mathscr{T}$ is a localizing subcategory.

DEFINITION 4. A filter F is called a *splitting filter* if F satisfies the condition above. Note that if F is a splitting filter, then it is idempotent by the condition (e).

PROPOSITION 3. *If F is a splitting filter, then $E(M/T(M))=E(M)/E(T(M))$ and $E(M)=E(T(M))\oplus E(M/T(M))$ for any A-module M.*

PROOF. First consider a canonical commutative diagram of modules with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & E(T(M))/T(M) \arrow[r] \arrow[u] & E(M)/M \arrow[r] \arrow[u] & L \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E(T(M)) \arrow[r] \arrow[u] & E(M) \arrow[r] \arrow[u, "f"] & E(M)/T(E(M)) \arrow[r] \arrow[u, "g"] & 0 \\
0 \arrow[r] & T(M) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & M/T(M) \arrow[r] \arrow[u, "h"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $E(M)/T(E(M))$ is injective, it suffices to show that homomorphism h in the diagram is essential. If h is not essential then, by virtue of the next Lemma 1, there exists a non-zero element x of $E(M)/T(E(M))$ such that $0(x)=0(g(x))$. And there exists an inverse image y of x in $E(M)$ such that $0(x)=0(y)$. Hence $0(y)=0(f(y))$, which contradicts the fact that $E(M)$ is essential over M.

LEMMA 1. *Let L be a submodule of an A-module M. Then L is not essential in M if and only if there exists a non-zero element x of M such that $0(x)=0(\bar{x})$,* *where $\bar{x}$ is the canonical image of x in M/L.*

PROOF. Clear.

PROPOSITION 4. *Suppose that F be a splitting filter and let F' be another filter of ideals of A. Then, for any A-module M, $T(D'(M))=D'(T(M))$, in which $D'(M)=\{x \in E(M); (M: x) \in F'\}$.*

PROOF. For any submodule L of M, we denote by $C'(L, M)$ the set of elements x in M such that $(L: x) \in F'$. Our assertion follows from the equalities for any module M: $TD'(M)=D'(M) \cap T(E(M))=C'(M, E(M)) \cap E(T(M))=C'(M \cap E(T(M)), E(T(M)))=C'(T(M), E(T(M)))=D'(T(M))$.

Now, with the same notations and assumptions in Prop. 4, can we see that $D'(M/T(M)) \simeq D'(M)/D'(T(M))$? The rest of this section will be devoted to examine into conditions for this equality.

Let E be an A-module and M, E' its submodules. Then we have a commutative diagram of modules with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{$*$}$$

in which $M'=E' \cap M$, and morphisms and modules are all canonical. From this diagram ($*$), we can construct directly another commutative diagram of modules with exact rows and columns except the middle row:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & T(N') & \longrightarrow & T(N) & \longrightarrow & T(N'') & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C(M', E') & \longrightarrow & C(M, E) & \longrightarrow & C(M'', E'') & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{$**$}$$

But, by consideration of the homology group at each module in the diagram ($**$), we have the following commutative diagram of modules with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & T(N') & \longrightarrow & T(N) & \longrightarrow & T(N'') & \longrightarrow & L & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & & \| & \\
0 \longrightarrow & C(M', E') & \longrightarrow & C(M, E) & \longrightarrow & C(M'', E'') & \longrightarrow & K & \longrightarrow 0
\end{array}$$

Hence we have

LEMMA 2. *With the same notations as above, if F is idempotent and $N' \in \mathscr{T}$, then $C(M'', E'') \simeq C(M, E)/C(M', E')$.*

More generally, using the notion of the right derived functors $R^nT\,(n \geq 0)$ of T, we have

LEMMA 3. *With the same notations as above, if $R^1T(N')=0$, then $C(M'', E'') \simeq C(M, E)/C(M', E')$.*

As a special case, we have

THEOREM 2. *Let F be a splitting filter and M an A-module. Then $D(T(M)) = T(D(M)) = E(T(M))$ and $D(M) \simeq D(T(M)) \oplus D(M/T(M))$.*

PROOF. Apply Lemma 2, putting $E' = T(E(M))$. Then our assertion follows directly since $D(T(M)) = E(T(M))$ is injective.

LEMMA 4. *Let F be a splitting filter and M an A-module. Then*

(a) *If $M \in \mathscr{T}$, then $R^nT(M)=0$ for $n \geq 1$.*

(b) *$R^nT(M) = R^nT(D(M)) = R^nT(D(M/T(M)))$ for $n \geq 2$.*

PROOF. (a) comes from (b) in Theorem 1. (b) follows from long exact sequences derived from R^nT's, using (a) and Theorem 2.

PROPOSITION 5. *Let F_t be a splitting filter with the associated left exact functor t, and let F be another splitting filter. Suppose that $R^2T(M)=0$ for any A-module M such that $t(M)=M$ and $M \in \mathscr{F}_d$. Then, for any module M, $D(M/t(M)) \simeq DM)/D(t(M))$.*

PROOF. Apply Lemma 3, putting $E' = t(E(M))$. Since $R^2T(t(M)) = R^1T(E(t(M))/t(M))$, it suffices to show that $R^2T(t(M))=0$. This last equality follows directly from our assumption and Lemma 4.

§3. Divisorial lattices

A lattice $C(A)$ of ideals of A will be said to be divisorial if it is closed under intersection and, for any ideal $\mathfrak{a}$ in $C(A)$ and any element x in A, $(\mathfrak{a}: x)$ lies also in $C(A)$. Let F be an idempotent filter of ideals of A. Then we say that an ideal

$\mathfrak{a}$ of A is F-closed if $c(\mathfrak{a})=\mathfrak{a}$. The set of F-closed ideals of A forms a divisorial lattice $C_F(A)$. Note that the closure $c(\mathfrak{a})$ of any ideal $\mathfrak{a}$ relative to F is the smallest F-closed ideal containing it.

Conversely, for each divisorial lattice $C(A)$ of ideals of A, we have a closure operation upon the lattice of ideals of A, defining the closure $\tilde{\mathfrak{a}}$ of an ideal $\mathfrak{a}$ by the smallest ideal in $C(A)$ containing $\mathfrak{a}$.

PROPOSITION 6. *With the same notations as above, the set F of ideals $\mathfrak{a}$ of A such that $\tilde{\mathfrak{a}}=A$ forms an idempotent filter.*

PROOF. First of all we show that F is a filter. Since $\mathfrak{a}\subseteq\tilde{\mathfrak{a}}$, and $\tilde{\mathfrak{a}}\subseteq\tilde{\mathfrak{b}}$ if $\mathfrak{a}\subseteq\mathfrak{b}$, it suffices to show that if $\mathfrak{a}$ and $\mathfrak{b}$ are in F, then so is $\mathfrak{a}\cdot\mathfrak{b}$. Suppose that $\mathfrak{a}\cdot\mathfrak{b}\notin F$. Then there exists a proper ideal $\mathfrak{c}$ in $C(A)$ containing $\mathfrak{a}\cdot\mathfrak{b}$. Since $\mathfrak{a}\nsubseteq\mathfrak{c}$, $(\mathfrak{c}\colon\mathfrak{a})$ is a proper ideal in $C(A)$ containing $\mathfrak{b}$, contrary to the hypothesis.

The fact that F is idempotent follows from the next

LEMMA 5. *With the same notations as above, let $\mathfrak{a}$ be an ideal of A. Then $c(\mathfrak{a})\subseteq\tilde{\mathfrak{a}}$. Thus, $c(\mathfrak{a})\in F$ if and only if $\tilde{\mathfrak{a}}=A$.*

PROOF. It suffices to show that, for any ideal $\mathfrak{a}$ and element x in A, if $(\mathfrak{a}\colon x)\in F$, then $x\in\tilde{\mathfrak{a}}$. If $x\notin\tilde{\mathfrak{a}}$, then $(\tilde{\mathfrak{a}}\colon x)$ is a proper ideal in $C(A)$ containing $(\mathfrak{a}\colon x)$, which shows that $(\mathfrak{a}\colon x)\notin F$.

PROPOSITION 7. *Let $C(A)$ be a divisorial lattice of ideals of A and F an associated idempotent filter as above. Then the following conditions for $C(A)$ are equivalent:*

(a) $C(A)=C_F(A)$.

(b) *For any ideal $\mathfrak{a}$ of A, $c(\mathfrak{a})=\tilde{\mathfrak{a}}$.*

(c) *For any ideal $\mathfrak{a}$ and element x in A, $(\tilde{\mathfrak{a}}\colon x)=\widetilde{(\mathfrak{a}\colon x)}$.*

PROOF. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c): Clear.

REMARK 2. With the above notations, consider the condition (d): $\tilde{\mathfrak{a}}\cap\tilde{\mathfrak{b}}=\widetilde{\mathfrak{a}\cap\mathfrak{b}}$ for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of A. It is easy to see that the condition in Prop. 7 implies (d). But the converse is not true. For example, let $C(A)$ be the set of ideals $\mathfrak{a}$ of A such that $\mathfrak{a}=\sqrt{\mathfrak{a}}$. Then $C(A)$ is a divisorial lattice with the condition (d) since $\tilde{\mathfrak{a}}=\sqrt{\mathfrak{a}}$ for any ideal $\mathfrak{a}$ of A. However $c(\mathfrak{a})=\mathfrak{a}$ for any ideal $\mathfrak{a}$ of A. Thus, unless A is regular in the sense of von-Neumann, $C(A)$ does not satisfy the condition in Prop. 7.

REMARK 3. With the above notations, suppose that A is an integral domain and $C(A)$ satisfies the conditions (d) and (e): $x\cdot\tilde{\mathfrak{a}}=\widetilde{x\mathfrak{a}}$ for any ideal $\mathfrak{a}$ and element x in A. Then $C(A)$ satisfies the condition in Prop. 7. In fact, for any $\mathfrak{a}$ and x, $x(\mathfrak{a}\colon x)=xA\cap\mathfrak{a}$. Therefore $x\widetilde{(\mathfrak{a}\colon x)}=xA\cap\tilde{\mathfrak{a}}=x(\tilde{\mathfrak{a}}\colon x)$ since $xA=\widetilde{xA}$.

EXAMPLE. Let A be an integral domain. Consider the set of ideals of A which are divisorial in the usual sense. Then it is a divisorial lattice with the condition (e). The associated filter consists of all ideals $\mathfrak{a}$ of A such that $\mathfrak{a}^{-1}=A$. Let $\mathfrak{a}$ be a non-zero ideal of A and K the fractional field of A. Then, since $K/A \in \mathscr{F}$ and thus $T(A/\mathfrak{a})=T(K/\mathfrak{a})$, $c(\mathfrak{a})=D(\mathfrak{a})$ with respect to the above filter.

In [8], the following proposition is proved. We shall prove it again rather easily.

PROPOSITION 8. *With the same situation in the above example, assume that A is completely integrally closed. Then, for any ideal $\mathfrak{a}$ of A, $c(\mathfrak{a})=\tilde{\mathfrak{a}}$.*

PROOF. Suppose that $x \in \tilde{\mathfrak{a}}$, namely that $x \cdot \mathfrak{a}^{-1} \in A$, then $x \cdot \mathfrak{a} \cdot \mathfrak{a}^{-1} \subseteq \mathfrak{a}$, thus $\mathfrak{a} \cdot \mathfrak{a}^{-1} \subseteq (\mathfrak{a}: x)$. By our assumption, $\widetilde{\mathfrak{a} \cdot \mathfrak{a}^{-1}}=A$, hence $x \in c(\mathfrak{a})$.

REMARK 4. To avoid the trivial case we assume that A is not a field. Let F be the filter in the above example. Then the associated (hereditary) torsion theory $(\mathscr{T}, \mathscr{F})$ is cogenerated by $E(K/A)$. That is, an A-module M belongs to $\mathscr{T}$ if and only if $\mathrm{Hom}_A(M, E(K/A))=0$ (cf. [6], [9] and [8, Prop. 5]). In fact, "only if" part is easy to see, so we shall show "if" part. If $\mathrm{Hom}_A(M, E(K/A))=0$, then $\mathrm{Hom}_A(Ax, K/A)=0$ for any element x of M. Hence it suffices to show that if $\mathrm{Hom}_A(A/\mathfrak{a}, K/A)=0$, then $\mathfrak{a} \in F$. Suppose that an ideal $\mathfrak{a}$ is not in F, then $\tilde{\mathfrak{a}} \neq A$ by Lemma 5. Hence we can take an element x of $\mathfrak{a}^{-1}-A$. Define $f: A \to K/A$ so that $f(a)=ax$ modulo A. Then $f(\mathfrak{a})=0$ and $f \neq 0$, which completes the proof.

§4. Relations with $\otimes$ and Hom

As before, we fix a filter F of ideals of A.

LEMMA 6. *Let M be an A-module. Then*
(a) *If M is in $\mathscr{T}$, then so are $\mathrm{Tor}_n^A(M, N)$ for any module N.*
(b) *If M is in $\mathscr{F}$, then so is $\mathrm{Hom}_A(N, M)$ for any module N.*

PROOF. Clear.

PROPOSITION 9. *If an A-module M is F-closed, then so is $\mathrm{Hom}_A(N, M)$ for any module N.*

PROOF. Let $\mathfrak{a}$ be an ideal in F and N an A-module. It suffices to show that $\mathrm{Hom}_A(A, \mathrm{Hom}_A(N, M)) \simeq \mathrm{Hom}_A(\mathfrak{a}, \mathrm{Hom}_A(N, M))$. Consider two exact sequences:

$$0 \longrightarrow \mathrm{Tor}_1^A(A/\mathfrak{a}, N) \longrightarrow \mathfrak{a} \otimes_A N \longrightarrow \mathfrak{a}N \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{a}N \longrightarrow N \longrightarrow N/\mathfrak{a}N \longrightarrow 0$$

Since $\mathrm{Tor}_1^A(A/\mathfrak{a}, N)$ is in $\mathscr{T}$, $\mathrm{Hom}_A(\mathfrak{a}N, M) \simeq \mathrm{Hom}_A(\mathfrak{a} \otimes_A N, M) \simeq \mathrm{Hom}_A(\mathfrak{a}, \mathrm{Hom}_A(N, M))$. Again since $N/\mathfrak{a}N$ is in $\mathscr{T}$ and M is F-injective, $\mathrm{Hom}_A(\mathfrak{a}N, M) \simeq \mathrm{Hom}_A(N, M)$. These isomorphisms are all natural, and hence the proof is complete.

PROPOSITION 10. *Let M and N be A-modules. If M is F-injective and if* $\mathrm{Tor}_1^A(A/\mathfrak{a}, N) = 0$ *for any ideal $\mathfrak{a}$ in F, then* $\mathrm{Hom}_A(N, M)$ *is F-injective.*

PROOF. Let $\mathfrak{a}$ be an ideal in F. Then we have a commutative diagram of modules with exact columns:

$$\begin{array}{ccccc}
\mathrm{Hom}_A(N, M) & \simeq & \mathrm{Hom}_A(A \otimes_A N, M) & \simeq & \mathrm{Hom}_A(A, \mathrm{Hom}_A(N, M)) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
\mathrm{Hom}_A(\mathfrak{a}N, M) & \overset{i}{\simeq} & \mathrm{Hom}_A(\mathfrak{a} \otimes_A N, M) & \simeq & \mathrm{Hom}_A(\mathfrak{a}, \mathrm{Hom}_A(N, M)) \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

In fact, since M is in $\mathscr{F}$, the columns are exact. And the isomorphism i can be obtained by our assumption for N. This diagram shows that p is epimorphic, which proves our assertion.

LEMMA 7. *Let M and N be A-modules. If M is F-closed, then* $\mathrm{Hom}_A(D(N), M) = \mathrm{Hom}_A(N, M)$.

PROOF. Clear.

COROLLARY 1. *Let N and M be A-modules. Assume that F is idempotent and that M is in $\mathscr{F}$. Then* $\mathrm{Hom}_A(D(N), D(M)) = \mathrm{Hom}_A(N, D(M))$.

PROOF. Since $D(M)$ is F-closed by Remark 1, our assertion follows directly from Lemma 7.

COROLLARY 2. *With the same assumptions as in* Cor. 1, *assume further that* $\mathrm{Hom}_A(N, D(M)/M) \in \mathscr{T}$. *Then* $\mathrm{Hom}_A(N, D(M))$ *is an F-injective hull of* $\mathrm{Hom}_A(N, M)$.

PROOF. By our assumption, $\mathrm{Hom}_A(N, M)$ is essential in $\mathrm{Hom}_A(N, D(M))$. Since the latter is F-injective, our assertion follows from the min-max property of an F-injective hull.

Let us now inquire into an A-module N such that $\mathrm{Hom}_A(N, L)$ belongs to $\mathscr{T}$ for any module L in $\mathscr{T}$. We shall say that such a module N is *of F-finite type.* It is easy to see that each module of finite type is always of F-finite type.

PROPOSITION 11. *If F is a splitting filter, then every submodule of an A-*

module of F-finite type is of F-finite type.

PROOF. Let N be an A-module of F-finite type and M its submodule. Then, for any module L in $\mathcal{T}$, $\mathrm{Hom}_A(M, L)$ is a submodule of $\mathrm{Hom}_A(M, E(L))$, which is a homomorphic image of $\mathrm{Hom}_A(N, E(L))$. The last module is in $\mathcal{T}$, since $E(L)$ is in $\mathcal{T}$ by our assumption. Thus M is of F-finite type because of the closedness of the class $\mathcal{T}$.

The above result generalizes Prop. 32 in [8]. But we can have more generalization as follows. At first, note that the class of modules of F-finite type is closed under image and group extension.

DEFINITION 5. Let F be a filter of ideals of A. We say that F is a *completely multiplicative filter* if it satisfies the conditions:

(i) For any ideals $\mathfrak{a}$ and $\mathfrak{b}$ in F; $\mathfrak{a}\cdot\mathfrak{b}$ belongs to F.

(ii) For any ideal $\mathfrak{a}$, $(\mathfrak{a}: c(\mathfrak{a}))$ belongs to F, or equivalently,

(ii)′ For any ideal $\mathfrak{a}$, there exists an ideal $\mathfrak{b}$ in F such that $(\mathfrak{a}: \mathfrak{b})=c(\mathfrak{a})$.

REMARK 5. In the above, the equivalence of (ii) and (ii)′ follows from the statement (5) in § 1.

PROPOSITION 12. *Let M be an A-module of finite type and suppose that F is a completely multiplicative filter. Then every submodule of M is of F-finite type.*

PROOF. We prove the assertion by induction on the number of generators of M. If M is cyclic, then its submodule is of the form $\mathfrak{a}/\mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A. It suffices to show that each ideal $\mathfrak{a}$ is of F-finite type. Let f be a homomorphism from an ideal $\mathfrak{a}$ into an A-module L in $\mathcal{T}$. Then $f(\mathfrak{a})\simeq\mathfrak{a}/\mathfrak{b}$ and $\mathfrak{b}\subseteq\mathfrak{a}\subseteq c(\mathfrak{b})$. By our assumption there exists an ideal $\mathfrak{c}$ in F such that $\mathfrak{c}\mathfrak{a}\subseteq\mathfrak{b}$, i.e., $\mathfrak{c}f=0$, which shows that $\mathrm{Hom}_A(\mathfrak{a}, L)\in\mathcal{T}$ if L is in $\mathcal{T}$.

If M is not cyclic, then we can write $M=M_1+M_2$, where M_1 and M_2 are generated by less elements than M is. Let N be a submodule of M. Then $N\cap M_1$ and $N/N\cap M_1=N+M_1/M_1$ are of F-finite type, by induction hypothesis, hence so is N, which completes the proof.

REMARK 6. Let F be a filter of ideals of A. It is easy to see that if F is completely multiplicative, then it is idempotent. Further, if F is of splitting type, then it is completely multiplicative, by the condition (e) in Theorem 1.

As a summary of the above results, we have

THEOREM 3. *Let M be an A-module with $M\in\mathcal{F}$ and N a submodule of an A-module of finite type. If F is a completely multiplicative filter, then* $\mathrm{Hom}_A(N,$

$D(M))$ *is an F-injective hull of* $\mathrm{Hom}_A(N, M)$.

EXAMPLE (continued). Let A be a completely integrally closed domain and F the set of ideals $\mathfrak{a}$ of A such that $\mathfrak{a}^{-1}=A$. Then F is completely multiplicative. In fact, for any non-zero ideal $\mathfrak{a}$ of A, $\mathfrak{a}\cdot\mathfrak{a}^{-1}\in F$ by our assumption on A. On the other hand since $\mathfrak{a}^{-1}=c(\mathfrak{a})^{-1}$, $\mathfrak{a}\cdot\mathfrak{a}^{-1}\subseteq(\mathfrak{a}\colon c(\mathfrak{a}))$, which shows our assertion.

By Theorem 3, for any A-lattices N and M, $D(\mathrm{Hom}_A(N, M))=\mathrm{Hom}_A(N, D(M))=\mathrm{Hom}_A(D(N), D(M))$.

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