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Positive Bounded Solutions for a Class of Linear Delay Differential Equations

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Let n be an integer, $n \geq 2$, let q be a continuous function from $[0, \infty)$ to $(0, \infty)$, and let G be the set to which g belongs if and only if g is a nondecreasing unbounded continuous function from $[0, \infty)$ to $[0, \infty)$ such that $g(t) \leq t$ whenever $t \geq 0$. Let G° be that subset of G to which g belongs if and only if g is in G and $g(t) < t$ whenever $t > 0$. We propose to study the differential equation

$$u^{(n)}(t) + (-1)^{n+1} q(t) u(g(t)) = 0, \tag{1}$$

for g in G. A function u from $[0, \infty)$ to $(-\infty, \infty)$ is called a solution of (1) if and only if there is $b \geq 0$ such that $u^{(n)}$ exists on (b, ∞) and (1) is true whenever $t > b$. A solution u of (1) is called oscillatory if and only if the set $\{t: t \geq 0$ and $u(t) = 0\}$ is unbounded. Otherwise, u is called nonoscillatory. Although the analogue of (1) without delay is known to have a positive bounded solution, several authors have shown that if the delay is large enough, i.e., g is small enough, then every bounded solution of (1) is oscillatory. In particular, if g is in G, if

$$\int_0^\infty t^{n-1} q(t) dt = \infty, \tag{2}$$

and if

$$\limsup_{t \to \infty} \int_{g(t)}^t (g(t) - g(s))^{n-1} q(s) ds > (n-1)!, \tag{3}$$

then G. Ladas, V. Lakshmikantham, and J. S. Papadakis [3] have shown that every bounded solution of (1) is oscillatory. M. Naito [7] has shown that if g is in G and

$$\limsup_{t \to \infty} \int_{g(t)}^t (s - g(t))^{n-1} q(s) ds > (n-1)!, \tag{4}$$

then every bounded solution of (1) is oscillatory. Note that although each of (3) and (4) implies (2), (3) and (4) are independent. Since the results of [3] and [7] are of the nature "if g is small enough then every bounded solution of (1) is oscillatory", the question arises: If g is large enough can we conclude the existence of a positive bounded solution? We shall give a result which answers this question affirmatively, and we shall also give a comparison result. R. Driver [1], [2] has given results, independent of the present study, which are related in the sense that they ensure that, in some circumstances, delay differential equations with small delays have behaviors similar to the corresponding ordinary differential equations without delay.

THEOREM 1: *Suppose that g is in G° and that*

$$v^{(n)}(t)+(-1)^{n+1}q(t)v(t) = (-1)^n(t-g(t))q(t) \tag{5}$$

has a positive bounded solution. Then (1) *has a positive bounded solution.*

THEOREM 2: *Suppose (g, h) is in $G^\circ \times G^\circ$ and $g(t)\leq h(t)$ whenever $t\geq 0$. Suppose also that there is a positive bounded solution of* (1). *Then there is a bounded positive solution of*

$$w^{(n)}(t)+(-1)^{n+1}q(t)w(h(t)) = 0. \tag{6}$$

Before proving Theorem 1, we need the following lemma.

LEMMA: *Suppose $c\geq 0$ and each of ϕ and ψ is a positive continuous function on $[c, \infty)$. Suppose also that $\psi(t)\leq\phi(t)$ whenever $t\geq c$, and that there is a positive bounded solution u of*

$$u^{(n)}(t)+(-1)^{n+1}q(t)u(t)=(-1)^n\phi(t) \tag{7}$$

on $[c, \infty)$. Then there is a positive bounded solution v of

$$v^{(n)}(t)+(-1)^{n+1}q(t)v(t) = (-1)^n\psi(t) \tag{8}$$

on $[c, \infty)$ such that $v(t)\leq u(t)$ whenever $t\geq c$.

PROOF: Since $u>0$ on $[c, \infty)$, (7) says that $u^{(n)}$ is eventually one-signed. Since $u^{(n)}$ is eventually one-signed, $u^{(n-1)}$ is eventually one-signed. Continuing this, we see that there is $d\geq c$ such that none of $u, u',\dots, u^{(n-1)}$ has a zero on $[d, \infty)$. With arguments similar to those of [4, Theorem 2], we see that $u^{(k)}u^{(k+1)}<0$ on $[d, \infty)$ if $k=0,\dots, n-1$. Thus $u^{(k)}>0$ on $[d, \infty)$ if k is even and $u^{(k)}<0$ on $[d, \infty)$ if k is odd. Since we now have $(-1)^k u^{(k)}(d)>0$ for $k=0,\dots, n-1$, arguments similar to those of [5, Lemma] show that $(-1)^k u^{(k)}(t)>0$ if $c\leq t\leq d$, $k=0,\dots, n-1$. Thus, $u^{(k)}>0$ on $[c, \infty)$ if k is even and $u^{(k)}<0$ on $[c, \infty)$ if k is odd. Arguments similar to those of [6, Lemma 2] now give

$$-u'(t) = \frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}q(s)u(s)ds+\frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}\phi(s)ds \tag{9}$$

and

(10) $$u(t) \geq \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)u(s)ds + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\phi(s)ds$$

if $t \geq c$. From (9) and (10) it follows that

(11) $$-u'(t) \geq \frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}q(s)u(s)ds + \frac{1}{(n-2)!}\int_t^\infty (s-t)^{n-2}\psi(s)ds$$

and

(12) $$u(t) \geq \frac{1}{(n-1)!}\int_t^\infty (s-)^{n-1}q(s)u(s)ds + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\psi(s)ds$$

if $t \geq c$. Now one can define a sequence $\{z_k\}_{k=1}^\infty$, each value of which is a positive continuous function on $[c, \infty)$, according to $z_1 = u$,

$$z_{k+1}(t) = \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)z_k(s)ds + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\psi(s)ds$$

if $k \geq 1$ and $t \geq c$. A straightforward induction argument shows that if k is a positive integer then $z_{k+1} \leq z_k \leq u$ on $[c, \infty)$. This and (11) say that $\{z_k\}_{k=1}^\infty$ is equicontinuous. Thus a subsequence converges locally uniformly, and, by monoticity of the sequence, $\{z_k\}_{k=1}^\infty$ converges locally uniformly. Call the limit v. Clearly $v(t) \leq u(t)$ whenever $t \geq c$, and the Dominated Convergence Theorem says

(13) $$v(t) = \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)v(s)ds + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\psi(s)ds$$

if $t \geq c$. Differentiating (13) yields (8) on $[c, \infty)$, and the proof is complete.

Note that (9), (10), and (13), and the facts that $v \leq u$ and $\psi \leq \phi$ on $[c, \infty)$, ensure that $v' < 0$ on $[c, \infty)$, and $-v' \leq -u'$ on $[c, \infty)$. This will be used in the proof of Theorem 1.

Proof of Theorem 1: Let w_1 be a bounded positive solution of (5). Since $w_1 > 0$, $w_1' < 0$, $w_1'' > 0$, we know that $w_1(\infty) = \lim_{t\to\infty} w_1(t)$ and $w_1'(\infty) = \lim_{t\to\infty} w_1'(t)$ both exist. Also, $w_1'(\infty) = 0$, for otherwise $w_1(\infty)$ and $w_1'(\infty)$ cannot both exist. Find $c \geq 0$ such that $|w_1'(s)| \leq 1$ if $s \geq g(c)$. Let $b > c$, and let λ and μ be continuous nonnegative functions on $[c, \infty)$ such that $\lambda(t) + \mu(t) = 1$ if $t \geq c$, such that $\lambda(t) = 1$ and $\mu(t) = 0$ if $t \geq b$, and such that $\mu(t) > 0$ if $c \leq t < b$. If $t \geq c$ then

$$|w_1(g(t)) - w_1(t)| \leq t - g(t)$$

since $|w_1'(s)| \leq 1$ whenever $s \geq g(c)$. Thus the lemma says there is a bounded positive solution w_2 on $[c, \infty)$ of

$$w_2^{(n)}(t)+(-1)^{n+1}q(t)w_2(t) = (-1)^n\mu(t)(t-g(t))q(t) + (-1)^n\lambda(t)q(t)(w_1(g(t))-w_1(t))$$

with $w_2 \leq w_1$ on $[c, \infty)$, and $-w_2' \leq -w_1'$ on $[c, \infty)$. Extend w_2 to $[g(c), \infty)$ by requiring $w_2(t)=w_2(c)$ if $g(c)\leq t\leq c$. Now our lemma says there is a bounded positive solution w_3 on $[c, \infty)$ of

$$w_3^{(n)}(t)+(-1)^{n+1}q(t)w_3(t) = (-1)^n\mu(t)(t-g(t))q(t) + (-1)^n\lambda(t)q(t)(w_2(g(t))-w_2(t))$$

with $w_3 \leq w_2$ and $-w_3' \leq -w_2'$ on $[c, \infty)$. Extend w_3 to $[g(c), \infty)$ by requiring $w_3(t)=w_3(c)$ if $g(c)\leq t\leq c$. Continuing, we see that there is a sequence $\{w_k\}_{k=1}^{\infty}$ of positive nonincreasing functions such that

$$w_{k+1} \leq w_k \leq w_1 \tag{14}$$

on $[c, \infty)$ if $k\geq 1$,

$$-w_{k+1}' \leq -w_k' \leq -w_1' \tag{15}$$

on $[c, \infty)$ if $k\geq 1$, $w_k(t)=w_k(c)$ if $g(c)\leq t\leq c$ and $k\geq 1$, and

$$w_{k+1}^{(n)}(t)+(-1)^{n+1}q(t)w_{k+1}(t) = (-1)^n\mu(t)(t-g(t))q(t) + (-1)^n\lambda(t)q(t)(w_k(g(t))-w_k(t)) \tag{16}$$

on $[c, \infty)$ if $k\geq 1$. By (14), $\{w_k\}_{k=1}^{\infty}$ converges pointwise, and (15) says the sequence is equicontinuous, so $\{w_k\}_{k=1}^{\infty}$ has a locally uniform limit. Call this limit u. Now (16) says $\{w_k^{(n)}\}_{k=1}^{\infty}$ converges locally uniformly, so $u^{(n)}$ exists on (c, ∞), $w_k^{(n)}\to u^{(n)}$ locally uniformly, and

$$u^{(n)}(t)+(-1)^{n+1}q(t)u(t) = (-1)^n\mu(t)(t-g(t))q(t) + (-1)^n\lambda(t)q(t)(u(g(t))-u(t)) \tag{17}$$

if $t>c$. From the hypotheses on λ and μ, (17) gives (1) on $[b, \infty)$, so u is a solution of (1), and clearly u is bounded, so it remains to show u is positive.

Clearly u is nonnegative and nondecreasing, so if $d\geq c$ and $u(d)=0$ then $u(t)=0$ whenever $t\geq d$. Suppose $c\leq d<b$ and $u(d)=0$. Now $u=0$ on $[d, \infty)$, so $u^{(n)}(d)=0$, and (17) is violated since $\mu(d)(d-g(d))q(d)>0$. Suppose $u>0$ on $[c, b)$, u has a zero, and d is the first such zero, i.e., $d\geq b$, $u>0$ on $[c, d)$, and $u(d)=0$. Now $u^{(n)}(d)=0$ and, since $g(d)<d$, $q(d)u(g(d)>0$; contradicting (1). Thus $u>0$ on $[c, \infty)$, and the proof is complete.

Note that, in the Proof of Theorem 1, the introduction of λ and μ, and the requirement that g be in the subset G° of G, ensured that u "starts off" positive, and the assumption that g is in G° ensured that, after (17) reduces to (1), u cannot have a zero.

PROOF OF THEOREM 2: Let u be a bounded positive solution of (1). Find $c \geq 0$ such that $u^{(n)}$ exists on (c, ∞), such that $u>0$ on $[g(c), \infty)$, and such that $(-1)^k u^{(k)}>0$ on $[c, \infty)$ for $k=1,\dots, n-1$. Let $b>c$ be such that $g(b)\geq c$. Let $\tilde{u}$ be given by $\tilde{u}(t)=u(b)$ if $t\leq b$ and $\tilde{u}(t)=u(t)$ if $t>b$. Let $v=u-\tilde{u}$. Now $v>0$ on $[g(b), b)$, and $v=0$ on $[b, \infty)$. Also,

$$\tilde{u}^{(n)}(t)+(-1)^{n+1}q(t)\tilde{u}(g(t))=(-1)^n q(t)v(g(t))$$

if $t>b$. Thus

$$\tilde{u}(t) = \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)\tilde{u}(g(s))ds + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)v(g(s))ds \tag{18}$$

if $t\geq b$. Since $\tilde{u}$ is nonincreasing, $\tilde{u}(g(t))\geq \tilde{u}(h(t))$ whenever $t\geq b$. Thus (18) yields

$$\tilde{u}(t) \geq \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)\tilde{u}(h(s))ds + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)v(g(s))ds \tag{19}$$

if $t\geq b$. Iteration as before yields a bounded nonnegative solution w of

$$w(t) = \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)w(h(s))ds + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)v(g(s))ds, \tag{20}$$

$$w^{(n)}(t)+(-1)^{n+1}q(t)w(h(t)) = (-1)_n q(t)v(g(t)),$$

on $[b, \infty)$. The positivity of v on $[g(b), b)$, and the fact that h is in G°, ensures as before that w has no zeros. If $d>b$ and $g(d)>b$, then (20) yields (6) whenever $t\geq d$, so the proof is complete.

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