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A Boundary Value Problem for Delay Differential Equtions

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1. Introduction

In this brief paper, we shall give a result on the existence of solutions of a boundary value problem for second order delay differential equations. We consider the delay differential equation

$$(1) \qquad (\rho(t)x'(t))' = f(t, x(t), x(t-g(t)), x'(t)), \qquad a \leqq t \leqq b$$

with boundary conditions

$$(2) \qquad x(t) = \phi(t), \qquad t \leqq a, \quad x(b) = A,$$

where $f(t, x, y, z)$ and $\phi(t)$ are continuous functions defined on $[a, b] \times R^3$ and $(-\infty, a]$, R being the real line, respectively, $\rho(t)$ is a positive, continuously differentiable function defined on $[a, b]$ and A is an arbitrary constant. In (1), the lag $g(t)$ is assumed to be a nonnegative continuous function defined on $[a, b]$.

Several authors have contributed to the establishment of the existence and uniqueness of solutions of such boundary value problems. Among them, K. de Nevers and K. Schmitt [1] have obtained an existence theorem of a unique solution under the assumption that $\rho(t) \equiv 1$ and the right hand member f of (1) does not depend on the fourth argument $x'(t)$ and satisfies the following condition:

$$(3) \qquad \begin{gathered} f(t, x, y) - f(t, \bar{x}, \bar{y}) \geqq p(t)(x-\bar{x}) - q(t)(y-\bar{y}) \\ \text{if} \quad x \geqq \bar{x},\ y \geqq \bar{y} \quad \text{and} \quad t \in [a, b], \end{gathered}$$

where $p(t)$ is a continuous function and $q(t)$ is a nonnegative continuous function. Their proof is based on the so-called shooting method.

We here apply the same shooting method to prove our existence theorem, though our hypotheses on f are somewhat different from those of K. de Nevers and K. Schmitt and, for instance, the condition (3) may be replaced by the following:

$$f(t, x, y, z) - f(t, \bar{x}, \bar{y}, \bar{z})$$

$$\geqq p(t)(x-\bar{x})^{\alpha}+q(t)(y-\bar{y})^{\beta}+r(t)(z-\bar{z})-s(t) \tag{4}$$

$$\text{if} \quad x \geqq \bar{x},\ y \geqq \bar{y} \quad \text{and} \quad (t, z, \bar{z}) \in [a, b] \times R^2,$$

where $p(t)$, $q(t)$ and $s(t)$ are nonnegative continuous functions, $r(t)$ is a continuous function and α, $\beta \geqq 1$.

2. An existence theorem

We shall first show a comparison theorem between solutions of initial value problems for two delay differential equations which plays an important role in the proof of our main theorem.

The initial value problem is to find a solution of the delay differential equation (1) with initial conditions

$$x(t) = \phi(t), \qquad t \leqq a, \tag{5}$$

$$x'(+a) = \gamma, \tag{6}$$

where $\phi(t)$ is a given continuous function and γ is a given number. We denote such a solution of the initial value problem (1), (5), (6) by $x(t, \phi, \gamma)$.

We can now prove the following

THEOREM 1. *Let $U(t, x, y, z)$ be a continuous function defined on $[a, b] \times R^3$ such that*

(i) *$U(t, x, y, z)$ is nondecreasing in x and y, respectively, for each fixed triple of the other variables,*

and

(ii) *for any number $\lambda > 1$, $\lambda U(t, x, y, z) < U(t, \lambda x, \lambda y, \lambda z)$ holds on $[a, b] \times R^3$.*

We shall assume:

(iii) *$U(t, x, y, z) \leqq f(t, x, y, z)$ on $[a, b] \times R^3$;*

(iv) *for some $\eta > 0$, a solution $u(t, 0, \eta)$ of the initial value problem for the delay differential equation*

$$(\rho(t)u'(t))' = U(t, u(t), u(t-g(t)), u'(t)), \qquad a \leqq t \leqq b, \tag{7}$$

subject to the initial conditions

$$u(t) = 0, \qquad t \leqq a, \tag{8}$$

$$u'(+a) = \eta, \tag{9}$$

exists in the whole interval $[a, b]$;

(v) *for any γ, a solution $x(t, 0, \gamma)$ of the initial value problem* (1), (5), (6),

putting $\phi(t)\equiv 0$, exists in the whole interval $[a, b]$.
Then, if $\gamma>\eta$, we have

$$x(t, 0, \gamma) \geqq (\gamma/\eta)\, u(t, 0, \eta) \quad \text{on} \quad [a, b]. \tag{10}$$

PROOF. Choose $\varepsilon>0$ so that $(\gamma-\varepsilon)/\eta>1$ and set

$$m(t) = x(t, 0, \gamma) - ((\gamma-\varepsilon)/\eta)\, u(t, 0, \eta).$$

We then obtain

$$m(t) > 0, \qquad t \in (a, a+\delta]$$

for a small positive number δ, since $m(a)=0$ and $m'(+a)=\varepsilon>0$. Now assume that there exists a point $t_1 \in (a, b]$ such that $m(t_1)=0$ and $m(t)>0$ for $t\in(a, t_1)$. Then $m(t)$ attains the positive maximum at some point t_0 in the interval (a, t_1). Then

$$m(t_0) = x(t_0, 0, \gamma) - \frac{\gamma-\varepsilon}{\eta} u(t_0, 0, \eta) > 0,$$

$$m'(t_0) = x'(t_0, 0, \gamma) - \frac{\gamma-\varepsilon}{\eta} u'(t_0, 0, \eta) = 0,$$

$$m''(t_0) = x''(t_0, 0, \gamma) - \frac{\gamma-\varepsilon}{\eta} u''(t_0, 0, \eta) \leqq 0.$$

Using the hypotheses, we have

$$\begin{aligned}
&f(t_0, x(t_0, 0, \gamma), x(t_0-g(t_0), 0, \gamma), x'(t_0, 0, \gamma)) \\
&\quad = \rho(t_0)x''(t_0, 0, \gamma) + \rho'(t_0)x'(t_0, 0, \gamma) \\
&\quad \leqq \frac{\gamma-\varepsilon}{\eta} \rho(t_0)u''(t_0, 0, \eta) + \frac{\gamma-\varepsilon}{\eta} \rho'(t_0)u'(t_0, 0, \eta) \\
&\quad = \frac{\gamma-\varepsilon}{\eta} U(t_0, u(t_0, 0, \eta), u(t_0-g(t_0), 0, \eta), u'(t_0, 0, \eta)) \\
&\quad < U\left(t_0, \frac{\gamma-\varepsilon}{\eta} u(t_0, 0, \eta), \frac{\gamma-\varepsilon}{\eta} u(t_0-g(t_0), 0, \eta), \frac{\gamma-\varepsilon}{\eta} u'(t_0, 0, \eta)\right) \\
&\quad \leqq U(t_0, x(t_0, 0, \gamma), x(t_0-g(t_0), 0, \gamma), x'(t_0, 0, \gamma)) \\
&\quad \leqq f(t_0, x(t_0, 0, \gamma), x(t_0-g(t_0), 0, \gamma), x'(t_0, 0, \gamma)),
\end{aligned}$$

which is a contradiction. It follows that $m(t)>0$ on $(a, b]$. Since ε can be chosen arbitrarily small, by letting $\varepsilon \downarrow 0$, we arrive at (10). This completes the proof.

REMARK. Suppose that the function $U(t, x, y, z)$ satisfies the properties (i), (ii) only for $(t, x, y, z) \in [a, b] \times R^+ \times R^+ \times R$, R^+ being the nonnegative real half-line, and that for some $\eta \neq 0$, there exists a positive solution $u(t, 0, \eta)$ in the whole interval $[a, b]$. Then we have the same result as in Theorem 1, replacing (iii) by the assumption

(iii)′ $$U(t, x, y, z) \leqq f(t, x, y, z) \qquad on \quad [a, b] \times R^+ \times R^+ \times R.$$

We are now in a position to state our main theorem.

THEOREM 2. *Assume that*

(I) *$f(t, x, y, z)$ and $U(t, x, y, z)$ are continuous functions defined on $[a, b] \times R^3$ such that*

$$f(t, x, y, z) - f(t, \bar{x}, \bar{y}, \bar{z}) \geqq U(t, x - \bar{x}, y - \bar{y}, z - \bar{z}),$$

if $x \geqq \bar{x}$, $y \geqq \bar{y}$ and $(t, x, y, z), (t, \bar{x}, \bar{y}, \bar{z}) \in [a, b] \times R^3$;

(II) *$U(t, x, y, z)$ satisfies the conditions* (i), (ii) *in Theorem* 1 *for $(t, x, y, z) \in [a, b] \times R^+ \times R^+ \times R$;*

(III) *for some $\eta > 0$, there exists a positive solution $u(t, 0, \eta)$ of the initial value problem* (7), (8), (9) *on $(a, b]$;*

(IV) *for any γ, the initial value problem* (1), (5), (6) *has a unique solution $x(t, \phi, \gamma)$ on $[a, b]$.*

Then, for an arbitrary function $\phi(t)$ and any constant A, there exists at least one solution of the boundary value problem (1), (2).

PROOF. Let $x(t, \phi, \Gamma)$ and $x(t, \phi, \gamma)$ be two solutions of the initial value problem (1), (5), (6) corresponding to the initial data (ϕ, Γ) and (ϕ, γ), respectively. We put

(11) $$X(t, \Gamma, \gamma) = x(t, \phi, \Gamma) - x(t, \phi, \gamma)$$

and then have

(12) $$\begin{aligned}(\rho(t)X'(t, \Gamma, \gamma))' &= (\rho(t)x'(t, \phi, \Gamma))' - (\rho(t)x'(t, \phi, \gamma))' \\ &= f(t, X(t, \Gamma, \gamma) + x(t, \phi, \gamma), X(t - g(t), \phi, \gamma) \\ &\quad + x(t - g(t), \phi, \gamma), X'(t, \Gamma, \gamma) + x'(t, \phi, \gamma)) \\ &\quad - f(t, x(t, \phi, \gamma), x(t - g(t), \phi, \gamma), x'(t, \phi, \gamma)), \\ &\qquad a \leqq t \leqq b,\end{aligned}$$

(13) $$X(t, \Gamma, \gamma) = 0, \qquad t \leqq a,$$

(14) $$X'(+a, \Gamma, \gamma) = \Gamma - \gamma.$$

From Theorem 1 it immediately follows that

(15) $$X((t, \Gamma, \gamma) \geqq \frac{\Gamma-\gamma}{\eta} u(t, 0, \eta) \quad \text{on } [a, b],$$

provided that $(\Gamma-\gamma)/\eta > 1$. Therefore, if $\Gamma-\gamma$ is sufficiently large, we have

(16) $$x(b, \phi, \Gamma) - x(b, \phi, \gamma) \geqq \frac{\Gamma-\gamma}{\eta} u(b, 0, \eta) > 0.$$

Keeping γ fixed and letting $\Gamma \to \infty$, we obtain $x(b, \phi, \Gamma) \to \infty$ and similarly, keeping Γ fixed and letting $\gamma \to -\infty$, we have $x(b, \phi, \gamma) \to -\infty$. Since the assumption (IV) implies that $x(b, \phi, \gamma) - A$ is a continuous function of γ (see J. K. Hale [2, Theorem 5.1]), there exists a γ_0 such that $x(b, \phi, \gamma_0) - A = 0$. We thus obtain a solution $x(t, \phi, \gamma_0)$ of the boundary value problem (1), (2).

REMARK. By a small extension of Kneser and Hukuhara's theorem for ordinary differential equations (see [3]) to that for delay differential equations, we can know that $S_c = \{(y(c), y'(c)) | y = y(t)$ is a solution of (1), (5), (6) on $[a, c]$, $(a \leqq c \leqq b)\}$ is a closed connected set, as long as all solutions of (1), (5), (6) exist on $[a, b]$. Therefore, we may replace the condition (IV) by

(IV)′ *for any* γ, *the initial value problem* (1), (5), (6) *has a solution on* $[a, b]$,

obtaining the same result as in Theorem 2.

3. Examples

As a function $U(t, x, y, z)$ in Theorem 2, we may take

$$U(t, x, y, z) = p(t)x^{\alpha} + q(t)y^{\beta} - s(t), \quad (t, x, y, z) \in [a, b] \times R^{+} \times R^{+} \times R,$$

where $p(t)$ and $q(t)$ are nonnegative functions, $s(t)$ is a positive function and α, $\beta \geqq 1$. We can easily examine that the function $U(t, x, y, z)$ satisfies the properties (i), (ii) stated in Theorem 1 on $[a, b] \times R^{+} \times R^{+} \times R$, and hence, we have only to prove that for some $\eta > 0$, there certainly exists a positive solution $u(t, 0, \eta)$ of the initial value problem (7), (8), (9).

For simplicity, we here consider

(17) $$(\rho(t)u'(t))' = p(t)u^{\alpha}(t) + q(t)u^{\beta}(t - g(t)) - s(t), \qquad 0 \leqq t \leqq 1,$$

(18) $$u(t) = 0, \qquad t \leqq 0$$

(19) $$u'(+0) = \eta > 0,$$

where $p(t)>0$ and $\alpha, \beta>1$. The following integro-differential equation is immediately obtained:

(20) $$\rho(t)u'(t) = \rho(0)\eta + \int_0^t \{p(\xi)u^{\alpha}(\xi) + q(\xi)u^{\beta}(\xi - g(\xi)) - s(\xi)\}d\xi, \quad 0 \leqq t \leqq 1.$$

From the initial conditions (18), (19), it follows that

$$u(t) > 0, \qquad t \in (0, \varepsilon]$$

for a small positive number ε. Suppose that there exists a point $t_1 \in (0, 1]$ such that $u(t_1)=0$ and $u(t)>0$ for $t \in (0, t_1)$. Then there exists $t_2 \in (0, t_1)$ such that $u'(t_2)=0$. On the other hand, the relation

(21) $$\rho(0)\eta - \int_0^{t_2} s(\xi)d\xi + \int_0^{t_2} \{p(\xi)u^{\alpha}(\xi) + q(\xi)u^{\beta}(\xi - g(\xi))\}d\xi = 0$$

means that $u(t)=0$ on $[0, t_2]$ under the assumption that $\eta \geqq \frac{1}{\rho(0)} \times \max_{0 \leqq t \leqq 1} s(t)$. Hence, if

(22) $$\eta \geqq \frac{1}{\rho(0)} \max_{0 \leqq t \leqq 1} s(t),$$

the solution $u(t, 0, \eta)$ never vanishes on $(0, 1]$. Now we shall show the global existence of solutions of (17), (18), (19). For that purpose, we have only to show the fact that any solution does not blow up. From (20), $u(t)$ is a nondecreasing function. If $u(t_0)=1$, we have

(23) $$u(t) \leqq \frac{1}{\rho}(\rho(0)\eta - S + M)t + \int_{t_0}^t \frac{1}{\rho}(t-\xi)Mu^N(\xi)d\xi, \quad t_0 \leqq t \leqq 1,$$

where we put

(24) $$M/2 = \max\{\max_{0 \leqq t \leqq 1} q(t), \quad \max q(t)\}, \quad N = \max_{0 \leqq t \leqq 1} (\alpha, \beta), \ \rho = \min_{0 \leqq t \leqq 1} \rho(t)$$

and $\quad S = \min_{0 \leqq t \leqq 1} s(t)$.

Moreover, we have

(25) $$\frac{u(t)}{t} \leqq \frac{1}{\rho}(\rho(0)\eta - S + M) + \int_{t_0}^t \frac{1}{\rho} M\xi^N \left(\frac{u(\xi)}{\xi}\right)^N d\xi, \quad t_0 \leqq t \leqq 1.$$

We here apply a generalization of Gronwall's inequality to (25). (See, for instance, D. Willett and J. S. W. Wong [4, Theorem 2].) We then obtain bounded solutions of (17), (18), (19) under the assumption that

$$-\frac{1}{\rho}M\frac{N-1}{N+1}+\left\{\frac{1}{\rho}(\rho(0)-S+M)\right\}^{1-N}\geqq 0. \tag{26}$$

Consequently, we obtain positive solutions of (17), (18), (19), if the conditions (22), (26) are assumed. Hence, we can always solve the boundary value problem of the form

$$(\rho(t)x'(t))' = h(t, x(t), x(t-g(t))), \qquad 0 \leqq t \leqq 1 \tag{27}$$

$$x(t) = \phi(t), \qquad t \leqq 0, \quad x(1) = A, \tag{28}$$

where the function $h(t, x, y)$ is a continuous function defined on $[0, 1]\times R^2$ such that

$$h(t, x, y)-h(t, \bar{x}, \bar{y}) \geqq p(t)(x-\bar{x})^{\alpha}+q(t)(y-\bar{y})^{\beta}-s(t) \tag{29}$$

holds for $x\geqq\bar{x}$ and $y\geqq\bar{y}$, if the initial value problem

$$(\rho(t)x'(t))' = h(t, x(t), x(t-g(t))), \tag{30}$$

$$x(t) = \phi(t), \qquad t \leqq 0 \tag{31}$$

$$x'(+0) = \gamma, \tag{32}$$

has a unique solution on $[a, b]$ for any γ.

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