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On the Radial Limits of Riesz Potentials at Infinity

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1. Introduction

In this paper, we shall study the limits of potentials on R^n along rays issuing from the origin. It is known that if U_2^μ is the Newtonian potential of a measure μ with finite energy, then $\lim_{r\to\infty} U_2^\mu(r\xi)=0$ for a. e. ξ with $|\xi|=1$ (see N. S. Landkof [2; Theorem 1.21]). We shall deal with the Riesz potential U_α^μ of order α, $0<\alpha<n$, of a measure μ whose energy may not be finite, and give an improvement of the above result (Theorem 1).

We shall then consider the functions of the form

$$F(x) = \int |x-y|^{\alpha-n}|y|^{\beta/p}f(y)dy,$$

where $\alpha>0$, $\beta\geqq 0$, $p>1$, $\alpha p+\beta<n$ and $f\in L^p(R^n)$. In special cases, e.g. in the case where $\alpha=1$, $\beta=0$ and $1<p<n$, *M.* Ohtsuka showed that $\lim_{r\to\infty}F(r\xi)=0$ for a.e. ξ with $|\xi|=1$ ([5; Theorems 9.6 and 9.12, Example 1 given after Theorem 3.21]). This result will be improved in Theorem 2.

Finally we shall be concerned with locally p-precise functions on R^n. We say that a function u is locally p-precise on R^n if u is p-precise on any bounded open set in R^n; for p-precise functions, see [7]. We also refer to [5; Chap. IV]. Let $1<p<n$ and u be a locally p-precise function on R^n such that

$$\int |\mathrm{grad}\, u|^p|x|^{-\beta}dx < \infty$$

for some non-negative number β smaller than $n-p$. Then we shall show in Theorem 3 that there are a constant c and a set $E\subset\Gamma=\{\xi\in R^n;\ |\xi|=1\}$ such that

$$\lim_{r\to\infty} u(r\xi) = c \qquad \text{if} \quad \xi\in\Gamma-E$$

and

$$C_p(E) = 0 \qquad \text{if} \quad p\leqq 2,$$

$$C_{p-\varepsilon}(E) = 0 \qquad \text{for any } \varepsilon \text{ with} \quad 0<\varepsilon<p \quad \text{if} \quad p>2,$$

where $C_\gamma(E)$ is the Riesz capacity of E of order γ. If, in addition, u is a Riesz potential of a non-negative measure with finite energy, then $c=0$ (cf. [5; Theorem

10.18]).

2. Preliminaries

Let R^n be the n-dimensional Euclidean space ($n \geqq 2$) with points x, y, etc. and let α be a number such that $0<\alpha<n$. For a non-negative (Radon) measure μ, the Riesz potential of μ of order α is defined by

$$U_\alpha^\mu(x) = \int |x-y|^{\alpha-n} d\mu(y).$$

The Riesz capacity of a Borel set $E \subset R^n$ of order α is defined by

$$C_\alpha(E) = \sup \mu(R^n),$$

where the supremum is taken over all non-negative measures μ such that S_μ (the support of μ) $\subset E$ and $U_\alpha^\mu \leqq 1$ on S_μ. By the definition of Riesz capacity and a maximum principle, we have

LEMMA 1. *Let μ be a non-negative measure on R^n and let $0<\alpha<n$. Set $E=\{x \in R^n;\ U_\alpha^\mu(x) \geqq 1\}$. Then*

$$C_\alpha(E) \leqq M\mu(R^n),$$

where $M=1$ if $\alpha \leqq 2$ and $M=2^{n-\alpha}$ if $\alpha>2$.

Let $1 \leqq p < \infty$. We denote by $L^p(R^n)$ the class of all measurable functions f on R^n such that

$$\|f\|_p = \left\{\int_{R^n} |f(x)|^p dx\right\}^{1/p} < \infty .$$

We denote by $L^p_{loc}(R^n)$ the class of all measurable functions f on R^n such that $\int_K |f(x)|^p dx < \infty$ for any compact set $K \subset R^n$.

We now let $1<p<n$. A set $E \subset R^n$ is said to be p-exceptional if there is a non-negative function $f \in L^p(R^n)$ such that $\int |x-y|^{1-n} f(y) dy = \infty$ for any $x \in E$. If a property is true on R^n except for a p-exceptional set, then we say that this property is true p-a.e. on R^n. We note that if u and v are locally p-precise functions on R^n such that $u=v$ a.e. on R^n, then $u=v$ p-a.e. on R^n. Furthermore, if u is a locally p-precise function on R^n, then $|\mathrm{grad}\, u|$ is defined a.e. on R^n and belongs to $L^p_{loc}(R^n)$. For these facts, see Ohtsuka [5; Chap. IV].

3. Radial limits of potentials of measures

We first show

THEOREM 1. *Let* $0<\alpha<n$ *and let* μ *be a non-negative measure such that*

$$(1) \qquad \int (1+|x|)^{\alpha-n} d\mu(x) < \infty .$$

Then there is a Borel set $E\subset\Gamma$ *such that* $C_\alpha(E)=0$ *and*

$$\lim_{r\to\infty} U_\alpha^\mu(r\xi) = 0 \qquad \text{if} \quad \xi\in\Gamma-E.$$

REMARK 1. Condition (1) is equivalent to $U_\alpha^\mu \not\equiv \infty$.

PROOF OF THEOREM 1. We decompose U_α^μ as U_1+U_2, where

$$U_1(x) = \int_{|x-y|\geqq|x|/2} |x-y|^{\alpha-n} d\mu(y),$$

$$U_2(x) = \int_{|x-y|<|x|/2} |x-y|^{\alpha-n} d\mu(y).$$

First we shall show that $U_1(x)\to 0$ as $|x|\to\infty$. Let $|x|=r^2$, $r>1$. If $|x-y|\geqq |x|/2$, then $|x-y|\geqq(1+|y|)/5$. If, in addition, $1+|y|\leqq r$, then $|x-y|\geqq r^2/2\geqq (r/2)(1+|y|)$. Hence

$$U_1(x) \leqq \left(\frac{r}{2}\right)^{\alpha-n} \int (1+|y|)^{\alpha-n} d\mu(y) + 5^{n-\alpha}\int_{1+|y|>r} (1+|y|)^{\alpha-n} d\mu(y),$$

which tends to zero as $r\to\infty$.

For a positive integer k, we set

$$a_k = \int_{2^{k-1}\leqq|y|<2^{k+2}} |y|^{\alpha-n} d\mu(y).$$

Since $\sum_{k=1}^\infty a_k<\infty$ by our assumption, there is a sequence $\{b_k\}$ of positive numbers such that $\lim_{k\to\infty} b_k=\infty$ and $\sum_{k=1}^\infty a_k b_k<\infty$. Set

$$E_k = \{x\in R^n;\ 2^k \leqq |x| < 2^{k+1},\ U_2(x) \geqq 1/b_k\}$$

for each positive integer k. If $x\in E_k$, then $|x-y|<|x|/2$ implies $2^{k-1}<|y|<2^{k+2}$, so that $\int_{2^{k-1}<|y|<2^{k+2}} |x-y|^{\alpha-n} d\mu(y)\geqq b_k^{-1}$. Hence we have by Lemma 1

$$C_\alpha(E_k) \leqq 2^{n-\alpha} b_k \int_{2^{k-1}<|y|<2^{k+2}} d\mu(y) \leqq 2^{n-\alpha} a_k b_k 2^{(k+2)(n-\alpha)}.$$

Denote by $\tilde{E}_k$ the set of all points $\xi\in\Gamma$ such that $r\xi\in E_k$ for some $r>0$. Then

$$C_\alpha(\tilde{E}_k) \leqq 2^{-k(n-\alpha)} C_\alpha(E_k)$$

for each positive integer k. Setting $\tilde{E}=\cap_{j=1}^\infty \cup_{k=j}^\infty \tilde{E}_k$, we see that $C_\alpha(\tilde{E})=0$ and

$\lim_{r\to\infty} U_2(r\xi)=0$ for $\xi\in\Gamma-\tilde{E}$. Thus $\tilde{E}$ is the required exceptional set.

REMARK 2. Theorem 1 is the best possible as to the size of the exceptional set; in fact, for a Borel set $E\subset\Gamma$ with $C_\alpha(E)=0$, there is a non-negative measure μ such that $U_\alpha^\mu\not\equiv\infty$ and $\limsup_{r\to\infty} U_\alpha^\mu(r\xi)=\infty$ for every $\xi\in E$. To show this fact, we set $\tilde{E}=\{j\xi;\ \xi\in E$ and j is a positive integer$\}$ and note that $C_\alpha(\tilde{E})=0$. Hence there is a non-negative measure μ such that $U_\alpha^\mu\not\equiv\infty$ but $U_\alpha^\mu(x)=\infty$ for each $x\in\tilde{E}$. Clearly, $\limsup_{r\to\infty} U_\alpha^\mu(r\xi)=\infty$ for each $\xi\in E$.

4. Radial limits of potentials of measures with density

The following two lemmas can be proved in the same manner as Lemmas 4 and 5 in [4] with slight modifications (also cf. [1; Lemma 4.3] for Lemma 2).

LEMMA 2 (*cf.* [4; *Lemma* 4]). *Let α and p be positive numbers such that $1<p\leqq 2$ and $\alpha p<n$. Let f be a non-negative function in $L^p(R^n)$ and set*

$$E=\left\{x\in R^n;\ \int|x-y|^{\alpha-n}f(y)dy\geqq 1\right\}.$$

Then there is a constant $M>0$ independent of f such that

$$C_{\alpha p}(E)\leqq M\|f\|_p^p.$$

LEMMA 3 (*cf.* [4; *Lemma* 5]). *Let α, p and ε be positive numbers such that $p>2$ and $\varepsilon<\alpha p<n$. For a positive number r and a non-negative function f in $L^p(R^n)$, we set*

$$E=\left\{x\in R^n;\ \int_{|y|<r}|x-y|^{\alpha-n}f(y)dy\geqq 1\right\}.$$

Then there is a constant $M>0$ independent of r and f such that

$$C_{\alpha p-\varepsilon}(E)\leqq Mr^{\varepsilon}\|f\|_p^p.$$

We now show

THEOREM 2. *Let α, β and p be numbers such that $\alpha>0$, $\beta\geqq 0$, $p>1$ and $\alpha p+\beta<n$. For a non-negative function f in $L^p(R^n)$, we set*

$$F(x)=\int|x-y|^{\alpha-n}|y|^{\beta/p}f(y)dy.$$

Then there is a Borel set $E\subset\Gamma$ such that

$$\lim_{r\to\infty}F(r\xi)=0 \qquad \text{for each}\quad \xi\in\Gamma-E,$$

$$C_{\alpha p}(E)=0 \qquad \text{if} \quad p \leqq 2$$

and

$$C_{\alpha p-\varepsilon}(E)=0 \quad \text{for any } \varepsilon \text{ with} \quad 0<\varepsilon<\alpha p \quad \text{if} \quad p>2.$$

Proof. We decompose F as F_1+F_2, where

$$F_1(x)=\int_{|x-y|\geqq|x|/2}|x-y|^{\alpha-n}|y|^{\beta/p}f(y)dy,$$

$$F_2(x)=\int_{|x-y|<|x|/2}|x-y|^{\alpha-n}|y|^{\beta/p}f(y)dy.$$

Since $|x-y|\geqq|x|/2$ implies $|y|\leqq 3|x-y|$, we have by Hölder's inequality

$$F_1(x)\leqq 3^{\beta/p}\int_{|x-y|\geqq|x|/2}|x-y|^{\alpha+\beta/p-n}f(y)dy$$

$$\leqq 3^{\beta/p}\left\{\int_{|x-y|\geqq|x|/2}|x-y|^{p'(\alpha+\beta/p-n)}dy\right\}^{1/p'}\|f\|_p,$$

where $1/p+1/p'=1$. Since $p'(\alpha+\beta/p-n)<-n$, this implies that $F_1(x)$ tends to zero as $|x|\to\infty$.

For a positive integer k, we set

$$E_k=\{x\in R^n;\ 2^k\leqq|x|<2^{k+1},\ F_2(x)\geqq 2^{k(\alpha p+\beta-n)/p}\}.$$

As in the proof of Theorem 1, we see that for $x\in E_k$

$$\int_{2^{k-1}<|y|<2^{k+2}}|x-y|^{\alpha-n}|y|^{\beta/p}f(y)dy\geqq 2^{k(\alpha p+\beta-n)/p}.$$

Hence we have by Lemmas 2 and 3

$$C_{\alpha p-\varepsilon}(E)\leqq M2^{\varepsilon(k+2)}2^{k(n-\alpha p-\beta)}\int_{2^{k-1}<|y|<2^{k+2}}|y|^{\beta}f(y)^p dy$$

$$\leqq M2^{2(\beta+\varepsilon)}2^{k(n-\alpha p+\varepsilon)}\int_{2^{k-1}<|y|<2^{k+2}}f(y)^p dy$$

for some constant $M>0$ independent of k and ε, where $\varepsilon=0$ if $p\leqq 2$ and $0<\varepsilon<\alpha p$ if $p>2$. Set

$$\tilde{E}_k=\{\xi\in\Gamma;\ r\xi\in E_k \quad \text{for some} \quad r>0\}.$$

Then

$$C_{\alpha p-\varepsilon}(\tilde{E}_k)\leqq 2^{-k(n-\alpha p+\varepsilon)}C_{\alpha p-\varepsilon}(E_k)$$

$$\leqq M2^{2(\beta+\varepsilon)}\int_{2^{k-1}<|y|<2^{k+2}} f(y)^p dy .$$

Consequently if we put $\tilde{E}=\cap_{j=1}^{\infty}\cup_{k=j}^{\infty}\tilde{E}_k$, then $C_{\alpha p-\varepsilon}(\tilde{E})=0$ and $\lim_{r\to\infty}F_2(r\xi)=0$ for $\xi\in\Gamma-\tilde{E}$. Thus the theorem is proved.

REMARK 3. Theorem 2 is also valid in case $p=1$ and $\alpha+\beta\leqq n$ on account of Theorem 1.

REMARK 4. Let $\alpha>0$, $p>1$ and $\alpha p<n$. Let E be a Borel set in Γ such that $C_{\alpha p}(E)=0$ if $p\geqq 2$ and $C_{\alpha p+\varepsilon}(E)=0$ for some $\varepsilon>0$ with $\alpha p+\varepsilon<n$ if $p<2$. Then there is a non-negative function $f\in L^p(R^n)$ such that $\limsup_{r\to\infty}\int|r\xi-y|^{\alpha-n}f(y)dy=\infty$ for every $\xi\in E$. To see this fact, setting $\tilde{E}=\{j\xi;\ \xi\in E$ and j is a positive integer$\}$, we note that $C_{\alpha p}(\tilde{E})=0$ if $p\geqq 2$ and $C_{\alpha p+\varepsilon}(\tilde{E})=0$ if $p<2$. In view of a result of B. Fuglede [1], there is a non-negative function f in $L^p(R^n)$ such that $\int|x-y|^{\alpha-n}f(y)dy=\infty$ for every $x\in\tilde{E}$. This shows that $\limsup_{r\to\infty}\int|r\xi-y|^{\alpha-n}|y|^{\beta/p}f(y)dy=\infty$ for any $\xi\in E$ and any number β.

5. Radial limits of locally p-precise functions

THEOREM 3. *Let β and p be numbers such that $\beta\geqq 0$, $p>1$ and $\beta+p<n$. Let u be a locally p-precise function on R^n such that*

$$\int|\mathrm{grad}\, u|^p|x|^{-\beta}dx<\infty .$$

Then there is a constant c such that $\lim_{r\to\infty}u(r\xi)=c$ except for ξ in a Borel set $E\subset\Gamma$ such that $C_p(E)=0$ if $p\leqq 2$ and $C_{p-\varepsilon}(E)=0$ for any ε with $0<\varepsilon<p$ if $p>2$.

To show Theorem 3, we shall establish the following integral representation of u.

LEMMA 4 (*cf.* [5; *Theorem* 9.11], [3; *Theorem* 4.1]). *Let β, p and u be as in Theorem 3. Then there are constants c_2 and c_2' such that*

$$u(x)=\begin{cases} c_1\sum_{j=1}^{n}\int\frac{\partial}{\partial x_j}(|x-y|^{2-n})\frac{\partial u}{\partial y_j}(y)\,dy+c_2 & (n\geqq 3), \\ c_1'\sum_{j=1}^{n}\int\frac{\partial}{\partial x_j}(\log|x-y|)\frac{\partial u}{\partial y_j}(y)\,dy+c_2' & (n=2) \end{cases}$$

holds for p-a.e. $x\in R^n$. Here c_1 and c_1' are the constants determined by $\Delta|x|^{2-n}=c_1^{-1}\delta$ if $n\geqq 3$ and $\Delta\log|x|=c_1'^{-1}\delta$ if $n=2$, where Δ is the Laplacian and δ is the Dirac measure.

PROOF. We shall prove only the case $n\geqq 3$ because the case $n=2$ is similarly proved. Put

$$G_u(x) = c_1 \sum_{j=1}^{n} \int \frac{\partial}{\partial x_j}(|x-y|^{2-n}) \frac{\partial u}{\partial y_j}(y) dy .$$

Since $p'(1-n)+\beta p'/p<-n$, we see that $\int |x-y|^{1-n}|\text{grad } u| dy \in L^1_{\text{loc}}(R^n)$. Consequently $G_u \in L^1_{\text{loc}}(R^n)$. We shall show that $\Delta(u-G_u)=0$ in the sense of distribution. Let φ be any infinitely differentiable function with compact support. Then we have by using Fubini's theorem

$$\begin{aligned}\int G_u(x)\Delta\varphi(x)dx &= c_1 \sum_{j=1}^{n} \int \frac{\partial u}{\partial y_j}\left\{\int -\frac{\partial}{\partial y_j}(|x-y|^{2-n})\Delta\varphi(x)dx\right\}dy \\ &= c_1 \sum_{j=1}^{n} \int \frac{\partial u}{\partial y_j}\left\{-\frac{\partial}{\partial y_j}\int |x-y|^{2-n}\Delta\varphi(x)dx\right\}dy \\ &= c_1 \sum_{j=1}^{n} \int \frac{\partial u}{\partial y_j}\left(-c_1^{-1}\frac{\partial \varphi}{\partial y_j}\right)dy \\ &= \int u(y)\Delta\varphi(y)dy ,\end{aligned}$$

which implies that $\Delta(u-G_u)=0$. According to Weyl's lemma, there is a harmonic function h such that

$$h(x) = u(x) - G_u(x) \tag{2}$$

holds for a.e. $x \in R^n$. If we use the following two lemmas, we see that h is constant and (2) holds for p-a.e. $x \in R^n$.

LEMMA 5 (*cf.* [5; *Lemma* 9.16]). *Let β and p be numbers such that $\beta<n$ and $p \geqq 1$. Let h be a harmonic function on R^n, and assume that $\int |\text{grad } h|^p |x|^{-\beta}dx<\infty$. Then h is constant.*

PROOF. Since $\partial h/\partial x_j$ is harmonic on R^n for each $j=1, 2,\ldots, n$, we have

$$\begin{aligned}\left|\frac{\partial h}{\partial x_j}(x)\right| &= c_n r^{-n}\left|\int_{|x|<r}\frac{\partial h}{\partial y_j}(y)dy\right| \\ &\leqq c'_n r^{(\beta-n)/p}\left\{\int |\text{grad } h|^p |y|^{-\beta}dy\right\}^{1/p} \longrightarrow 0\end{aligned}$$

as $r\to\infty$, where c_n and c'_n are constants depending only on n. Thus $\partial h/\partial x_j=0$ on R^n for each j, so that h is constant.

The proof of the following lemma will be given in the next section.

LEMMA 6. *Let β and p be as in Theorem 3. For an integer j, $1 \leqq j \leqq n$, and a function $f \in L^p(R^n)$, we set*

$$F(x) = \int \frac{x_j - y_j}{|x-y|^n} |y|^{\beta/p} f(y) dy .$$

Then F is locally p-precise on R^n and

$$\int |\mathrm{grad}\, F|^p |x|^{-\beta} dx \leqq M \|f\|_p^p ,$$

where M is a constant independent of f.

PROOF OF THEOREM 3. By Lemma 4, there are constants c_1, c_2 and a p-exceptional set $E_1 \subset R^n$ such that for $x \in R^n - E_1$

$$u(x) = c_1 \sum_{j=1}^{n} \int \frac{x_j - y_j}{|x-y|^n} \frac{\partial u}{\partial y_j}(y) dy + c_2 .$$

According to [1], $C_p(E_1)=0$ if $p \leqq 2$ and $C_{p-\varepsilon}(E_1)=0$ for any ε with $0<\varepsilon<p$ if $p>2$. Set

$$\tilde{E}_1 = \{\xi \in \Gamma;\ r\xi \in E_1 \quad \text{for some} \quad r \geqq 1\} .$$

By Theorem 2 there is a Borel set $E_2 \subset \Gamma$ such that

$$\lim_{r \to \infty} \int |r\xi - y|^{1-n} |\mathrm{grad}\, u| dy = 0 \qquad \text{if} \quad \xi \in \Gamma - E_2$$

and

$$C_p(E_2) = 0 \qquad \text{if} \quad p \leqq 2,$$

$$C_{p-\varepsilon}(E_2) = 0 \qquad \text{for any } \varepsilon \text{ with } 0 < \varepsilon < p \quad \text{if} \quad p > 2.$$

It is easy to check that $\tilde{E}_1 \cup E_2$ is the required exceptional set.

COROLLARY. *Let $0<\alpha<n$ and $1<p<n$. Let μ be a non-negative measuresuch that $\int U_\alpha^\mu d\mu < \infty$. Assume that U_α^μ is locally p-precise on R^n and $\int |\mathrm{grad}\, U_\alpha^\mu|^p |x|^{-\beta} dx < \infty$ for some non-negative number β with $\beta < n-p$. Then there is a Borel set $E \subset \Gamma$ such that*

$$\lim_{r \to \infty} U_\alpha^\mu(r\xi) = 0 \qquad \textit{for} \quad \xi \in \Gamma - E$$

and

$$C_p(E) = 0 \qquad \textit{if} \quad p \leqq 2,$$

$$C_{p-\varepsilon}(E) = 0 \qquad \textit{for any } \varepsilon \textit{ with} \quad 0 < \varepsilon < p \quad \textit{if} \quad p > 2 .$$

This is an easy consequence of Theorem 3 and the following lemma.

LEMMA 7 (*cf.* [5; *Theorem* 10.18]). *Let* $0<\alpha<n$ *and let* μ *be a non-negative measure such that* $\int U_\alpha^\mu d\mu<\infty$. *Assume that* $\lim_{r\to\infty} U_\alpha^\mu(r\xi)$ *is a constant* c *for a.e.* $\xi\in\Gamma$. *Then* $c=0$.

6. Proof of Lemma 6

We may suppose that f is non-negative on R^n. Noting that $(1+|y|)^{1-n}|y|^{\beta/p}\in L^{p'}(R^n)$, $p'=p/(p-1)$, we have

$$(3) \qquad \int(1+|y|)^{1-n}|y|^{\beta/p}f(y)dy < \infty .$$

We set $\kappa_\varepsilon(x)=x_j(|x|^2+\varepsilon^2)^{-n/2}$, $\varepsilon>0$, and define

$$F_\varepsilon(x) = \int\kappa_\varepsilon(x-y)\,|y|^{\beta/p}f(y)dy ,$$

$$G_\varepsilon(x) = \int\kappa_\varepsilon(x-y)f(y)dy .$$

From (3) we see that $F_\varepsilon\in C^\infty(R^n)$ and

$$\frac{\partial F_\varepsilon}{\partial x_i}(x) = \int\frac{\partial\kappa_i}{\partial x_i}(x-y)\,|y\,|^{\beta/p}f\,(y)dy$$

for any $i=1, 2,\dots, n$. From the proof of [3; Lemma 3.2] we derive that

$$(4) \qquad \|D_iG_\varepsilon\|_p \leqq M_1\|f\|_p,$$

where $D_i=\partial/\partial x_i$, $i=1,\dots, n$, and M_1 is a constant independent of ε and f. On the other hand

$$(5) \qquad |\,|x|^{-\beta/p}\,D_iF_\varepsilon-D_iG_\varepsilon| \leqq M_2\int\frac{|1-(|y|/|x|)^{\beta/p}|}{|x-y|^n}f(y)\,dy .$$

We write $x=R\xi$ and $y=r\eta$, where $R=|x|$ and $r=|y|$. Setting $H(x)=\int|1-(|y|/|x|)^{\beta/p}|\,|x-y|^{-n}f(y)dy$, we have by Hölder's inequality

$$H(x) \leqq \int_0^\infty |1-(r/R)^{\beta/p}|\,r^{n-1}\left\{\int_{|\eta|=1}\frac{dS(\eta)}{|R\xi-r\eta|^n}\right\}^{1/p'}$$

$$\times\left\{\int_{|\eta|=1}\frac{f(r\eta)^p}{|R\xi-r\eta|^n}dS(\eta)\right\}^{1/p}dr .$$

For simplicity, we set

(6) $$I(R, r) = \int_{|\eta|=1} \frac{dS(\eta)}{|R\xi - r\eta|^n} .$$

This is independent of $\xi \in \Gamma$ and

(7) $$I(R, r) = \sigma_n |R^2 - r^2|^{-1} \{\max(R, r)\}^{2-n},$$

where σ_n is the area of Γ. By (6),

$$H(x) \leqq \int_0^\infty |1-(r/R)^{\beta/p}| r^{n-1} I(R, r)^{1/p'} \left\{ \int_{|\eta|=1} \frac{f(r\eta)^p}{|R\xi - r\eta|^n} dS(\eta) \right\}^{1/p} dr .$$

Using Minkowski's inequality ([6; Appendix A.1]), we have

$$\left\{ \int_{|\xi|=1} H(r\xi)^p dS(\xi) \right\}^{1/p} \leqq \int_0^\infty |1-(r/R)^{\beta/p}| r^{n-1} I(R, r)^{1/p'}$$

$$\times \left[\int_{|\eta|=1} f(r\eta)^p \left\{ \int_{|\xi|=1} \frac{dS(\xi)}{|R\xi - r\eta|^n} \right\} dS(\eta) \right]^{1/p} dr$$

$$\leqq R^{(1-n)/p} \int_0^\infty K(R, r) g(r) dr ,$$

where

$$g(r) = r^{(n-1)/p} \left\{ \int_{|\eta|=1} f(r\eta)^p dS(\eta) \right\}^{1/p},$$

$$K(R, r) = R^{(n-1)/p} r^{(n-1)/p'} I(R, r) |1-(r/R)^{\beta/p}| .$$

Note that $K(R, r)$ is homogeneous of degree -1, that is, $K(\lambda R, \lambda r) = \lambda^{-1} K(R, r)$ for $\lambda > 0$ and that $\int_0^\infty K(1, r) r^{-1/p} dr < \infty$ on account of (7). Hence we can apply Appendix A.3 in [6] and obtain

$$\int H(x)^p dx = \int_0^\infty \left\{ \int_{|\xi|=1} H(R\xi)^p dS(\xi) \right\} R^{n-1} dR$$

$$\leqq \int_0^\infty \left\{ \int_0^\infty K(R, r) g(r) dr \right\}^p dR$$

$$\leqq M_3 \int_0^\infty g(r)^p dr = M_3 \|f\|_p^p ,$$

where M_3 is a constant independent of f. Therefore (4) and (5) give

(8) $$\| |x|^{-\beta/p} D_i F_\varepsilon \|_p \leqq M_2 M_3{}^{1/p} \|f\|_p + \|D_i G_\varepsilon\|_p \leqq M \|f\|_p,$$

where $M = M_1 + M_2 M_3{}^{1/p}$.

Let $N > 0$. We write $F = F_{1,N} + F_{2,N}$, where

$$F_{1,N}(x) = \int_{|y| \leqq 2N} \frac{x_j - y_j}{|x-y|^n} |y|^{\beta/p} f(y) dy ,$$

$$F_{2,N}(x) = \int_{|y| > 2N} \frac{x_j - y_j}{|x-y|^n} |y|^{\beta/p} f(y) dy .$$

From [3; Lemma 3.3] it follows that $F_{1,N}$ is locally p-precise on R^n and for any i

$$D_i \int_{|y| \leqq 2N} \kappa_\varepsilon(x-y) |y|^{\beta/p} f(y) dy \longrightarrow D_i F_{1,N} \quad \text{in} \quad L^p(R^n) \quad \text{as} \quad \varepsilon \longrightarrow 0 .$$

Furthermore $F_{2,N}$ is continuously differentiable on $\{x \in R^n; |x| < N\}$ and $D_i \int_{|y| > 2N} (x_j - y_j) |x-y|^{-n} |y|^{\beta/p} f(y) dy$ converges to $D_i F_{2,N}$ uniformly on $\{x \in R^n; |x| < N\}$ as $\varepsilon \to 0$ for any i. Thus F is locally p-precise on R^n and

$$\| |x|^{-\beta/p} D_i F \|_p \leqq M \|f\|_p, \qquad i = 1, 2, \dots, n,$$

by (8) and Fatou's lemma. These complete the proof of Lemma 6.

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