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On the Existence of Non-tangential Limits of Harmonic Functions

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1. Introduction and statement of results

In this paper, we let R^n be the n-dimensional Euclidean space ($n \geqq 2$). We use the notation:

$$x = (x', x_n) \in R^{n-1} \times R^1,$$

$$R_+^n = \{x = (x', x_n) \in R^n;\ x_n > 0\},$$

$$R_0^n = \{x = (x', x_n) \in R^n;\ x_n = 0\}.$$

For a positive number a and a point $\xi \in R_0^n$, we set

$$\Gamma(\xi; a) = \{x = (x', x_n) \in R_+^n;\ |(x', 0) - \xi| < a x_n\}.$$

Let u be a function on R_+^n. We say that u has a non-tangential limit at $\xi \in R_0^n$ if

$$\lim_{\Gamma(\xi; a) \ni x \to \xi} u(x)$$

exists and is finite for any positive number a. Our aim is to show

THEOREM 1. *Let $1 < p < \infty$ and $-\infty < \alpha < p$. If u is a harmonic function on R_+^n satisfying*

(1) $$\iint_\Omega |\mathrm{grad}\, u|^p x_n^\alpha dx' dx_n < \infty \quad \textit{for any bounded open set} \quad \Omega \subset R_+^n,$$

then there is a Borel set $E \subset R_0^n$ such that $B_{1-\alpha/p, p}(E) = 0$ and u has a non-tangential limit at each $\xi \in R_0^n - E$.

Here $B_{1-\alpha/p,p}(E)$ denotes the Bessel capacity of E of index $(1-\alpha/p,\ p)$ (cf. [1]). By [3; Theorem A], [4; Theorems 2.4, 3.2 and Proposition 3.1] and our theorem, we have

COROLLARY. *Let α, p and u be as in Theorem 1. Then u has a non-tangential limit at each $\xi \in R_0^n - E$, where E is a Borel set in R_0^n such that*

$$C_{p-\alpha}(E) = 0 \qquad \textit{if} \quad p \leqq 2 \quad \textit{and} \quad p - \alpha \leqq n,$$

$$C_{p-\alpha-\varepsilon}(E) = 0 \qquad \textit{for any } \varepsilon \textit{ with } \quad 0 < \varepsilon < p-\alpha$$

$$\textit{if} \quad 2 < p \leqq n+\alpha$$

and

$$E \textit{ is empty} \qquad \textit{if} \quad p-\alpha > n.$$

Here $C_\beta(E)$ is the Riesz capacity of E of order β. This corollary is a generalization of a result of H. Wallin [7; Theorem 3] and a result of T. Murai [5; Theorem 2]. We note that A. A. Bagarshakyan [2] evaluated the size of the exceptional set in our problem by means of a capacity of different type.

In case $-1<\alpha<p-1$, Theorem 1 is the best possible as to the size of the exceptional set in the following sense:

THEOREM 2. *Let* $1<p<\infty$ *and* $-1<\alpha<p-1$. *Let* E *be a set in* R_0^n *with* $B_{1-\alpha/p,p}(E) = 0$. *Then there is a harmonic function* u *on* R_+^n *such that* $\int_{R_+^n} |\mathrm{grad}\, u|^p x_n^\alpha dx < \infty$ *and* $\lim_{R_+^n \ni x \to \xi} u(x) = \infty$ *for every* $\xi \in E$.

2. Proof of Theorem 1

Let α, p and u be as in Theorem 1. Given $M>0$, let us consider the existence of non-tangential limits of u at points in $B_M = \{\xi \in R_0^n;\ |\xi| < M\}$. Set

$$f(x) = \begin{cases} x_n^{\alpha/p} |(\mathrm{grad}\, u)(x)|, & \text{if} \quad x = (x', x_n) \in R_+^n \quad \text{and} \quad |x| < 2M, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in L^p(R^n)$ by our assumption. We denote by g_β, $0<\beta<\infty$, the Bessel kernel of order β, which has the following property (cf. [1; p. 878]): There is a constant $c_1 > 0$ such that for all $x \in R^n$ with $|x| < 2M$

(i) $$g_\beta(x) \geqq c_1 |x|^{\beta-n} \qquad \text{if} \quad 0 < \beta < n,$$

(ii) $$g_\beta(x) \geqq c_1 \qquad \text{if} \quad \beta \geqq n.$$

Setting

$$E = \{x \in R^n; \int g_{1-\alpha/p}(x-y) f(y) dy = \infty\},$$

we see that $B_{1-\alpha/p,p}(E) = 0$. Let $\xi \in B_M - E$ be fixed. In the case where $1-\alpha/p < n$, we have

$$\infty > \int_{\Gamma(\xi;a)} g_{1-\alpha/p}(\xi - y) f(y) dy$$

$$\geqq c_1 \int_{S(a)} \int_0^\infty r^{-\alpha/p} f(\xi + r\sigma) dr dS(\sigma)$$

$$\geqq \min\{1, (a+1)^{-\alpha/p}\} c_1 \int_{S(a)} \int_0^{\varepsilon_0} |(\operatorname{grad} u)(\xi + r\sigma)| dr dS(\sigma),$$

where $r=|y-\xi|$, $\sigma=(y-\xi)/r$, $S(a)=\{x \in \Gamma(O;\, a);\, |x|=1\}$, dS is the surface element and $\varepsilon_0>0$ is chosen so that $|\xi+r\sigma|<2M$ whenever $0<r<\varepsilon_0$ and $\sigma \in S(a)$. Hence there is $\sigma^* \in S(a)$ such that

$$A_{\sigma^*} = \int_0^{\varepsilon_0} |(\operatorname{grad} u)(\xi + r\sigma^*)| dr < \infty .$$

Since $\int_0^{\varepsilon_0} |\partial u(\xi + r\sigma^*)/\partial r| dr \leqq A_{\sigma^*}$, $\lim_{r \downarrow 0} u(\xi + r\sigma^*)$ exists and is finite. For $x=(x', x_n) \in \Gamma(\xi;\, a)$, we denote by x_{σ^*} the point on $\{\xi + r\sigma^*;\, r>0\}$ whose n-th coordinate is x_n, and by L_x the line segment between x and x_{σ^*}. Since $\partial u/\partial x_j$, $j=1, 2,\dots, n$, are harmonic on R_+^n,

$$\frac{\partial u}{\partial x_j}(x) = c_2 x_n^{-n} \int_{|x-y|<x_n/2} \frac{\partial u}{\partial y_j}(y) dy, \qquad j=1, 2,\dots, n, \tag{2}$$

with a constant $c_2>0$ independent of $x \in \Gamma(\xi;\, a)$. Noting that $y_n \leqq |\xi - y| < (a+3/2)x_n < (2a+3)y_n$ whenever $x \in \Gamma(\xi;\, a)$ and $|x-y|<x_n/2$, we obtain from (2) that for $x \in \Gamma(\xi;\, a)$ sufficiently close to ξ

$$\left|\frac{\partial u}{\partial x_j}(x)\right| \leqq c_2 x_n^{-n} \int_{|x-y|<x_n/2} |\operatorname{grad} u| dy$$

$$\leqq c_3 x_n^{-1} \int_{|\xi-y|<(a+3/2)x_n} |\xi - y|^{1-\alpha/p-n} f(y) dy$$

$$\leqq c_4 x_n^{-1} \int_{|\xi-y|<(a+3/2)x_n} g_{1-\alpha/p}(\xi - y) f(y) dy,$$

where $c_3 = c_2(a+3/2)^{n-1} \max\{1, (2a+3)^{\alpha/p}\}$ and $c_4 = c_1^{-1} c_3$. Consequently,

$$|u(x) - u(x_{\sigma^*})| \leqq |x - x_{\sigma^*}| \sup_{L_x} |\operatorname{grad} u|$$

$$\leqq 2a\sqrt{n}\, c_4 \int_{|\xi-y|<(a+3/2)x_n} g_{1-\alpha/p}(\xi - y) f(y) dy,$$

which tends to zero as $x_n \downarrow 0$. Therefore $\lim_{\Gamma(\xi;a) \ni x \to \xi} u(x)$ exists and is finite. We can show the case $1-\alpha/p \geqq n$ in the same way as above by using (ii) instead of (i). Since M is arbitrary, we obtain the theorem.

3. Proof of Theorem 2

By our assumption that $B_{1-\alpha/p,p}(E)=0$, there is a non-negative function $f\in L^p(R^n)$ such that $\int g_{1-\alpha/p}(\xi-y)f(y)dy=\infty$ for every $\xi\in E$. We denote by F the restriction of $\int g_{1-\alpha/p}(x-y)f(y)dy$ to R^{n-1}, i.e.,

$$F(x')=\int g_{1-\alpha/p}((x',0)-y)f(y)dy, \qquad x'\in R^{n-1}.$$

We note that F belongs to the Lipschitz space $\Lambda_{1-(\alpha+1)/p}^{p,p}(R^{n-1})$ (cf. [6; Chap. 6, §4.3]). Let u be the Poisson integral of F with respect to R_+^n. In view of [6; Chap. 5, Proposition 7′, Lemma 4′][1],

$$\int_{R_+^n}|\operatorname{grad} u|^p x_n^\alpha dx<\infty.$$

Moreover we see from a property of the Poisson integral and the lower semicontinuity of F that $\lim_{R_+^n\ni x\to\xi}u(x)=\infty$ for every $\xi\in E$. Thus u satisfies all the conditions in the theorem.

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1) In the inequalities (61) and (62), $\alpha-1$ should be replaced by $1-\alpha$.