HIROSHIMA MATH. J.
7 (1977), 153–160

On the Existence of Boundary Values of p-Precise Functions

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(Received September 6, 1976)

1. Introduction and statement of results

Let R^n be the n-dimensional Euclidean space ($n \geqq 2$). We use the notation:

$$x = (x', x_n) \in R^{n-1} \times R^1,$$

$$R^n_+ = \{x = (x', x_n) \in R^n;\ x_n > 0\},$$

$$R^n_0 = \{x = (x', x_n) \in R^n;\ x_n = 0\}.$$

Throughout this paper let $1 < p < \infty$. We say that a function u is locally p-precise on an open set $G \subset R^n$ if u is p-precise on any relatively compact open subset of G; for p-precise functions, see [10]. For a real number α, we consider a locally p-precise function u on R^n_+ such that

$$\iint_{R^n_+} |\mathrm{grad}\, u|^p x_n^\alpha dx' dx_n < \infty . \tag{1}$$

In case $\alpha \geqq 0$ and $1+\alpha < p < n+\alpha$, we have already discussed the existence of $\lim u(x', x_n)$ as $x_n \downarrow 0$ ([6]). In the present paper, we shall discuss it in more general cases. We denote by C_ℓ $(0 < \ell < n)$ the Riesz capacity of order ℓ (which refers to the kernel $|x|^{\ell-n}$), by C_n the logarithmic capacity and by $B_{\ell,p}$ $(0 < \ell < \infty)$ the Bessel capacity of index (ℓ, p) (cf. [4]).

First we state

THEOREM 1. *Let u be a locally p-precise function on R^n_+ satisfying* (1). *Then there is a Borel set $E \subset R^n_0$ such that*

$B_{1,p}(E) = 0$ *if* $\alpha \leqq 0$,

$C_{p-\alpha}(E) = 0$ *if* $\alpha > 0$ *and* $1+\alpha < p \leqq 2$,

$C_{p-\alpha-\varepsilon}(E) = 0$ *for any* ε *with* $0 < \varepsilon < p-\alpha$
if $\alpha > 0$, $p > 2$ *and* $1+\alpha < p \leqq n+\alpha$,

E is empty *if* $0 < \alpha < p-n$

and $\lim_{x_n \downarrow 0} u(x', x_n)$ *exists and is finite for every* $(x', 0) \in R_0^n - E$.

Theorem 1 is good as to the size of the exceptional set in the following sense.

THEOREM 2. *Let E be a set in* R_0^n *such that*

$$B_{1,p}(E) = 0 \text{ and } E \text{ is compact} \quad \text{if} \quad \alpha \leqq 0,$$

$$B_{1-\alpha/p,p}(E) = 0 \quad \text{if} \quad \alpha > 0 \text{ and } 1+\alpha < p \leqq n+\alpha.$$

Then there is a C^∞*-function u on* R_+^n *satisfying* (1) *such that* $\overline{\lim}_{x_n \downarrow 0} u(x', x_n) = \infty$ *for any* $(x', 0) \in E$.

REMARK 1. Theorems 1 and 2 do not deal with the case $p \leqq 1+\alpha$. In this case there is a C^∞-function u on R_+^n satisfying (1) such that $\lim_{R_+^n \ni y \to (x',0)} u(y) = \infty$ for every $x' \in R^{n-1}$. For example, the function
$u(x) = u(x', x_n) = \exp(-|x|^2)\{(\log x_n)^2 + 1\}^{\varepsilon/2}, \quad 0 < \varepsilon < 1 - 1/p,$
has the required properties.

In case $n=2$, we are concerned with oblique limits. For a function u on R_+^2, $\xi \in R_0^2$ and $0<\theta<\pi$, we set

$$u(\xi, \theta) = \lim_{r \downarrow 0} u(\xi + (r\cos\theta, r\sin\theta))$$

if the limit exists.

THEOREM 3. *Let u be a locally p-precise function on* R_+^2 *satisfying* (1). *Then there is a Borel set* $E \subset R_0^2$ *such that*

$$B_{1-\alpha/p,p}(E) = 0 \quad \text{if} \quad \alpha < p \leqq 2+\alpha,$$

$$E \text{ is empty} \quad \text{if} \quad 0 \leqq \alpha < p-2,$$

$$E \text{ is at most countable} \quad \text{if} \quad \alpha < 0 \quad \text{and} \quad \alpha < p-2$$

and for each $\xi \in R_0^2 - E$ *there is a constant* c_ξ *satisfying that*

$$u(\xi, \theta) = c_\xi \quad \text{for} \quad a.e.\ \theta \in (0, \pi).$$

In view of [2; Theorem A] and [5; Theorems 2.4 and 3.2], Theorem 3 implies

COROLLARY 1. *Let u be as in Theorem 3. Then there is a Borel set* $E \subset R_0^2$ *such that*

$$C_{p-\alpha}(E) = 0 \quad \text{if} \quad p \leqq 2 \quad \text{and} \quad \alpha < p \leqq 2+\alpha,$$

$$C_{p-\alpha-\varepsilon}(E) = 0 \quad \text{for any } \varepsilon \text{ with } 0 < \varepsilon < p-\alpha$$

$$\text{if} \quad p > 2 \quad \text{and} \quad \alpha < p \leqq 2+\alpha,$$

$$E \text{ is empty} \qquad \text{if} \quad 0 \leqq \alpha < p-2,$$

$$E \text{ is at most countable} \qquad \text{if} \quad \alpha < 0 \quad \text{and} \quad \alpha < p-2$$

and for each $\xi \in R_0^2 - E$ *there is a constant* c_ξ *satisfying that*

$$u(\xi, \theta) = c_\xi \qquad \text{for} \quad \text{a.e. } \theta \in (0, \pi).$$

REMARK 2. For $0 < \ell \leqq 1$, $C_\ell(R_0^2) = 0$, so that in case $p \leqq \alpha + 1$, E may be the whole of R_0^2 (cf. Remark 1).

Combining Corollary 1 with Theorem 1, we have

COROLLARY 2. *Suppose* $\alpha \geqq 0$ *and* $1+\alpha < p \leqq 2+\alpha$. *Let* u *be a locally p-precise function on* R_+^2 *satisfying* (1). *Then there is a Borel set* $E \subset R_0^2$ *such that*

$$C_{p-\alpha}(E) = 0 \qquad \text{if} \quad p \leqq 2,$$

$$C_{p-\alpha-\varepsilon}(E) = 0 \qquad \text{for any } \varepsilon \text{ with } 0 < \varepsilon < p-\alpha$$

$$\text{if} \quad p > 2$$

and $u(\xi, \pi/2)$ *is finite and* $u(\xi, \theta) = u(\xi, \pi/2)$ *for a.e.* $\theta \in (0, \pi)$ *if* $\xi \in R_0^2 - E$.

In case $p=2$, Corollary 2 gives the result corresponding to [3; Theorem 1].

2. Proof of Theorem 1

In case $\alpha > 0$ and $1+\alpha < p < n+\alpha$, Theorem 1 has already been shown in [6; Theorem 1]. If $\alpha \leqq 0$, then

$$\int_G |\mathrm{grad}\, u|^p dx < \infty$$

for any bounded open set $G \subset R_+^n$. Hence Theorem 1 is a consequence of [10; Theorem 4.4] or [9; Theorem 2]. In case $\alpha > 0$ and $p - \alpha = n$, by using the following lemmas instead of [6; Lemmas 2 and 3], we can show Theorem 1 in the same way as [6; Theorem 1].

LEMMA 1. *Let* $n=2$ *and* $0 \leqq \gamma < 1$. *Then*

$$\int_{|x-y| \geqq \eta,\ |y| \leqq 2a} |x-y|^{\gamma-2} |y_2|^{-\gamma} dy \leqq M \log(2a/\eta)$$

whenever $0 < \eta < a$ *and* $|x| < a$, *where* M *is a constant independent of* x, η *and* a.

PROOF. For simplicity, we denote by $I(x)$ the left-hand side of the above inequality. We note first that $I(x) \leqq I((0, x_2)) = I((0, |x_2|))$, where $x = (x_1, x_2)$. Hence we may assume that $x = (0, x_2)$, $x_2 > 0$. We divide the domain of integration into three parts, that is,

$$D_1 = \{y = (y_1, y_2);\ \eta < |y| \leqq 2a,\ |x-y| \geqq \eta,\ y_2 \geqq x_2/2\},$$

$$D_2 = \{y = (y_1, y_2);\ \eta < |y| \leqq 2a,\ |x-y| \geqq \eta,\ y_2 < x_2/2\},$$

$$D_3 = \{y = (y_1, y_2);\ |y| \leqq \eta,\ |x-y| \geqq \eta\}.$$

Then we note that

$$\int_{D_1} \{|x-y|^{\gamma-2} - |y|^{\gamma-2}\}|y_2|^{-\gamma} dy \leqq \int_{D_2} \{|y|^{\gamma-2} - |x-y|^{\gamma-2}\}|y_2|^{-\gamma} dy.$$

Hence

$$\begin{aligned} I(x) &= \int_{D_1 \cup D_2} |x-y|^{\gamma-2}|y_2|^{-\gamma} dy + \int_{D_3} |x-y|^{\gamma-2}|y_2|^{-\gamma} dy \\ &\leqq \int_{D_1 \cup D_2} |y|^{\gamma-2}|y_2|^{-\gamma} dy + \eta^{\gamma-2}\int_{D_3} |y_2|^{-\gamma} dy \\ &\leqq \int_{\eta < |y| \leqq 2a} |y|^{\gamma-2}|y_2|^{-\gamma} dy + \eta^{\gamma-2}\int_{|y| \leqq \eta} |y_2|^{-\gamma} dy \\ &\leqq M_1(\log(2a/\eta) + 1) \\ &\leqq M_2 \log(2a/\eta) \end{aligned}$$

for some constants M_1 and M_2 independent of a and η with $0 < \eta < a$.

LEMMA 2. *Let $n = 2$ and $0 \leqq \gamma < 1$. Then*

$$\int_{|y| \leqq 2a} |x-y|^{\gamma/2-1}|z-y|^{\gamma/2-1}|y_2|^{-\gamma} dy \leqq M \log(4a/|x-z|)$$

for $|x| < a$ and $|z| < a$, where M is a constant independent of a, x and z.

PROOF. Set $\eta = |x-z|/2$. Then $0 \leqq \eta < a$. We divide the domain of integration into four parts, that is, (i) $|x-y| \leqq \eta$, $|y| \leqq 2a$, (ii) $|z-y| \leqq \eta$, $|y| \leqq 2a$, (iii) $|x-y| > \eta$, $|x-y| \leqq |z-y|$, $|y| \leqq 2a$ and (iv) $|z-y| > \eta$, $|x-y| > |z-y|$, $|y| \leqq 2a$. The corresponding integrals are denoted by I_1, I_2, I_3 and I_4, respectively. Since $|x-y| \leqq \eta$ implies $|z-y| \geqq \eta$,

$$I_1 \leqq \eta^{\gamma/2-1}\int_{|x-y| \leqq \eta} |x-y|^{\gamma/2-1}|y_2|^{-\gamma} dy.$$

From [6; Lemma 1] it follows that

$$I_1 \leqq M_1 \eta^{\gamma/2-1} \eta^{-\gamma/2+1} = M_1,$$

where M_1 is a constant independent of x and η. Similarly $I_2 \leqq M_1$. For I_3, we have

$$I_3 \leqq \int_{|x-y| \geqq \eta,\ |y| \leqq 2a} |x-y|^{\gamma-2} |y_2|^{-\gamma} dy .$$

By Lemma 1, there is a constant $M_2 > 0$ independent of a, η and x such that

$$I_3 \leqq M_2 \log(2a/\eta) .$$

Similarly $I_4 \leqq M_2 \log(2a/\eta)$. Thus we have

$$\begin{aligned} \int_{|y| \leqq 2a} |x-y|^{\gamma/2-1} |z-y|^{\gamma/2-1} |y_2|^{-\gamma} dy &\leqq 2(M_1 + M_2 \log(2a/\eta)) \\ &\leqq M \log(4a/|x-z|) \end{aligned}$$

for some constant $M > 0$ independent of a, x and z.

In case $0 \leqq \alpha < p-n$, we have the following proposition.

PROPOSITION 1. *Suppose* $0 \leqq \alpha < p-n$. *Then any locally p-precise function u on* R_+^n *satisfying* (1) *is continuous on* R_+^n *and has a continuous extension to the whole space.*

PROOF. First, we note that $p > n$. Since all locally p-precise functions on R_+^n are continuous if $p > n$, u is continuous on R_+^n. If we show that for any $\varphi \in C_0^\infty(R^n)$, φu in R_+^n has a continuous extension to the whole space, then we see that the function $\tilde{u}$ defined as follows is a continuous extension of u to the whole space:

$$\tilde{u}(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n > 0, \\ \lim_{R_+^n \ni (y', y_n) \to (x', 0)} u(y', y_n) & \text{if } x_n = 0, \\ u(x', -x_n) & \text{if } x_n < 0 . \end{cases}$$

Choose a number r such that $1 < r < p/(\alpha+1)$. Then by (1) and Hölder's inequality we see that for any bounded open set G in R_+^n

$$\int_G |\text{grad}\, u|^r dx < \infty .$$

Hence by [8; Theorem 5.6] there exists an extension $\bar{u}$ of u to the whole space so that $\bar{u}$ is locally r-precise on R^n. Noting that $\varphi \bar{u}$ is r-precise, we have the following integral representation of $\varphi \bar{u}$ by virtue of [5; Theorem 3.1]:

(2) $$\varphi\bar{u}(x)=\sum_{i=1}^{n}\int\frac{x_i-y_i}{|x-y|^n}|y_n|^{-\alpha/p}f_i(y)dy \quad \text{for a.e. } x\in R^n,$$

where f_i is a function in L^p with compact support. Since $|x_i-y_i|\,|x-y|^{-n}|y_n|^{-\alpha/p}\in L^{p'}_{loc}$ and

$$\int_{|x-y|<\delta}|x-y|^{p'(1-n)}|y_n|^{-\alpha p'/p}dy \leqq \int_{|y|<\delta}|y|^{p'(1-n)}|y_n|^{-\alpha p'/p}dy \longrightarrow 0$$

as $\delta\downarrow 0$, $1/p+1/p'=1$, the right-hand side of (2) is continuous on R^n, and hence it is a continuous extension of φu in R^n_+ to the whole space. Thus the proposition is proved.

3. Proof of Theorem 2

Let $\alpha\leqq 0$ and E be a compact set in R^n_0 with $B_{1,p}(E)=0$. Choose a sequence $\{r_j\}$ of positive numbers such that $r_j\downarrow 0$ as $j\to\infty$ and set

$$E_j=\{x+(0,r_j);\ x\in E\}$$

for each j. Then $B_{1,p}(E_j)=0$. Using [5; Theorem 2.4], for each j we choose a function $u_j\in C_0^\infty(R^n)$ such that $u_j\geqq j$ on E_j, the support of u_j is contained in $\{x=(x',x_n)\in R^n;\ (r_j+r_{j+1})/2<x_n<(r_{j-1}+r_j)/2\}$ and $\int|\text{grad } u_j|^p|x_n|^\alpha dx<2^{-j}$. Then $u=\sum_{j=1}^\infty u_j$ has the required properties. The remaining part of Theorem 2 follows from [7; Theorem 2] (see also [6; Theorem 2′]).

4. Proof of Theorem 3

Let $\alpha<p$ and u be a locally p-precise function on R^2_+ satisfying (1). Set

$$f(x_1,x_2)=\begin{cases} x_2^{\alpha/p}|\text{grad } u(x_1,x_2)| & \text{if } x_2>0,\\ 0 & \text{otherwise.}\end{cases}$$

Then $f\in L^p(R^2)$ by (1). We denote by g_ℓ the Bessel kernel of order ℓ and consider the set

$$F=\{x\in R^2;\ \int g_{1-\alpha/p}(x-y)f(y)dy=\infty\}.$$

Then $B_{1-\alpha/p,p}(F)=0$. Let $\xi\in R^2_0-F$. We note that the Bessel kernels have the following properties (cf. [4; p. 279]):

$$g_\ell(x)\geqq\begin{cases} c|x|^{\ell-2} & \text{for } |x|<1 \text{ if } 0<\ell<2,\\ c & \text{for } |x|<1 \text{ if } \ell\geqq 2,\end{cases}$$

where c is a positive constant. Consequently,

$$\infty > \int_{|\xi-x|<1, x_2>0} g_{1-\alpha/p}(\xi-x)f(x)dx$$

$$\geqq c\int_0^\pi (\sin\theta)^{\alpha/p}d\theta\int_0^1 |(\mathrm{grad}\, u)(\xi+(r\cos\theta,\ r\sin\theta))|dr\,,$$

so that

$$\int_0^1 \left|\frac{\partial u}{\partial r}(\xi+(r\cos\theta,\ r\sin\theta))\right| dr$$

$$\leqq \int_0^1 |(\mathrm{grad}\, u)(\xi+(r\cos\theta,\ r\sin\theta))|dr < \infty$$

for *a.e.* $\theta\in(0, \pi)$. On the other hand, $u(\xi+(r\cos\theta,\ r\sin\theta))$ is an absolutely continuous function of $r\in(0, \infty)$ for *a.e.* $\theta\in(0, \pi)$. Thus $u(\xi, \theta)$ exists and is finite for *a.e.* $\theta\in(0, \pi)$. We put

$$F' = \left\{ \xi\in R_0^2-F;\ \begin{array}{l} u(\xi, \theta_1) \text{ and } u(\xi, \theta_2) \text{ are finite and} \\ \text{distinct for some } \theta_1 \text{ and } \theta_2\in(0, \pi) \end{array} \right\}.$$

By Bagemihl's theorem, F' is at most countable (see [1; Chap. 4]). Set $E=F'\cup(F\cap R_0^2)$. In case $\alpha<p\leqq 2+\alpha$, any single point x of R^2 has $B_{1-\alpha/p,p}(\{x\})=0$, so that $B_{1-\alpha/p,p}(E)=0$. If $2+\alpha<p$, then F is empty. If, in addition, $\alpha\geqq 0$, then F' is empty by Proposition 1. Thus E has the required properties.

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