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A Remark on Parabolic Index of Infinite Networks

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In the preceding paper [1], M. Yamasaki introduced the notion of parabolic index of infinite networks. He proposed (orally) a problem to determine the parabolic index of the infinite network N_d formed by the lattice points and the segments parallel to coordinate axes in the d-dimensional euclidean space. The purpose of the present paper is to show that the parabolic index of N_d is equal to the dimension d. This is a discrete analogue of the well-known fact that

$$\inf\int_{\mathbf{R}^d}|\mathrm{grad} f|^p dx = 0$$

if and only if $p \geqq d$, where the infimum is taken over all C^1-functions f on $\mathbf{R}^d$ with compact support such that $f \geqq 1$ on a fixed ball in $\mathbf{R}^d$.

For notation and terminologies, we mainly follow [1].

1. Description of the network

Let $\mathbf{R}^d$ be the d-dimensional euclidean space ($d \geqq 1$). Let $X^{(d)}$ be the set of all lattice points, i.e.,

$$X^{(d)} = \mathbf{Z}^d \qquad (\mathbf{Z}\text{: the set of integers}).$$

Let $e_1, \ldots, e_d$ be the standard base of $\mathbf{R}^d$, i.e., the k-th component of e_j is 1 for $k=j$ and 0 for $k \neq j$. For $a, b \in \mathbf{R}^d$, let $[a, b]$ denote the directed line segment from a to b. For each j $(=1, \ldots, d)$, set

$$S_{j,+}^{(d)} = \{[x, x+e_j];\ x = (v_1, \ldots, v_d) \in X^{(d)},\ v_j \geqq 0\},$$

$$S_{j,-}^{(d)} = \{[x, x-e_j];\ x = (v_1, \ldots, v_d) \in X^{(d)},\ v_j \leqq 0\}$$

and

$$S_j^{(d)} = S_{j,+}^{(d)} \cup S_{j,-}^{(d)}.$$

We define $Y^{(d)}$ by

$$Y^{(d)} = \bigcup_{j=1}^{d} S_j^{(d)}.$$

For $x \in X^{(d)}$ and $y = [x_1, x_2] \in Y^{(d)}$, let

$$K(x, y) = \begin{cases} 1, & \text{if } x_2 = x, \\ -1, & \text{if } x_1 = x, \\ 0, & \text{if } x_1 \neq x \text{ and } x_2 \neq x. \end{cases}$$

With $r(y) \equiv 1$, $N_d = \{X^{(d)}, Y^{(d)}, K, r\}$ is an infinite network in the sense of [1]. What we shall prove is

THEOREM. $\operatorname{Ind} N_d = d$.

Here $\operatorname{Ind} N_d$ is the parabolic index of N_d (see [1, §5]). The case $d=1$ is proved in [1, Example 4.1]. The proof for $d \geqq 2$ consists of two parts:

(I) If $p \geqq d$, then N_d is of parabolic type of order p;

(II) If $1 < p < d$, then N_d is of hyperbolic type of order p.

For simplicity, we shall omit the superscript (d) in the notation. For $x = (v_1, \dots, v_d) \in X$, we write $|x| = (|v_1|, \dots, |v_d|)$ and $\|x\| = \max_j |v_j|$. For $y = [x_1, x_2] \in Y$, the point x_1 will be denoted by $a(y)$; if $y \in S_j$, then the index j will be denoted by $j(y)$.

2. Proof of (I)

Let

$$X_n = \{x \in X;\ \|x\| \leqq n\}, \qquad n = 0, 1, \dots$$

and

$$Y_n = \{[x_1, x_2] \in Y;\ x_1, x_2 \in X_n\}, \qquad n = 1, 2, \dots.$$

Then $\{<X_n, Y_n>\}$ is an exhaustion of N_d. It is elementary to see that

$$\operatorname{Card} Y_n = 2dn(2n+1)^{d-1}, \qquad n = 1, 2, \dots.$$

(Here, Card stands for the cadinal.) Hence, if we put $Z_n = Y_n - Y_{n-1}$ ($Y_0 = \emptyset$), then

$$\operatorname{Card} Z_n = \operatorname{Card} Y_n - \operatorname{Card} Y_{n-1} \leqq 2d^2(2n+1)^{d-1}, \qquad n = 1, 2, \dots.$$

Since $r(y) \equiv 1$,

$$\mu_n^{(p)} \equiv \sum_{y \in Z_n} r(y)^{1-p} = \operatorname{Card} Z_n \leqq 2d^2(2n+1)^{d-1}, \qquad n = 1, 2, \dots.$$

Hence, if $p \geqq d$ and $1/p + 1/q = 1$, then

$$\sum_{n=1}^{\infty} (\mu_n^{(p)})^{1-q} \geqq (2d^2)^{1-q} \sum_{n=1}^{\infty} (2n+1)^{(d-1)(1-q)} = +\infty,$$

since $(d-1)(1-q) \geqq -1$. Therefore, by [1, Corollary 1 to Theorem 4.1], N_d is of parabolic type of order p if $p \geqq d$.

3. Proof of (II)

We shall prove (II) in several steps. Let $\mathscr{P}$ be the set of all permutations of $\{1,\ldots, d\}$ and for $x=(v_1,\ldots, v_d)$ and $\pi \in \mathscr{P}$, let $\pi^* x=(v_{\pi(1)},\ldots, v_{\pi(d)})$.

(i) For $y, y' \in Y$, if there is $\pi \in \mathscr{P}$ such that $\pi^*|a(y)|=|a(y')|$ and $j(y)=\pi(j(y'))$, then we say that y and y' are equivalent and write $y \sim y'$. Obviously, this is an equivalence relation in Y. Now we put

$$X^* = \{x^* = (\mu_1,\ldots, \mu_d) \in X;\ \mu_1 \geqq \cdots \geqq \mu_d \geqq 0\}$$

and

$$Y^* = \{[x_1, x_2] \in Y;\ x_1, x_2 \in X^*\}.$$

Observe that for $x^*=(\mu_1,\ldots, \mu_d) \in X^*$, $x^*+e_j \in X^*$ (resp. $x^*-e_j \in X^*$) if and only if

$$j = \min\{k;\ \mu_k = \mu_j\} \qquad (\text{resp. } \mu_j \neq 0 \text{ and } j = \max\{k;\ \mu_k = \mu_j\}).$$

Using this fact, we can easily see that Y^* is a set of representatives with respect to the equivalence relation $\sim$, i.e., for every $y \in Y$, there is exactly one $y^* \in Y^*$ such that $y^* \sim y$.

(ii) In order to construct a flow from $\{0\}$ to ∞, we consider the following values defined inductively:

$$(1)\quad \begin{cases} F(n;\,1) = (2n-1)^{-d}, \\ F(n;\,j+1) = 2^{-1}j\,(d-j)^{-1}\{(2n-1)F(n;\,j)-(2n+1)^{1-d}\}, \quad j = 1, 2,\ldots, d-1, \end{cases}$$

$n=1, 2,\ldots$. In a closed form, $F(n;j)$ is expressed as

$$F(n;\,j) = \binom{d-1}{j-1}^{-1} (2n+1)^{1-d} \sum_{k=0}^{d-j} \binom{d-1}{k+j-1} 2^k (2n-1)^{-k-1},$$

which is verified by induction on j. Observing that

$$\binom{d-1}{j-1}^{-1} \binom{d-1}{k+j-1} \leqq \binom{d-j}{k}, \quad k=0, 1,\ldots, d-j,$$

we obtain

$$(2)\qquad F(n;j) \leqq (2n-1)^{-d}, \qquad j = 1,\ldots, d;\ n = 1, 2,\ldots.$$

(iii) Given $x^*=(\mu_1,\dots,\mu_d)\in X^*$, let $k(x^*)$ denote the largest integer k such that $\mu_k=\mu_1$. We define a function w on Y as follows:

(a) If $y=[x^*, x^*+e_1]\in Y^*$ and $x^*=(\mu_1,\dots,\mu_d)$, then

$$w(y) = (2d)^{-1}(2\mu_1+1)^{1-d};$$

(b) If $y=[x^*, x^*+e_j]\in Y^*$ with $j>1$ and $x^*=(\mu_1,\dots,\mu_d)$, then

$$w(y) = (2d)^{-1}(2\mu_j+1)F(\mu_1;\, k(x^*)+1);$$

(c) If $y\in Y-Y^*$, then $w(y)=w(y^*)$ for $y^*\in Y^*$ such that $y^*\sim y$.

Now we shall prove that w is a flow (see [1, §4]) from $\{0\}$ to ∞ with strength $I(w)=1$.

(iii–1) Every $y\in Y(0)$ is equivalent to $y^*=[0, e_1]\in Y^*$ (see [1, (1.3)] for $Y(x)$). Therefore, $w(y)=w(y^*)=(2d)^{-1}$ for all $y\in Y(0)$, and hence

$$I(w) = -\sum_{y\in Y(0)} K(0, y)w(y) = -2d(-1)\frac{1}{2d} = 1.$$

(iii–2) Let $x=(\nu_1,\dots,\nu_d)\in X$ and $x\neq 0$. Choose $x^*=(\mu_1,\dots,\mu_d)\in X^*$ such that $\pi^*|x|=x^*$ for some $\pi\in\mathscr{P}$. For each j, let us compute $\sum_{y\in S_j} K(x, y)w(y) = \sum_{y\in Y(x)\cap S_j} K(x, y)w(y)$.

If $\nu_j>0$ (resp. $\nu_j<0$), then

$$Y(x)\cap S_j = \{[x, x+e_j], [x-e_j, x]\} \qquad (\text{resp.} = \{[x, x-e_j], [x+e_j, x]\}).$$

For $y_1=[x, x+e_j]$ (resp. $[x, x-e_j]$), choose π so that

$$\pi^{-1}(j) = \min\{k;\, |\nu_{\pi(k)}| = |\nu_j|\}.$$

Then $y_1^*=[x^*, x^*+e_m]$ $(m=\pi^{-1}(j))$ belongs to Y^* and is equivalent to y_1. Since $\mu_m=|\nu_j|$ and $m=1$ if and only if $|\nu_j|=\mu_1$,

$$w(y_1) = w(y_1^*) = \begin{cases} (2d)^{-1}(2\mu_1+1)^{1-d}, & \text{if } |\nu_j| = \mu_1, \\ (2d)^{-1}(2|\nu_j|+1)F(\mu_1;\, k(x^*)+1), & \text{if } |\nu_j| < \mu_1. \end{cases}$$

For $y_2=[x-e_j, x]$ (resp. $[x+e_j, x]$), choose π so that

$$\pi^{-1}(j) = \max\{k;\, |\nu_{\pi(k)}| = |\nu_j|\}.$$

Then $y_2^*=[x^*-e_m, x^*]$ $(m=\pi^{-1}(j))$ belongs to Y^* and is equivalent to y_2. The m-th component of x^*-e_m is equal to $|\nu_j|-1$; $m=1$ if and only if $|\nu_j|=\mu_1$ and $k(x^*)=1$. Furthermore,

$$k(x^*-e_m) = \begin{cases} k(x^*)-1, & \text{if } |\nu_j| = \mu_1 \text{ and } k(x^*) > 1, \\ k(x^*), & \text{if } |\nu_j| < \mu_1. \end{cases}$$

Hence

$$w(y_2)=w(y_2^*)=\begin{cases}(2d)^{-1}(2\mu-1)^{1-d}=(2d)^{-1}(2\mu_1-1)F(\mu_1;\,k(x^*)), & \\ \qquad \text{if } |v_j|=\mu_1 \text{ and } k(x^*)=1, & \\ (2d)^{-1}(2\mu_1-1)F(\mu_1;\,k(x^*)), \quad \text{if } |v_j|=\mu_1 \text{ and } k(x^*)>1, & \\ (2d)^{-1}(2|v_j|-1)F(\mu_1;\,k(x^*)+1), \quad \text{if } |v_j|<\mu_1 . & \end{cases}$$

Therefore, in case $v_j\neq 0$, using (1) we have

$$(3)\quad \sum_{y\in S_j}K(x,\,y)w(y)=w(y_2)-w(y_1)$$
$$=\begin{cases}(2d)^{-1}\{(2\mu_1-1)F(\mu_1;\,k(x^*))-(2\mu_1+1)^{1-d}\} & \\ \qquad =\dfrac{d-k(x^*)}{dk(x^*)}F(\mu_1;\,k(x^*)+1)\,, \quad \text{if } |v_j|=\mu_1, & \\ -\,d^{-1}F(\mu_1;\,k(x^*)+1), \quad \text{if } |v_j|<\mu_1 . & \end{cases}$$

If $v_j=0$, then

$$Y(x)\cap S_j=\{[x,\,x+e_j],\,[x,\,x-e_j]\}$$

and by an argument similar to the above, we see that both $y_1=[x,\,x+e_j]$ and $y_2=[x,\,x-e_j]$ are equivalent to $y^*=[x^*,\,x^*+e_m]\in Y^*$. Since $v_j=0$, $m\neq 1$. Hence, in case $v_j=0$, we have

$$(4)\quad \sum_{y\in S_j}K(x,\,y)w(y)=-w(y_1)-w(y_2)=-2w(y^*)=-d^{-1}F(\mu_1;\,k(x^*)+1)\,.$$

Combining (3) and (4), we have in any case

$$\sum_{y\in S_j}K(x,\,y)w(y)=\begin{cases}\dfrac{d-k(x^*)}{dk(x^*)}F(\mu_1;\,k(x^*)+1), & \text{if } |v_j|=\mu_1\,, \\ -\dfrac{1}{d}F(\mu_1;\,k(x^*)+1), & \text{if } |v_j|<\mu_1\,. \end{cases}$$

Since Card $\{j;\,|v_j|=\mu_1\}=k(x^*)$,

$$\sum_{y\in Y}K(x,\,y)w(y)=\sum_{j=1}^{d}\sum_{y\in S_j}K(x,\,y)w(y)$$
$$=k(x^*)\frac{d-k(x^*)}{dk(x^*)}F(\mu_1;\,k(x^*)+1)\;-\{d-k(x^*)\}d^{-1}F(\mu_1;\,k(x^*)+1)$$
$$=0\,.$$

Therefore, w is a flow from $\{0\}$ to ∞.

(iv) Finally, we show that if $1<p<d$ and $1/p+1/q=1$, then

$$\sum_{y\in Y} w(y)^q < +\infty.$$

If $y \sim y^* \in Y^*$ and $a(y^*)=(\mu_1,\dots,\mu_d)$, then $\mu_1=\|a(y)\|$. Hence if $j(y^*)=1$ and $\mu_1 \geqq 1$, then

$$w(y) = w(y^*) = (2d)^{-1}(2\mu_1+1)^{1-d} \leqq (2d)^{-1}(2\|a(y)\|-1)^{1-d}$$

and if $j(y^*)>1$, then $\mu_{j(y^*)} \leqq \mu_1 - 1$, so that by virtue of (2),

$$\begin{aligned} w(y) = w(y^*) &= (2d)^{-1}(2\mu_{j(y^*)}+1)F(\mu_1;\, k(a(y^*))+1) \\ &\leqq (2d)^{-1}(2\mu_1-1)(2\mu_1-1)^{-d} = (2d)^{-1}(2\|a(y)\|-1)^{1-d}. \end{aligned}$$

Now let $T_n=\{y\in Y;\ \|a(y)\|=n\}$, $n=0, 1,\dots$. Then $Y=\cup_{n=0}^{\infty} T_n$. For each $x \in X$ with $\|x\|=n$, there are at most $2d$ elements y in Y such that $a(y)=x$. Hence,

$$\mathrm{Card}\, T_0 = 2d \quad \text{and} \quad \mathrm{Card}\, T_n \leqq 2d\, \mathrm{Card}\,(X_n - X_{n-1}), \qquad n = 1, 2,\dots.$$

On the other hand, $\mathrm{Card}\, X_n=(2n+1)^d$. Hence

$$\mathrm{Card}\, T_n \leqq 4d^2(2n+1)^{d-1}, \qquad n = 1, 2,\dots.$$

Therefore

$$\begin{aligned} \sum_{y\in Y} w(y)^q &= \sum_{n=0}^{\infty} \sum_{y\in T_n} w(y)^q \\ &\leqq 2d(2d)^{-q} + \sum_{n=1}^{\infty} 4d^2(2n+1)^{d-1}(2d)^{-q}(2n-1)^{(1-d)q} < +\infty, \end{aligned}$$

since $p<d$ implies $d-1+(1-d)q<-1$.

(v) Now the statement of (II) follows from [1, Theorem 4.3].

Reference

[1] M. Yamasaki, Parabolic and hyperbolic infinite networks, this journal., 135–146.

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