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Parabolic and Hyperbolic Infinite Networks

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Introduction

We shall classify in this paper a set of infinite networks into parabolic networks and hyperbolic networks of order p.

More precisely, let $N=\{X, Y, K, r\}$ be an infinite network which is connected and locally finite and which has no self-loop, let $\mathbf{D}^{(p)}(N)$ be the set of all real functions on X with finite Dirichlet integrals $D_p(u)$ of order p $(1<p\leqq\infty)$ and let $L_0(X)$ be the set of all real functions on X with finite supports. We say that N is of parabolic type of order p if there exists a nonempty finite subset A of X such that the value $d_p(A, \infty)$ of the following extremum problem (*) on N relative to A and the ideal boundary ∞ of N vanishes:

$$(*)\qquad \text{Find}\quad d_p(A, \infty) = \inf\{D_p(u);\ u\in L_0(X) \text{ and } u = 1 \text{ on } A\}.$$

We say that N is of hyperbolic type of order p if it is not of parabolic type of order p.

We shall prove in § 3 that N is of parabolic type of order p $(1<p<\infty)$ if and only if any one of the following conditions is fulfilled: (C. 1) $1\in\mathbf{D}_0^{(p)}(N)$, (C. 2) $\mathbf{D}^{(p)}(N)=\mathbf{D}_0^{(p)}(N)$, where $\mathbf{D}_0^{(p)}(N)$ is the closure of $L_0(X)$ in $\mathbf{D}^{(p)}(N)$ with respect to the norm $\|u\|_p=[D_p(u)+|u(x_0)|^p]^{1/p}$ $(x_0\in X)$. Some practical criteria which assure that the network is of parabolic type of order p will be given in § 4 by means of some results in [4] and [6] concerning the extremal length of an infinite network.

In case $p=2$, this classification problem, which was partially studied by C. Blanc [2], is very analogous to the classification problem of Riemann surfaces (see for instance [1], [3] and [5]).

It will be shown in § 5 that if N is of parabolic type of order p_1 and if $1<p_1<p_2$, then N is of parabolic type of order p_2. By this fact, we define a parabolic index $\operatorname{ind} N$ of N as the infimum of $p>1$ for which N is of parabolic type of order p. Some geometric meaning of $\operatorname{ind} N$ will be shown by two examples.

§ 1. Some definitions related to an infinite network

Let X and Y be countable (infinite) sets and K be a function on $X\times Y$ satisfy-

ing the following conditions:
(1.1) The range of K is $\{-1, 0, 1\}$.
(1.2) For each $y \in Y$, $e(y)=\{x \in X;\ K(x, y) \neq 0\}$ consists of exactly two points x_1, x_2 and $K(x_1, y)K(x_2, y)=-1$.
(1.3) For each $x \in X$, $Y(x)=\{y \in Y;\ K(x, y) \neq 0\}$ is a nonempty finite set.
(1.4) For any $x, x' \in X$, there are $x_1,\dots, x_n \in X$ and $y_1,\dots, y_{n+1} \in Y$ such that $e(y_j)=\{x_{j-1}, x_j\}$, $j=1,\dots, n+1$ with $x_0=x$ and $x_{n+1}=x'$.

Let r be a strictly positive function on Y. Then $N=\{X, Y, K, r\}$ is called an infinite network.

Let X' and Y' be subsets of X and Y respectively and let K' and r' be the restrictions of K and r onto $X' \times Y'$ and Y' respectively. Then $N'=\{X', Y', K', r'\}$ is called a subnetwork of the network N if conditions (1.2)–(1.4) are fulfilled replacing X, Y and K by X', Y' and K' respectively. Let us put for simplicity $<X', Y'>=N'$. In case X' (or Y') is a finite set, $<X', Y'>$ is a finite subnetwork.

A sequence $\{<X_n, Y_n>\}$ of finite subnetworks of N is called an exhaustion of N if $X=\bigcup_{n=1}^{\infty} X_n$, $Y=\bigcup_{n=1}^{\infty} Y_n$ and $Y(x) \subset Y_{n+1}$ for all $x \in X_n$.

For a subset A of X, denote by ε_A the characteristic function of A, i.e., $\varepsilon_A(x)=1$ if $x \in A$ and $\varepsilon_A(x)=0$ if $x \in X-A$. Throughout this paper, let $1<p\leqq\infty$ and $1/p+1/q=1$ $(1<p<\infty)$.

§2. Functional spaces on an infinite network

Denote by $L(X)$ the set of all real functions on X. For $u \in L(X)$, its support Su and its Dirichlet integral $D_p(u)$ of order p are defined by

$$Su = \{x \in X;\ u(x) \neq 0\},$$

$$D_p(u) = \sum_{y\in Y} r(y)^{1-p} \Big|\sum_{x\in X} K(x, y)u(x)\Big|^p \qquad (1 < p < \infty),$$

$$D_\infty(u) = \sup_{y\in Y} r(y)^{-1} \Big|\sum_{x\in X} K(x, y)u(x)\Big|.$$

Let us put

$$L_0(X) = \{u \in L(X);\ Su \text{ is a finite set}\},$$

$$\mathbf{D}^{(p)}(N) = \{u \in L(X);\ D_p(u) < \infty\}.$$

For a nonempty subset A of X, we set

$$\mathbf{D}^{(p)}(N; A) = \{u \in \mathbf{D}^{(p)}(N);\ u = 0 \text{ on } A\},$$

$$L_0^A(X) = \{u \in L_0(X);\ u = 0 \text{ on } A\}.$$

Note that $\mathbf{D}^{(p)}(N; A)$ $(1<p<\infty)$ is a reflexive Banach space with respect to the norm $[D_p(u)]^{1/p}$ (cf. [4]) and that $\mathbf{D}^{(\infty)}(N; A)$ is a Banach space with respect to the norm $D_\infty(u)$. Denote by $\mathbf{D}_0^{(p)}(N; A)$ the closure of $L_0^A(X)$ in $\mathbf{D}^{(p)}(N; A)$ with respect to the above norm.

Let $x_0 \in X$ be fixed. We define $\|u\|_p$ by

$$\|u\|_p = [D_p(u)+|u(x_0)|^p]^{1/p} \qquad (1 < p < \infty), \tag{2.1}$$

$$\|u\|_\infty = D_\infty(u)+|u(x_0)|. \tag{2.2}$$

By the same reasoning as in the proof of Lemma 1 in [6], we can prove

LEMMA 2.1. *For every finite subset F of X, there exists a constant $M(F)$ such that*

$$\sum_{x\in F}|u(x)| \leqq M(F)\|u\|_p$$

for all $u \in \mathbf{D}^{(p)}(N)$.

COROLLARY. *If $u_n, u \in \mathbf{D}^{(p)}(N)$ and $\|u-u_n\|_p \to 0$ as $n\to\infty$, then $\{u_n\}$ converges pointwise to u, i.e., $u_n(x)\to u(x)$ as $n\to\infty$ for each $x\in X$.*

We can prove by a standard argument

PROPOSITION 2.1. *$\mathbf{D}^{(p)}(N)$ is a Banach space with respect to the norm $\|u\|_p$. Moreover $\mathbf{D}^{(p)}(N)$ $(1<p<\infty)$ is reflexive.*

Denote by $\mathbf{D}_0^{(p)}(N)$ the closure of $L_0(X)$ in $\mathbf{D}^{(p)}(N)$ with respect to the norm $\|u\|_p$, i.e., $u\in\mathbf{D}_0^{(p)}(N)$ if and only if there is a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u-f_n\|_p\to 0$ as $n\to\infty$.

LEMMA 2.2. *For a nonempty finite subset A of X, $\mathbf{D}_0^{(p)}(N; A)=\mathbf{D}^{(p)}(N; A)\cap\mathbf{D}_0^{(p)}(N)$.*

PROOF. Clearly $\mathbf{D}_0^{(p)}(N; A)\subset\mathbf{D}^{(p)}(N; A)\cap\mathbf{D}_0^{(p)}(N)$. Let $u\in\mathbf{D}^{(p)}(N; A)\cap\mathbf{D}_0^{(p)}(N)$. We can find a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u-f_n\|_p\to 0$ as $n\to\infty$. Then $f_n(x)\to 0$ as $n\to\infty$ for each $x\in A$ by Lemma 2.1. Define $g_n \in L(X)$ by $g_n=0$ on A and $g_n=f_n$ on $X-A$. Then $g_n\in L_0^A(X)$ and $\|g_n-f_n\|_p\to 0$ as $n\to\infty$, since A is a finite set. Therefore $\|u-g_n\|_p\to 0$ as $n\to\infty$, and hence $u\in\mathbf{D}_0^{(p)}(N; A)$. Thus $\mathbf{D}^{(p)}(N; A)\cap\mathbf{D}_0^{(p)}(N)\subset\mathbf{D}_0^{(p)}(N; A)$.

REMARK 2.1. Let $a\in X$ and denote by $\mathbf{D}_0^{(p)}(N, a)$ the closure of $L_0(X)$ with respect to the norm which is defined by (2.1) and (2.2) replacing x_0 by a. Then $\mathbf{D}_0^{(p)}(N, a)=\mathbf{D}_0^{(p)}(N)$.

We shall call a function T on the real line R into itself a normal contraction of R if $T0=0$ and $|Tx_1-Tx_2|\leqq|x_1-x_2|$ for any $x_1, x_2\in R$. Define $Tu\in L(X)$ for $u\in L(X)$ by $(Tu)(x)=Tu(x)$.

We can easily prove

LEMMA 2.3. *Let T be a normal contraction of R and $u \in \mathbf{D}^{(p)}(N)$. Then $Tu \in \mathbf{D}^{(p)}(N)$, $D_p(Tu) \leqq D_p(u)$ and $\|Tu\|_p \leqq \|u\|_p$.*

We shall use the following normal contractions:

$$T^{(+)}x = \max(x, 0) \quad \text{and} \quad T^{(-)}x = \max(-x, 0).$$

§3. Classification of infinite networks

For a nonempty finite subset A of X, let us consider the following extremum problem relative to the ideal boundary ∞ of N:

$$\text{Find} \quad d_p(A, \infty) = \inf\{D_p(u);\ u \in L_0(X) \text{ and } u = 1 \text{ on } A\}.$$

First we shall prove

THEOREM 3.1. *Let A be a nonempty finite subset of X. Then $d_p(A, \infty) = 0$ if and only if $1 \in \mathbf{D}_0^{(p)}(N)$.*

PROOF. Assume that $d_p(A, \infty) = 0$. Then there exists a sequence $\{u_n\}$ in $L_0(X)$ such that $u_n = 1$ on A and $D_p(u_n) \to 0$ as $n \to \infty$. Since $\|u_n - u_n(x_0)\|_p \to 0$ as $n \to \infty$, we see by Proposition 2.1 and the corollary of Lemma 2.1 that $\{u_n\}$ converges pointwise to 1. Therefore $\|u_n - 1\|_p \to 0$ as $n \to \infty$, i.e., $1 \in \mathbf{D}_0^{(p)}(N)$. Next we assume that $1 \in \mathbf{D}_0^{(p)}(N)$. There is a sequence $\{f_n\}$ in $L_0(X)$ such that $\|1 - f_n\|_p \to 0$ as $n \to \infty$. Notice that $\{f_n\}$ converges pointwise to 1 and $D_p(f_n) \to 0$ as $n \to \infty$. Define $g_n \in L(X)$ by $g_n = 1$ on A and $g_n = f_n$ on $X - A$. Then $g_n \in L_0(X)$ and $\|g_n - f_n\|_p \to 0$ as $n \to \infty$, since A is a finite set. Therefore $D_p(g_n - f_n) \to 0$ as $n \to \infty$, and hence $D_p(g_n) \to 0$ as $n \to \infty$. Since $d_p(A, \infty) \leqq D_p(g_n)$, we conclude that $d_p(A, \infty) = 0$.

COROLLARY. *Let A and A' be nonempty finite subsets of X. Then $d_p(A, \infty) = 0$ if and only if $d_p(A', \infty) = 0$.*

On account of this result, we can classify the set of all infinite networks as follows:

DEFINITION 3.1. We say that an infinite network $N = \{X, Y, K, r\}$ is of parabolic type of order p if there exists a nonempty finite subset A of X such that $d_p(A, \infty) = 0$. We say that N is of hyperbolic type of order p if it is not of parabolic type of order p, i.e., $d_p(A, \infty) > 0$ for any nonempty finite subset A of X.

We prepare

LEMMA 3.1. *Let $1 < p < \infty$ and let $u \in \mathbf{D}^{(p)}(N)$ be non-negative.* If

$\{v_n\}$ is a sequence in $\mathbf{D}^{(p)}(N)$ such that $v_n \geqq 0$ on X, $\{v_n\}$ converges pointwise to ∞ and $D_p(v_n) \to 0$ as $n \to \infty$, then $\|u - \min(u, v_n)\|_p \to 0$ as $n \to \infty$.

PROOF. Let us put $u_n = \min(u, v_n)$, $B_n = \{x \in X;\ u(x) > v_n(x)\}$ and $V_n = \cup\{Y(x);\ x \in B_n\}$. Notice that for every finite subset F of Y, there exists n_0 such that $F \subset Y - V_n$ for all $n \geqq n_0$, since $v_n(x) \to \infty$ as $n \to \infty$ for every $x \in X$. If $y \in Y - V_n$, then

$$\sum_{x \in X} K(x, y)[u(x) - u_n(x)] = 0.$$

If $y \in V_n$ and $e(y) = \{a, b\}$, then

$$\begin{aligned} |\sum_{x \in X} K(x, y)[u(x) - u_n(x)]| &= |\{u(a) - u_n(a)\} - \{u(b) - u_n(b)\}| \\ &\leqq |u(a) - u(b)| + |v_n(a) - v_n(b)|. \end{aligned}$$

Thus we have

$$\begin{aligned} D_p(u - u_n) &= \sum_{y \in V_n} r(y)^{1-p} |\sum_{x \in X} K(x, y)[u(x) - u_n(x)]|^p \\ &\leqq 2^{p-1} D_p(v_n) + 2^{p-1} \sum_{y \in V_n} r(y)^{1-p} |\sum_{x \in X} K(x, y) u(x)|^p \to 0 \end{aligned}$$

as $n \to \infty$. Since $\{u_n\}$ converges pointwise to u, we conclude that $\|u - u_n\|_p \to \infty$ as $n \to \infty$.

Now we shall prove

THEOREM 3.2. *Let $1 < p < \infty$. An infinite network N is of parabolic type of order p if and only if any one of the following conditions is fulfilled:*

(C. 1) $\quad 1 \in \mathbf{D}_0^{(p)}(N)$.

(C. 2) $\quad \mathbf{D}_0^{(p)}(N) = \mathbf{D}^{(p)}(N)$.

PROOF. On account of Theorem 3.1, we have only to prove that (C. 1) implies (C. 2). Assume that $1 \in \mathbf{D}_0^{(p)}(N)$. Then we can find a sequence $\{f_n\}$ in $L_0(X)$ such that $\|1 - f_n\|_p < 1/n^2$ for all n. By Lemma 2.3, we may assume that $f_n \geqq 0$ on X. Put $v_n = n f_n$. Then $v_n \geqq 0$ on X, $v_n(x) \to \infty$ as $n \to \infty$ for every $x \in X$ and $D_p(v_n) = n^p D_p(f_n) \leqq 1/n^p \to 0$ as $n \to \infty$. For any $u \in \mathbf{D}^{(p)}(N)$ which is non-negative, $\|u - \min(u, v_n)\|_p \to 0$ as $n \to \infty$ by Lemma 3.1. Since $\min(u, v_n) \in L_0(X)$, $u \in \mathbf{D}_0^{(p)}(N)$. By Lemma 2.3, every $u \in \mathbf{D}^{(p)}(N)$ belongs to $\mathbf{D}_0^{(p)}(N)$. Therefore $\mathbf{D}_0^{(p)}(N) = \mathbf{D}^{(p)}(N)$.

COROLLARY. *Let $1 < p < \infty$ and let A be a nonempty finite subset of X. An infinite network N is of parabolic type of order p if and only if any one of the following conditions is fulfilled:*

(C. 3) $\quad \varepsilon_{X-A} \in \mathbf{D}_0^{(p)}(N; A)$.

(C. 4) $\quad \mathbf{D}_0^{(p)}(N; A) = \mathbf{D}^{(p)}(N; A)$.

PROOF. (C. 2) implies (C. 4) by Lemma 2.2. Clearly (C. 4) implies (C. 3). Assume that (C. 3) holds. Since $\varepsilon_A \in \mathbf{D}_0^{(p)}(N)$, we have $1 = \varepsilon_{X-A} + \varepsilon_A \in \mathbf{D}_0^{(p)}(N)$. Namely (C. 1) holds. Our corollary is thus proved.

REMARK 3.1. In case $p = \infty$, (C. 1) does not imply (C. 2) in general. This will be shown by Example 4.1 in the next section.

§4. Practical criteria

In order to obtain some practical criteria which assure that N is of parabolic type of order p, we recall the extremal distance and the extremal width of an infinite network studied in [4] and [6].

Denote by $L(Y)$ the set of all real functions on Y. For $w \in L(Y)$, its energy $H_p(w)$ of order p is defined by

$$H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p \qquad (1 < p < \infty),$$

$$H_\infty(w) = \sup\{|w(y)|;\ y \in Y\}.$$

Denote by $L_p(Y; r)$ the set of all $w \in L(Y)$ such that $H_p(w) < \infty$ and by $L_p^+(Y; r)$ the subset of $L_p(Y; r)$ which consists of non-negative functions.

For a nonempty finite subset A of X, denote by $\mathbf{P}_{A,\infty}$ the set of all paths from A to the ideal boundary ∞ of N and by $\mathbf{Q}_{A,\infty}$ the set of all cuts between A and ∞ (cf. [4] for definitions).

The extremal distance $EL_p(A, \infty)$ of order p of N relative to A and ∞ is defined by

$$EL_p(A, \infty)^{-1} = \inf\{H_p(W);\ W \in E_p(\mathbf{P}_{A,\infty})\},$$

where $E_p(\mathbf{P}_{A,\infty}) = \{W \in L_p^+(Y; r);\ \sum_P r(y)W(y) \geqq 1$ for all $P \in \mathbf{P}_{A,\infty}\}$. The extremal width $EW_p(A, \infty)$ of order p $(1 < p < \infty)$ of N relative to A and ∞ is defined by

$$EW_p(A, \infty)^{-1} = \inf\{H_p(W);\ W \in E_p^*(\mathbf{Q}_{A,\infty})\},$$

where $E_p^*(\mathbf{Q}_{A,\infty}) = \{W \in L_p^+(Y; r);\ \sum_Q W(y)^{p-1} \geqq 1$ for all $Q \in \mathbf{Q}_{A,\infty}\}$.

We proved in [4] the following three lemmas:

LEMMA 4.1. $EL_p(A, \infty) = [EW_p(A, \infty)]^{1-p}$ $(1 < p < \infty)$.

LEMMA 4.2. *Let* $1 < p < \infty$ *and let* $\{\langle X_n, Y_n \rangle\}$ *be an exhaustion of* N *such that* $A \subset X_1$ *and put*

$$\mu_n^{(p)} = \sum_{y \in Z_n} r(y)^{1-p} \quad with \quad Z_n = Y_n - Y_{n-1} \quad (Y_0 = \phi).$$

Then $EL_p(A, \infty)^{q-1} \geqq \sum_{n=1}^{\infty} [\mu_n^{(p)}]^{1-q}$.

LEMMA 4.3. *Let* $1 < p < \infty$. *For any path* $P \in \mathbf{P}_{A,\infty}$, $EL_p(A, \infty) \leqq [\sum_P r(y)]^{p-1}$.

We shall prove

LEMMA 4.4. $EL_\infty(A, \infty) = \inf\{\sum_P r(y);\ P \in \mathbf{P}_{A,\infty}\}$.

PROOF. Let us put $\beta = \inf\{\sum_P r(y);\ P \in \mathbf{P}_{A,\infty}\}$. Then $\beta > 0$. For any $W \in E_\infty(\mathbf{P}_{A,\infty})$ and $P \in \mathbf{P}_{A,\infty}$,

$$1 \leqq \sum_P r(y)W(y) \leqq H_\infty(W) \sum_P r(y),$$

so that $EL_\infty(A, \infty) \leqq \sum_P r(y)$. Thus $EL_\infty(A, \infty) \leqq \beta$. On the other hand, for any t with $0 < t < \beta$, define $W \in L(Y)$ by $W(y) = 1/t$ on Y. Then $W \in E_\infty(\mathbf{P}_{A,\infty})$ and $EL_\infty(A, \infty)^{-1} \leqq H_\infty(W) = 1/t$. By the arbitrariness of t, we have $\beta \leqq EL_\infty(A, \infty)$.

LEMMA 4.5. $d_p(A, \infty) = EL_p(A, \infty)^{-1}$ *for all* p, $1 < p \leqq \infty$.

PROOF. Let $\{<X_n, Y_n>\}$ be an exhaustion of N such that $A \subset X_1$. The extremal distance $EL_p(A, X - X_n)$ of order p of N relative to A and $X - X_n$ is defined by

$$EL_p(A, X - X_n)^{-1} = \inf\{H_p(W);\ W \in E_p(\mathbf{P}_{A, X - X_n})\},$$

where $E_p(\mathbf{P}_{A,X-X_n})$ is defined as above replacing $\mathbf{P}_{A,\infty}$ by the set $\mathbf{P}_{A,X-X_n}$ of all paths from A to $X - X_n$. First we consider the case where $1 < p < \infty$. We have $EL_p(A, X - X_n) \to EL_p(A, \infty)$ as $n \to \infty$ by Theorem 2.2 in [4] and

$$\begin{aligned} EL_p(A, X - X_n)^{-1} &= \inf\{D_p(u);\ u \in \mathbf{D}^{(p)}(N; A) \text{ and } u = 1 \text{ on } X - X_n\} \\ &= \inf\{D_p(u);\ u \in \mathbf{D}^{(p)}(N; X - X_n) \text{ and } u = 1 \text{ on } A\} \end{aligned}$$

by Theorem 2.1 in [4]. It follows that $d_p(A, \infty) = EL_p(A, \infty)^{-1}$ if $1 < p < \infty$. Since the proofs of Theorems 2.1 and 2.2 in [4] are still effective in the case where $p = \infty$, we can similarly prove $d_\infty(A, \infty) = EL_\infty(A, \infty)^{-1}$.

On account of Lemmas 4.1 and 4.5, we obtain

THEOREM 4.1. *Let* $1 < p < \infty$ *and let* A *be a nonempty finite subset of* X. *An infinite network* N *is of parabolic type of order* p *if and only if any one of the* *following conditions is fulfilled:*

(C. 5) $EL_p(A, \infty) = \infty$.

(C. 6) $EW_p(A, \infty) = 0$, *i.e.*, $E_p^*(\mathbf{Q}_{A,\infty}) = \phi$.

By this theorem and Lemma 4.2, we have

COROLLARY 1. *Let* $1<p<\infty$. *If there exists an exhaustion* $\{<X_n, Y_n>\}$ *of* N *such that* $\sum_{n=1}^{\infty} [\mu_n^{(p)}]^{1-q} = \infty$ (*cf.* *Lemma* 4.2), *then* N *is of parabolic type of order* p.

By this theorem and the definition of $E_p^*(\mathbf{Q}_{A,\infty})$, we have

COROLLARY 2. *If* N *is of parabolic type of order* p $(1<p<\infty)$, *then* $\inf\{\sum_Q W(y)^{p-1};\ Q \in \mathbf{Q}_{A,\infty}\} = 0$ *for every* $W \in L_p^+(Y; r)$ *and every nonempty finite subset* A *of* X.

On account of Lemmas 4.4 and 4.5, we obtain

THEOREM 4.2. *An infinite network* N *is of parabolic type of order* ∞ *if and only if there exists a nonempty finite subset* A *of* X *such that* $\sum_P r(y) = \infty$ *for all* $P \in \mathbf{P}_{A,\infty}$.

By this theorem and Lemma 4.3, we have

COROLLARY. *If* N *is of hyperbolic type of order* ∞, *then* N *is of hyperbolic type of order* p *for all* $p>1$.

We say that N is totally hyperbolic if it is of hyperbolic type of order ∞.

In order to obtain another practical criterion, we consider the set $F(A, \infty)$ of flows from a nonempty finite subset A of X to the ideal boundary ∞ of N: $w \in F(A, \infty)$ if and only if $w \in L(Y)$ and $\sum_{y \in Y} K(x, y)w(y) = 0$ for all $x \in X - A$. For $w \in F(A, \infty)$, we define the strength $I_A(w)$ of w by

$$I_A(w) = -\sum_{x \in A} \sum_{y \in Y} K(x, y)w(y).$$

We shall prove

THEOREM 4.3. *Let* $1<p<\infty$. *If there exists* $w \in F(A, \infty)$ *such that* $I_A(w) = 1$ *and* $H_q(w) < \infty$, *then* N *is of hyperbolic type of order* p.

PROOF. Put $W(y) = |w(y)|^{1/(p-1)}$. For any $Q \in \mathbf{Q}_{A,\infty}$ such that $Q = Q(A) \ominus Q(\infty)$ (see [4]), define $u_Q \in L(X)$ by $u_Q = 1$ on $Q(A)$ and $u_Q = 0$ on $Q(\infty)$. Since $u_Q \in L_0(X)$, we have

$$-1 = \sum_{x \in A} \sum_{y \in Y} K(x, y)w(y) = \sum_{x \in X} u_Q(x) \sum_{y \in Y} K(x, y)w(y)$$
$$= \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y)u_Q(x),$$

so that

$$1 \leqq \sum_{y\in Y} |w(y)| \,|\sum_{x\in X} K(x, y)u_Q(x)| = \sum_Q W(y)^{p-1}.$$

Since $H_p(W)=H_q(w)<\infty$, we conclude that $W\in E_p^*(\mathbf{Q}_{A,\infty})$. Therefore N is of hyperbolic type of order p.

Finally we shall prove Remark 3.1 by

EXAMPLE 4.1. Denote by J the set of all positive integers. Let us take

$$X = \{x_n;\, n\in J\}, \quad Y = \{y_n;\, n\in J\},$$

$$K(x_n, y_n) = -1, \quad K(x_{n+1}, y_n) = 1 \qquad \text{for all} \quad n\in J,$$

$$K(x, y) = 0 \qquad \text{for any other pair } (x, y).$$

Let $r=1$ on Y. Then $N=\{X, Y, K, r\}$ is an infinite network. It follows from Corollary 1 of Theorem 4.1 and Theorem 4.2 that N is of parabolic type of order p, $1<p\leqq\infty$. By Theorem 3.1, $1\in\mathbf{D}_0^{(p)}(N)$. We show that $\mathbf{D}_0^{(\infty)}(N) \neq \mathbf{D}^{(\infty)}(N)$. In fact, let $x_0=x_1$ and consider $u\in L(X)$ defined by $u(x_n)=n$ for all $n\in J$. Then $\|u\|_\infty=2$ and $u\in\mathbf{D}^{(\infty)}(N)$. For any $f\in L_0(X)$, $\|u-f\|_\infty\geqq D_\infty(u-f)\geqq 1$, so that $u\notin\mathbf{D}_0^{(\infty)}(N)$.

§5. A parabolic index of an infinite network

THEOREM 5.1. *Let $1<p_1<p_2$. If N is of hyperbolic type of order p_2, then N is of hyperbolic type of order p_1.*

PROOF. Assume that N is of hyperbolic type of order p_2. In case $p_2=\infty$, our theorem follows from the corollary of Theorem 4.2. Let $p_2<\infty$ and let A be a nonempty finite subset of X. Then there exists $W\in E_{p_2}^*(\mathbf{Q}_{A,\infty})$ by Theorem 4.1. Since $\min(W, 1)$ belongs to $E_{p_2}^*(\mathbf{Q}_{A,\infty})$, we may suppose that $W(y)\leqq 1$ on Y. Let $W'=W^{p_2/p_1}$. Then $W'\in L_{p_1}^+(Y; r)$ and $(W')^{p_1-1}=W^{p_2-p_2/p_1}\geqq W^{p_2-1}$ on Y, and hence $W'\in E_{p_1}^*(\mathbf{Q}_{A,\infty})$. Therefore N is of hyperbolic type of order p_1.

On account of Theorem 5.1, we can define a parabolic index $\operatorname{ind} N$ of an infinite network N which is not totally hyperbolic by

$$\operatorname{ind} N = \inf\{p > 1;\, N \text{ is of parabolic type of order } p\}.$$

A geometric meaning of $\operatorname{ind} N$ may be seen by the following examples:

EXAMPLE 5.1. Let $\{t_n\}$ be a sequence of positive integers and denote by J the set of all positive integers. Let us take

$$X = \{x_n;\ n \in J\}, \quad Y = \{y_1^{(n)}, y_2^{(n)}, \dots, y_{t_n}^{(n)};\ n \in J\},$$

$$K(x_n, y_1^{(n)}) = \cdots = K(x_n, y_{t_n}^{(n)}) = -1 \qquad \text{for} \quad n \in J,$$

$$K(x_{n+1}, y_1^{(n)}) = \cdots = K(x_{n+1}, y_{t_n}^{(n)}) = 1 \qquad \text{for} \quad n \in J,$$

$$K(x, y) = 0 \qquad \text{for any other pair } (x, y).$$

Let $r=1$ on Y. Then $N=\{X, Y, K, r\}$ is an infinite network. Let α be a non-negative number and let t_n be the greatest integer less than or equal to n^α. If $\alpha=0$, then $t_n=1$ for all n and N is the network given in Example 4.1. In this case $\operatorname{ind} N=1$. Next let $\alpha>0$. Consider an exhaustion $\{<X_n, Y_n>\}$ of N defined by

$$X_n = \{x_j;\ j = 1, 2, \dots, n+1\},$$

$$Y_n = \{y_1^{(j)}, y_2^{(j)}, \dots, y_{t_j}^{(j)};\ j = 1, 2, \dots, n\}.$$

Then $\mu_n^{(p)}=t_n$ (cf. Lemma 4.2) and

$$\sum_{n=1}^{\infty} [\mu_n^{(p)}]^{1-q} \geqq \sum_{n=1}^{\infty} n^{\alpha(1-q)} = \infty$$

if $1<q\leqq 1+1/\alpha$. Therefore N is of parabolic type of order $p\geqq\alpha+1$. We consider the case where $1<p<\alpha+1$. Define $W\in L(Y)$ by $W(y_1^{(1)})=1$ and

$$W(y_1^{(n)}) = \cdots = W(y_{t_n}^{(n)}) = (n^\alpha-1)^{1-q} \qquad \text{for} \quad n \geqq 2.$$

Since $q>1+1/\alpha$, we have

$$H_p(W) = 1 + \sum_{n=2}^{\infty} t_n(n^\alpha-1)^{p(1-q)} \leqq 1 + \sum_{n=2}^{\infty} n^\alpha(n^\alpha-1)^{-q} < \infty.$$

Let $A=\{x_1\}$ and $Q\in \mathbf{Q}_{A,\infty}$. There exists an n such that $Z_n=Y_n-Y_{n-1}\subset Q$, so that

$$\sum_Q W(y)^{p-1} \geqq \sum_{Z_n} W(y)^{p-1} = (n^\alpha-1)^{-1}t_n \geqq 1$$

for $n\geqq 2$. Therefore $W\in E_p^*(\mathbf{Q}_{A,\infty})$. Thus N is of hyperbolic type of order p, $1<p<\alpha+1$. Namely $\operatorname{ind} N=\alpha+1$.

Next we consider the case where $t_n=2^n$. Define $W\in L(Y)$ by

$$W(y_1^{(n)}) = \cdots = W(y_{t_n}^{(n)}) = 2^{n(1-q)} \qquad \text{for} \quad n \in J.$$

Then we can prove in the same way as above that $W\in E_p^*(\mathbf{Q}_{A,\infty})$ with $A=\{x_1\}$, so that N is of hyperbolic type of order p for all p, $1<p<\infty$. Clearly N is of parabolic type of order ∞. Thus $\operatorname{ind} N=\infty$.

EXAMPLE 5.2. Let $X=\bigcup_{n=0}^{\infty} C_n$ and $Y=\bigcup_{n=1}^{\infty} Z_n$, where $C_n=\{x_i^{(n)};\ i=1,$

$2,\dots, 2^n\}$ and $Z_n=\{y_i^{(n)};\ i=1, 2,\dots, 2^n\}$. For each $n\in J$, we define

$$K(x_i^{(n)}, y_i^{(n)}) = 1 \qquad \text{for} \quad i = 1, 2,\dots, 2^n,$$

$$K(x_i^{(n-1)}, y_i^{(n)}) = K(x_i^{(n-1)}, y_{2^{n-1}+i}^{(n)}) = -1 \qquad \text{for} \quad i = 1, 2,\dots, 2^{n-1}.$$

For any other pair (x, y), we set $K(x, y)=0$. Let $\{r_n;\ n\in J\}$ be a set of positive numbers and define $r\in L(Y)$ by $r(y)=r_n$ on Z_n for each $n\in J$. Then $N=\{X, Y, K, r\}$ is an infinite network which may be called a binary tree stemmed from $x_1^{(0)}$. Let $1<p<\infty$. We shall prove that N is of parabolic type of order p if and only if $\sum_{n=1}^{\infty} 2^{n(1-q)}r_n=\infty$. Define $w\in L(Y)$ by $w(y)=2^{-n}$ on Z_n $(n\in J)$. Then w is a flow from $A=\{x_1^{(0)}\}$ to the ideal boundary ∞ of N such that $I_A(w)=1$ and

$$H_q(w) = \sum_{n=1}^{\infty} r_n \sum_{Z_n} |w(y)|^q = \sum_{n=1}^{\infty} 2^{n(1-q)}r_n.$$

Therefore the "only if" part follows from Theorem 4.3. On the other hand, consider an exhaustion $\{<X_n, Y_n>\}$ of N defined by

$$X_n = \bigcup_{j=0}^{n} C_j \quad \text{and} \quad Y_n = \bigcup_{j=1}^{n} Z_j.$$

Then we have $\mu_n^{(p)}=2^n(r_n)^{1-p}$ and

$$\sum_{n=1}^{\infty} [\mu_n^{(p)}]^{1-q} = \sum_{n=1}^{\infty} 2^{n(1-q)}r_n,$$

so that the "if" part follows from Corollary 1 of Theorem 4.1.

Now we can calculate $\operatorname{ind} N$ for several choices of $\{r_n;\ n\in J\}$. In case $r_n=1$ for all $n\in J$, $\operatorname{ind} N=\infty$. Let α be a positive number. In case $r_n=2^{n/\alpha}$ for $n\in J$, $\operatorname{ind} N=\alpha+1$ and N is of parabolic type of order $\operatorname{ind} N$. In case $r_n=n^{-2}2^{n/\alpha}$ for $n\in J$, $\operatorname{ind} N=\alpha+1$ and N is of hyperbolic type of order $\operatorname{ind} N$. In case $r_n=2^{n^2}$ for $n\in J$, $\operatorname{ind} N=1$.

References

[1] L. V. Ahlfors and L. Sario: Riemann surfaces, Princeton Univ. Press, Princeton, 1960.

[2] C. Blanc: Les réseaux Riemanniens, Comm. Math. Helv. **13** (1940), 54–67.

[3] C. Constantinescu and A. Cornea: Ideale Ränder Riemannscher Flächen, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[4] T. Nakamura and M. Yamasaki: Generalized extremal length of an infinite network, Hiroshima Math. J. **6** (1976), 95–111.

[5] L. Sario and M. Nakai: Classification theory of Riemann surfaces, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[6] M. Yamasaki: Extremum problems on an infinite network, Hiroshima Math. J. **5** (1975), 223–250.

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