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Vanishing Nonoscillations of Lienard Type Retarded Equations

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1. Introduction

Our purpose in this paper is to study the asymptotic nature of nonoscillatory solutions of the equation

$$(1)\qquad (r(t)x'(t))'+p(t)x'(t)+q(t)x(t)+a(t)h(x(g(t)))=f(t)$$

under the assumptions:

(i) $r(t)$, $p(t)$ and $g(t)$ are nonnegative, real valued and continuous on the whole real line R;

(ii) $a(t), f(t), q(t)$: $R \to R$ and continuous;

(iii) $r(t)$, $p(t)$ and $g(t)$ are $C'[A, \infty)$ for some $A>0$;

(iv) $r'(t)\geq 0$, $0<g'(t)\leq S$ for some $S>0$, $r(t)\geq K>0$ and bounded; let $R(t)=\int^t 1/r(s)ds$;

(v) h: $R\to R$, increasing, $\operatorname{sign}(h(t))=\operatorname{sign} t$, $h(-t)=-h(t)$, and if $t\to 0$, then $h(t)\to 0$, $h(t)/t$ is bounded;

(vi) $g(t)\leq t$ and $g(t)\to\infty$ as $t\to\infty$.

A function $y(t)\in C[A, \infty)$ is said to be nonoscillatory, if it eventually assumes a constant sign for arbitrarily large values of t; otherwise it is called oscillatory.

The existence of the continuously extendable solutions of equation (1) will be taken for granted. From here on the term "solution" applies only to such solutions on $[A, \infty)$.

Recently T. Kusano and H. Onose [4] studied the equation

$$(2)\qquad (r(t)y'(t))' + a(t)h(y(g(t))) = f(t)$$

under practically similar assumptions and showed that bounded nonoscillatory solutions of (2) would approach to zero if

$$\int^\infty R(t)a^-(t)dt < \infty, \qquad \int^\infty R(t)a^+(t)dt = \infty$$

and $\int^\infty R(t)|f(t)|dt<\infty$. It will be shown in this manuscript that these conditions are strong enough to cause all nonoscillatory solutions of (2) to approach zero. In the process known results of Hammett [3], Grimmer [2], Londen [5], and this author and Dahiya [7] are generalized. It is interesting to note that Hammett's study of the equation

$$(r(t)y'(t))' + a(t)h(y(t)) = f(t) \tag{3}$$

via a theorem of Bhatia [1] does not extend to equations of the type (2), let alone equation (1), as was observed by Travis [11]. Also see this author [8].

Equations of type (1) are encountered in the study of perturbed combustion inside a rocket engine. Norkin [6] gives the specific equation

$$x''(t) + \alpha x'(t) + \beta x(t) + \delta x(t - \tau(t)) = \gamma, \tag{4}$$

$\alpha > 0$, $\beta \geq 0$, $\delta > 0$. $\tau(t) > 0$ indicates the delay in combustion in the chamber after fuel injection. $x(t)$ represents the perturbation of injection velocity.

2. Main results

Our first lemma puts a bound on the growth of nonoscillatory solutions of (1).

LEMMA 1. *Suppose*

$$q(t) - p'(t) \geq 0, \qquad t \in [A, \infty), \tag{5}$$

$$\int^{\infty} |f(s)| ds < \infty, \tag{6}$$

$$\int^{\infty} s\, a^{-}(s) ds < \infty. \tag{7}$$

Let $x(t)$ be a nonoscillatory solution of equation (1), *then $|x(t)|/t$ is bounded in* $[A, \infty)$.

PROOF. Without any loss of generality we can assume that A is large enough so that for $t \geq A$, $x(t)$ is of constant sign and for that matter we can assume that $x(g(t)) > 0$ and $x(t) > 0$ for $t \geq A$. Let $T > A$; integrating equation (1) between T and t we have

$$r(t)x'(t) - r(T)x'(T) + \int_T^t p(s)x'(s)ds + \int_T^t q(s)x(s)ds + \int_T^t a(s)h(x(g(s)))ds = \int_T^t f(s)ds, \tag{8}$$

which gives on further integration and rearrangement of terms

$$r(t)x'(t) - r(T)x'(T) + p(t)x(t) - p(T)x(T) + \int_T^t (q(s) - p'(s))x(s)ds \tag{9}$$

$$+\int_T^t a(s)h(x(g(s)))ds \le \int_T^t |f(s)|ds.$$

Let $a^+(t)=\max[a(t), 0]$, $a^-(t)=\max[-a(t), 0]$. From (9) we have

$$r(t)x'(t) \le K_0+\int_T^t a^-(s)h(x(g(s)))ds+\int_T^t |f(s)|ds \tag{10}$$

since $q(t)-p'(t)\ge 0$, $x(t)>0$ and we have set $K_0=r(T)x'(T)+p(T)x(T)$. Dividing (10) by $r(t)$ and integrating between T and $g(t)$ we get

$$\begin{aligned}x(g(t)) &\le x(T)+K_0\int_T^{g(t)} 1/r(s)ds+\int_T^{g(t)} 1/r(s)\int_T^s a^-(y)h(x(g(y)))dyds \\ &\quad +\int_T^{g(t)} 1/r(s)\int_T^s |f(y)|dyds \\ &\le x(T)+K_0\int_T^t 1/r(s)ds+\int_T^t\int_T^s (a^-(y)h(x(g(y))))/r(y)dyds \\ &\quad +\int_T^t\int_T^s |f(y)|/r(y)dyds.\end{aligned}$$

Therefore

$$\begin{aligned}x(g(t)) &\le K_1+K_2t+K_3\int_T^t ((t-s)a^-(s)x(g(s))/r(s)ds \\ &\quad +\int_T^t (|f(s)|(t-s))/r(s)ds,\end{aligned} \tag{11}$$

where

$$K_1=x(T),\ \left(\int_T^t 1/r(s)ds\right)/t\le K_2$$

by condition (iv) of this lemma and $h(t)/t\le K_3$ as assumed earlier. From (11) we obtain

$$\begin{aligned}x(g(t))/t &\le K_1/t+K_2+K_3\int_T^t (a^-(s)x(g(s))/r(s)ds \\ &\quad +\int_T^t |f(s)|/r(s)ds,\end{aligned}$$

which gives

$$x(g(t))/t\le K_4+K_3\int_T^t \frac{sa^-(s)}{r(s)}\ \frac{x(g(s))}{s}ds, \tag{12}$$

where

$$K_4 \geq K_1/t + K_2 + \int_T^t |f(s)|/r(s)ds, \qquad t \geq T.$$

Since $r(t) \geq K$, from (12) by Groanwall's inequality we obtain (in a manner of Y. P. Singh [10]),

$$x(g(t))/t \leq K_5, \tag{13}$$

where $K_5 > 0$ is some constant. The lemma is proved.

REMARK. It is interesting to note the control over the negative nature of $a(t)$, controls the growth of the nonoscillatory trajectories of equation (1).

LEMMA 2. *Suppose conditions* (5)–(7) *of Lemma* 1 *hold. Further suppose that $p(t)$ is bounded and*

(14) *there exists a constant K_6 such that* $\liminf_{t\to\infty} \int_t^{t+k_6} a^+(t)dt \geq E$, *where $E > 0$ and $K_6 > 0$.*

Let $x(t)$ be a nonoscillatory solution of equation (1). *Suppose* $\lim_{t\to\infty} x(t) \neq 0$. *Then* $\lim_{t\to\infty} (r(t)x'(t) + p(t)x(t)) = 0$.

PROOF. Without any loss of generality, suppose $x(t)$ is eventually positive. Let $T > A > 0$ be large enough so that for $t \geq T$, $x(t) > 0$, $x(g(t)) > 0$ and (5) holds. From (9) in the proof of Lemma 1, we have

$$r(t)x'(t) - r(T)x'(T) + p(t)x(t) - p(T)x(T) + \int_T^t (q(s) - p'(s))x(s)ds$$
$$+ \int_T^t a(s)h(x(g(s)))ds \leq \int_T^t |f(s)|ds.$$

This gives

$$r(t)x'(t) - r(T)x'(T) + p(t)x(t) - p(T)x(T) + \int_T^t a^+(s)h(x(g(s)))ds \tag{15}$$
$$+ \int_T^t (q(s) - p'(s))x(s)ds \leq \int_T^t |f(s)|\, ds$$
$$+ \int_T^t sa^-(s)\frac{h(x(g(s)))}{x(g(s))}(x(g(s)))/s ds.$$

Now $h(t)/t$ is bounded and

$$\int_T^t (s\, a^-(s))ds < \infty \qquad \text{as} \quad t \to \infty.$$

Since the conclusion of Lemma 1 and conditions (5)–(7) make the right hand side of (15) bounded we must have

(16) $$\int_T^\infty a^+(t)h(x(g(t)))dt < \infty,$$

(17) $$\int_T^\infty (q(t)-p'(t))x(t)dt < \infty$$

because if either of (16) and (17) is infinite then $r(t)x'(t)\to-\infty$ as $t\to\infty$. Since $r(t)$ is bounded $x'(t)\to-\infty$ and $x(t)<0$ eventually, a contradiction. Hence (16) and (17) hold. Condition (14) implies

(18) $$\int_T^\infty a^+(t)dt=\infty.$$

From (16) and (18) we have

(19) $$\liminf_{t\to\infty} x(t) = \liminf_{t\to\infty} x(g(t)) = 0.$$

Now $x'(t)$ must be oscillatory. In fact if $x'(t)$ is nonoscillatory then from (19) we have $\lim_{t\to\infty} x(t)=0$, a contradiction since by hypothesis $\lim_{t\to\infty} x(t)$ does not approach zero. Let

(20) $$\limsup_{t\to\infty} x(t) = \limsup_{t\to\infty} x(g(t)) > 2d > 0$$

for some $d>0$. Let $0<p(t)\le M$. Let $e>0$ be small enough so that $e/M<d$. There exists large point $T_0>T$ such that

(21) $$\int_{T_0}^\infty a^-(s)\cdot s ds < e/4L, \qquad h(t)/t \le L,$$

(22) $$\int_{T_0}^\infty a^+(t)h(x(g(t)))dt < e/4,$$

(23) $$\int_{T_0}^\infty (q(t)-p'(t))x(t)dt < e/4,$$

(24) $$\int_{T_0}^\infty |f(t)|dt < e/4.$$

Due to (19) and (20) there exist points T_1 and T_2, $T_2>T_1>T_0$ such that

(25) $$x(T_1) > e/M,$$

(26) $$x(T_2) < e/4M.$$

Let $[S_1, S_2]$ be the largest interval around T_2 such that

(27) $$x(t) < e/3M, \qquad t\in(S_1, S_2),$$

and

(28) $$x(S_1) = x(S_2) = e/3M.$$

Then

(29) $$S_2 > S_1 \geq T_1 > T_0.$$

Let

(30) $$x(S_3) = \min x(t), \quad t \in [S_1, S_2], \quad x'(S_3) = 0, \quad S_3 > T_0.$$

Replacing T by S_3 in (15) we get

(31) $$|r(t)x'(t) + p(t)x(t)| \leq \int_{S_3}^{t} (q(s) - p'(s))x(s)ds + \int_{S_3}^{t} a^+(s)h(x(g(s)))ds$$
$$+ p(S_3)x(S_3) + LK_5\int_{S_3}^{\infty} sa^-(s)ds + \int_{S_3}^{t} |f(s)|ds,$$

where

$$\frac{h(x(g(s)))}{x(g(s))} \leq L, \text{ and } \frac{x(g(s))}{s} \leq K_5 \text{ by Lemma 1.}$$

From (31) using (21)–(24) we get

$$|r(t)x'(t) + p(t)x(t)| \leq e/4 + e/4 + (e/4M) \cdot M + 2e/4 \cdot K_5$$
$$= e(1 + K_5).$$

Hence

$$\lim_{t \to \infty} (r(t)x'(t) + p(t)x(t)) = 0,$$

and the proof is complete.

THEOREM 1. *Suppose* (5)–(7) *of Lemma* 1 *and* (14) *of Lemma* 2 *hold. Suppose $p(t)$ is bounded. Then all nonoscillatory solutions of equation* (1) *approach to zero asymptotically.*

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1). We proceed as in the proof of Lemma 2 and arrive at conclusions (16), (17), (18), (19) and (20) for $t \geq T > A$. We shall use the proof of theorem 1 of this author [8, p. 267] with minor modifications. From (19) and (20) there exists a sequence $\{t_n\}_{n=0}^{\infty}$ such that

(32) $$t_n \to \infty \quad \text{as} \quad n \to \infty, \qquad t_n \geq T \qquad \text{for} \quad n \geq 0;$$

(33) $$x(g(t_n)) > 2d, \qquad n \geq 1;$$

(34) for each $n \geq 1$, there exists t_n' such that $t_{n-1} < t_n' < t_n$ and $x(g(t_n')) < d/2$.

Let $[\alpha_n, \beta_n]$ be the largest interval around t_n such that for $n \geq 1$

$$x(g(\alpha_n)) = x(g(\beta_n)) = d, \quad x(g(t)) > d, \quad t \in (\alpha_n, \beta_n). \tag{35}$$

Now in the interval $[\alpha_n, t_n]$, there exists a number S_n such that

$$x'(g(S_n))g'(S_n) = [x(g(t_n)) - x(g(\alpha_n))]/(t_n - \alpha_n). \tag{36}$$

Multiplying by $r(S_n)$ and adding $p(S_n)x(g(S_n))g'(S_n)$ on the left in (36) we have

$$\begin{aligned} r(S_n)x'(g(S_n))g'(S_n) + p(S_n)x(g(S_n))g'(S_n) \\ \geq \frac{x(g(t_n)) - x(g(\alpha_n))}{t_n - \alpha_n} r(S_n). \end{aligned} \tag{37}$$

From (37) we have

$$(r(S_n)x'(g(S_n)) + (p(S_n)x(g(S_n))))g'(S_n) \geq (2d-d)K/(\beta_n - \alpha_n) \tag{38}$$

since $\beta_n - \alpha_n \geq t_n - \alpha_n$, $x(g(t_n)) \geq 2d$, $x(g(\alpha_n)) = d$ and $r(t) \geq K$. Since $g'(S_n)$ is bounded and nonnegative and since the lefthand side of (38) tends to zero as $t \to \infty$, we have from (38)

$$\lim_{n \to \infty} (\beta_n - \alpha_n) = \infty. \tag{39}$$

From (16) and the fact that $h(t)$ is increasing we have

$$\begin{aligned} \infty &> \int_T^\infty a^+(t)h(x(g(t)))dt \\ &> \int_{\alpha_n}^{\beta_n} a^+(t)h(x(g(t)))dt \\ &\geq h(d)\int_{\alpha_n}^{\beta_n} a^+(t) \to \infty \qquad \text{as } n \to \infty \end{aligned}$$

in view of condition (14) and conclusion (39). This contradiction proves the theorem.

EXAMPLE 1. Consider the equation

$$y''(t) + a(t)y((3\log t)^{1/3}) = f(t), \tag{40}$$

where

$$a(t) = \begin{cases} \sin t, & n\pi \leq t \leq (n+1)\pi, \\ \exp(-t)\sin t, & (n+1)\pi \leq t \leq (n+2)\pi, \end{cases}$$

$$f(t) = \begin{cases} (\sin t)/t^3 + 9t^4 e^{-t^3} - 6te^{-t^3}, & n\pi \leq t \leq (n+1)\pi, \\ 9t^4 e^{-t^3} - 6te^{-t^3} + (e^{-t}\sin t)/t^3, & (n+1)\pi \leq t \leq (n+2)\pi. \end{cases}$$

Then

$$a^+(t) = \begin{cases} \sin t, & n\pi \le t \le (n+1)\pi, \\ 0, & \text{elsewhere}, \end{cases}$$

$$a^-(t) = \begin{cases} -\exp(-t)\sin t, & (n+1)\pi \le t \le (n+2)\pi, \\ 0, & \text{elsewhere}, \end{cases}$$

$n=2, 3, 4, \dots$.

All conditions of theorem 1 are satisfied. Hence all nonoscillatory solutions of equation (40) approach zero as $t \to \infty$. In fact $y(t)=e^{-t^3}$ is one such solution of (40).

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