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On Distributions Measured by the Riemann-Liouville Operators Associated with Homogeneous Convex Cones

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Introduction

This article deals with Riemann-Liouville operators associated with homogeneous convex cones V which have been studied by M. Riesz [5], L. Gårding [2] and S. G. Gindikin [3], and sets up a theory of distributions measured by the operators $\mathscr{P}^{\rho}_{V\pm}$ (as for the definition, see (1–5) of § 1).

È. B. Vinberg, S. G. Gindikin, and I. I. Pyateckiĭ-Šapiro [9] have proved that every complex bounded homogeneous domain is analytically equivalent to an affine-homogeneous Siegel domain of the first or second kind, and then it is easy to prove that any affine homogeneous real domain is affine-equivalent to a convex linear homogeneous cone or a real Siegel domain (cf. [3], [8]).

We shall define a homogeneous distribution associated with a homogeneous domain D. Let G be a group to act transitively on D. Then G operates on the Schwartz space defined on D such that

$$(0\text{–}1) \qquad (f, g) \longmapsto f^g \quad \text{defined by} \quad f^g(x) = f(gx),$$

and by the duality, on the distribution space such that

$$(0\text{–}2) \qquad (g, \Delta) \longmapsto g\Delta \quad \text{defined by} \quad (g\Delta)(f) = \Delta(f^g).$$

Let ω be a one-dimensional representation on G. If the distribution Δ satisfies the relation

$$(0\text{–}3) \qquad g\Delta = \omega(g)^{-1}\Delta$$

for any $g \in G$, it is called a homogeneous distribution associated with D (cf. [10]). Then the homogeneous distribution Δ is extended to the whole space such that Δ_+ is equal to Δ on D, and to 0 for other else. If a set M of homogeneous distributions depends on a parameter, and for any Δ_α, Δ_β in M the convolution operator $(\Delta_\alpha)_+ * (\Delta_\beta)_+$ is well defined and equals $(\Delta_{\alpha+\beta})_+$ in M, an operator $(\Delta_{\alpha+} f) \longmapsto \Delta_{\alpha+} * f$ is called the Riemann-Liouville operator associated with the domain D. Therefore the operator $\mathscr{P}^{\rho}_{V+}$ is one of canonical Riemann-Liouville operators, and satisfies the Huygens principle (cf. [7]).

The distribution theory of L. Schwartz is developed for each linear form T on $C_0^\infty(\Omega)$ such that to every compact set $K\subset\Omega$ there exist constants C and k satisfying

(0–4) $$|<T, \phi>| < C \sum_{|\alpha|\leq k} \sup|D^\alpha \phi| \qquad \phi\in C_0^\infty(K).$$

On the other hand, we define the distribution measured by the Riemann-Liouville operator $\mathscr{P}_{V+}^{\alpha}$. Namely, we consider the linear form T on $C_0^\infty(\Omega)$ satisfying

(0–5) $$|<T, \phi>| < C \sum_{\{\alpha^*\}<k} \sup|\mathscr{P}_{V+}^{\alpha}\phi| \qquad \phi\in C_0^\infty(K).$$

We shall prove the local representation theorem and the Paley-Wiener theorem attached to this distribution.

As is well-known, the local representation theorem states that distributions of finite order with compact support are represented by a sum of partial derivatives to all directions of some L^2-functions. In this paper, we obtain that distributions of V-order finite with compact support are represented by a sum of partial derivatives by $\mathscr{P}_{V+}^{\rho}$ of some L^2-functions. For this purpose, the "Sobolev inequality" obtained by the author [12] plays an essential role.

The Paley-Wiener theorem states that the duality of the Fourier-Laplace transform between the distributions of finite order with support in the sphere

(0–6) $$S_r = \{x;\ |x| < r\}$$

and the entire functions $F(\omega)$ estimated by

(0–7) $$(1 + |\omega|)^n \exp r|\mathrm{Im}\,\omega| \qquad n > 0 \text{ integer}.$$

The duality of the Fourier-Laplace transform between the distributions of V-order finite with support in the compact set

(0–8) $$[a, b] = \{x\in\mathbf{R}^n;\ x - a\in\overline{V},\ b - x\in\overline{V}\}$$

for certain vectors a, $b\in\mathbf{R}^n$, and the entire functions $F(\omega)$ estimated by

(0–9) $$\begin{cases} \sum_{\{-\alpha^*\}<m} |(-i\omega)_*^{-\alpha^*}|\exp(b, \mathrm{Im}\,\omega) & \mathrm{Im}\,\omega\in V^*, \\ \sum_{\{-\alpha^*\}<m} |(-i\omega)_*^{-\gamma^*}|\exp(a, \mathrm{Im}\,\omega) & \mathrm{Im}\,\omega\in -V^*. \end{cases}$$

Here $(-i\omega)_*^{-\alpha^*}$ is the polynomial associated with the dual cone V^* of V (see § 1). Therefore, in our theorem, we characterize the entire functions having the following properties:

(a) it is estimated in the subdomain $\mathbf{R}^n \pm iV^*$.
(b) it is estimated by the only polynomial $(-i\omega)_*^{-\alpha^*}$.

However we note that the Paley-Wiener-Schwartz theorem characterizes the entire functions estimated by the polynomial of $|\omega|$ in $\mathbf{C}^n$.

We turn now to a survey of the contents of the present article section by section.

Section 1 deals with a definition of distributions measured by $\mathscr{P}^{\rho}_{V+}$. It is called distributions of V-order finite.

Section 2 deals with the local representation theorem.

Section 3 presents a relation between distributions of V-order finite with compact support and entire functions. It is called the Paley-Wiener type theorem.

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§ 1. Definition of a distribution of V-order finite

Let V be a convex linear homogeneous cone in $\mathbf{R}^n$ not containing straight lines, and let Ω be an open set in $\mathbf{R}^n$. In this section we shall distinguish among distributions defined on Ω a class of distributions which can be measured by means of the Riemann-Liouville operators $\mathscr{P}^{\rho}_{V+}$ associated with the cone V. We shall call elements of this class distributions of V-order finite.

The precise formulation of the Riemann-Liouville operator associated with V has been given in the work [3] of S. G. Gindikin (cf. also M. Riesz [5] and L. Gårding [2]). We begin with summarizing his formulation of the Riemann-Liouville operator.

È. B. Vinberg [8] has proved that in the group of linear transformations of V it is always possible to select a simply transitive subgroup $G(V)$ whose elements can be represented by triangular matrices in a suitable basis. Then by fixing a particular point $e\in V$, it is possible to transfer to V the multiplicative structure of the group $G(V)$ by setting:

$$x_1x_2=g(x_1)x_2,\quad \text{where}\quad g(x_1)e=x_1\quad g(x_1)\in G(V). \tag{1-1}$$

We call functions satisfying the condition

$$f(x_1x_2)=f(x_1)f(x_2) \tag{1-2}$$

compound power functions. They form a multiplicative group in which we can choose l generators $\chi_i(x)$, l being the dimension of the diagonal subgroup of $G(V)$ (i.e. the rank of the cone V). Each compound power function $f(x)$, normalized by the condition $f(e)=1$, is specified by

$$f(x)=x^{\rho}=\prod_{i=1}^{l}(\chi_i(x))^{\rho_i}. \tag{1-3}$$

It is essential that $\chi_i(x)>0$ when $x\in V$.

If $L(x)$ is a linear functional on $\mathbf{R}^n$, the function $\exp(L(x))$ serves as an analogue of the exponential function. A particular form $L_0(x)=(e, x)$ is fixed henceforth.

Now the Siegel integral of the second kind (the gamma function for the cone V) is defined by the formula

$$(1\text{–}4) \qquad \Gamma_V(\rho) = \int_V \exp-(e, x)x^{\rho+d}dx, \qquad \rho \in \mathbf{C}^l,$$

where dx is the Euclidean measure and $x^d dx$ is the invariant measure on $\mathbf{R}^n$ with respect to $G(V)$. The function $\Gamma_V(\rho)$, which is unique to within a factor not depending on ρ, is a product of one-dimensional gamma functions.

Now we can construct the Riemann-Liouville operator associated with the cone V:

$$(1\text{–}5) \qquad (\mathscr{P}^\rho_{V\pm}f)\ (x) = \frac{x_\pm^{\rho+d}}{\Gamma_V(\rho)} *f \qquad f \in \mathscr{S}(\mathbf{R}^n),$$

where the function $x_+^{\rho+d}$ equals $x^{\rho+d}$ for $x \in V$, zero for other else, and $x_-^{\rho+d}$ equals $(-x)_+^{\rho+d}$. If the operator (1–5) is a partial differential operator in a usual sense, that is, its Fourier-Laplace transform is a multiplication by a polynomial, the vector ρ is called V-integral.

We shall finish to summarize it after noting an equation universal in application.

When we denote by x_*^ρ the compound power function associated with the dual cone V^* of V, it can be continued to the Siegel domain of the first kind in virtue of the equation

$$(1\text{–}6) \quad \int_V \exp(-(z, x))x^{\rho+d}dx = \Gamma_V(\rho)z_*^{-\rho^*} \qquad \operatorname{Re} z \in V^*$$

$$\operatorname{Re} \rho_i > \frac{m_i}{2} \qquad \rho^* = (\rho_l, \dots, \rho_1),$$

where m_i is a positive integer associated with V.

Now we shall define a distribution of V-order finite:

DEFINITION 1–1. A distribution T in $\mathscr{D}'(\Omega)$ is called to be of V-order m in $\mathscr{D}'(\Omega)$ if to every compact set K in Ω there exist a constant C and a positive integer m such that

$$(1\text{–}7) \qquad |\langle T, \phi\rangle| < C \sum_{\{-\alpha^*\}<m} \sup |\mathscr{P}^\alpha_{V+}\phi|, \qquad \phi \in C_0^\infty(K).$$

DEFINITION 1–2. A distribution T in $\mathscr{E}'(\Omega)$ is called to be of V-order m in $\mathscr{E}'(\Omega)$ if to some compact set K in Ω there exist a constant C and a positive integer m such that

(1-8) $$|<T, \phi>| < C \sum_{\{-\alpha^*\}<m} \sup_K |\mathscr{P}^{\alpha}_{V+}\phi|, \qquad \phi \in \mathscr{E}(\Omega).$$

Here the representation $\{-\alpha^*\}<m$ denotes a set of all V-integral vectors α such that $|-\alpha^*| = -\sum_{i=1}^{l} \alpha_i$ is less than the positive integer m.

REMARK. If a distribution in $\mathscr{E}'(\Omega)$ is of V-order m in $\mathscr{E}'(\Omega)$, it is of V-order m in $\mathscr{D}'(\Omega)$.

§2. Representation theorem

In this section we shall obtain a representation theorem of distributions of V-order finite in $\mathscr{E}'(\Omega)$. For this purpose we introduce some function space:

DEFINITION 2–1. We denote by $\mathring{H}^m_V(\Omega)$ the closure of $C_0^\infty(\Omega)$ with the norm

(2-1) $$\|f\|^2_{V,m} = \sum_{\{-\alpha^*\}<m} \|\mathscr{P}^{\alpha}_{V+}f\|^2_{L^2(\Omega)} \qquad < \infty.$$

We easily see that the space $\mathring{H}^m_V(\Omega)$ is a Hilbert space with the inner product $(f, g)_{V,m} = \sum_{\{-\alpha^*\}<m} (\mathscr{P}^{\alpha}_{V+}f, \mathscr{P}^{\alpha}_{V+}g)_{L^2(\Omega)}$ for f and g in $\mathring{H}^m_V(\Omega)$. This space $\mathring{H}^m_V(\Omega)$ may be considered as the Sobolev space measured by the Riemann-Liouville operators, and has some properties similar to $\mathring{H}^m(\Omega)$.

REMARK. Any element T in the dual space $(\mathring{H}^m_V(\Omega))'$ is represented by

(2-2) $$T = \sum_{\{-\alpha^*\}<m} \mathscr{P}^{\alpha}_{V+}f_\alpha \qquad \text{for some} \quad f_\alpha \in L^2(\Omega).$$

Conversely the right term of (2-2) is an element in $(\mathring{H}^m_V(\Omega))'$.

We begin with proving a proposition about a convergent sequence in $\mathring{H}^m_V(\Omega)$.

PROPOSITION 2–2. *Suppose a sequence $\{f_j\}$ in $\mathring{H}^m_V(\Omega)$ is weakly convergent to some element f in $\mathring{H}^m_V(\Omega)$. Then for any ϕ in $L^2(\Omega)$*

(2-3) $$\lim_{j\to\infty} (\mathscr{P}^{\alpha}_{V+}f_j, \phi)_{L^2(\Omega)} = (\mathscr{P}^{\alpha}_{V+}f, \phi)_{L^2(\Omega)} \qquad \{-\alpha^*\} < m.$$

Conversely if for any ϕ in $L^2(\Omega)$ and a sequence $\{f_j\}$ in $\mathring{H}^m_V(\Omega)$, there exist some elements f_α in $L^2(\Omega)$ satisfying

(2-4) $$\lim_{j\to\infty} (\mathscr{P}^{\alpha}_{V+}f_j, \phi)_{L^2(Q)} = (f, \phi)_{L^2(\Omega)} \qquad \{-\alpha^*\} < m,$$

there exists an element f in $\mathring{H}^m_V(\Omega)$ such that

(2-5) $$f_\alpha = \mathscr{P}^{\alpha}_{V+}f \qquad \{-\alpha^*\} < m$$

and the sequence $\{f_j\}$ *is weakly convergent to* f.

We can prove the proposition by an argument similar to the case of the usual Sobolev space, so we omit the proof (cf. S. Mizohata [4] p. 78).

In the following we suppose that Ω is a bounded open set. When we shall study the distribution of V-order finite, the next inequality, which is obtained by the author (see [12], (1–11)), is essential and performs the role analogous to the Sobolev inequality.

LEMMA 2–3. *Let* Ω *be a bounded domain in* $\mathbf{R}^n$, *and let* a_0 *be a vector satisfying* $\Omega+a_0\subset V$. *Then we have for any* f *in* $L^2(\Omega)$ *and a fixed vector* $-\eta$ *in* V^*.

$$\text{(2–6)}\qquad \sup|e^{2(x,\eta)}(\mathscr{P}^{\alpha}_{V+}f)(x-a_0)| \leqq \left(\frac{\Gamma_V(2\alpha+d)}{\Gamma^2_V(\alpha)}(-2\eta)_*^{-2\alpha^*-d^*}\right)^{1/2}$$

$$\times\|f\|_{L^2(\Omega)},\ \alpha_i>\frac{1}{2}\left(-d_i+\frac{m_i}{2}\right)\qquad i=1,\dots,l.$$

PROPOSITION 2–4. *Suppose* K_0 *is a compact subset in* Ω. *Then we have the following assertions*:

i) *Let* B *be a bounded set of* V*-order* m *in* $\mathscr{E}'(K_0)$. *Then there exists a positive integer* m_0 *such that* B *is a bounded set in* $(\mathring{H}^{m+m_0}_V(\Omega))'$. *Also the converse is true.*

ii) *If a sequence* $\{T_j\}$ *of distributions and* T *in* $\mathscr{E}'(\Omega)$ *are of* V*-order* m *in* $\mathscr{E}'(K_0)$ *and the sequence* $\{T_j\}$ *converges to* T *in a sense of* V*-order* m, *there exists a positive integer* m_0 *such that the subsequence* $\{T_j\}$ *and* T *are contained in* $(\mathring{H}^{m+m_0}_V(\Omega))'$ *and* $\{T_j\}$ *is* $(\mathring{H}^{m+m_0}_V(\Omega))'$*-simply convergent to* T. *Also the converse is true.*

In this proposition the bounded set of V-order m means that there exists a compact set K in K_0 and a positive number η such that

$$\text{(2–7)}\qquad \sup|<T,\phi>|<1$$

for any $\phi\in\mathscr{E}(K_0)$ satisfying $\sum_{\{-\alpha^*\}<m}\sup_K|\mathscr{P}^{\alpha}_{V+}\phi|<\eta$. If we observe that a set $C^\infty_0(\Omega)$ is dense in $\mathring{H}^m_V(\Omega)$, we see the proposition from Lemma 2–3.

Now we obtain the main result in this section which represents the local property of the distribution of V-order finite.

THEOREM 2–5. *Let* K_0 *be a compact subset in* Ω. *Then we have the following assertions*:

i) *If a set* B *is bounded in a topology of* V*-order* m *in* $\mathscr{E}'(K_0)$, *there exists a positive integer* m_0 *such that any element* T *in* B *is represented by* $\sum_{\{-\alpha^*\}<m+m_0}\mathscr{P}^{\alpha}_{V+}f_\alpha$

for some f_α *in* $L^2(\Omega)$ *and also the set of all* f_α *corresponding to* B *is bounded in* $L^2(\Omega)$.

ii) *Suppose a sequence* $\{T_j\}$ *is convergent of V-order* m *to an element* T *in* $\mathscr{E}'(K_0)$. *If we set* $T_j = \sum_{\{-\alpha^*\}<m+m_0} \mathscr{P}^{\alpha}_{V+} f^{(j)}_{\alpha}$ *and* $T = \sum_{\{-\alpha^*\}<m+m_0} \mathscr{P}^{\alpha}_{V+} f_\alpha$, *the sequence* $f^{(j)}_{\alpha}$ *is weakly convergent to* f_α *in* $L^2(\Omega)$.

PROOF. i) For any ϕ in $C^\infty_0(\Omega)$ we see from Proposition 2–4 and the Riesz theorem that there exists a unique g in $\mathring{H}^{m+m_0}_V(\Omega)$ which satisfies

$$
\text{(2–8)} \qquad < T, \phi > = (\phi, g)_{V,m+m_0} = \sum_{\{-\alpha^*\}<m+m_0} (\mathscr{P}^{\alpha}_{V+}\phi, \mathscr{P}^{\alpha}_{V+} g)_{L^2(\Omega)}
$$

$$
= \sum_{\{-\alpha^*\}<m+m_0} < \mathscr{P}^{\alpha}_{V+}\mathscr{P}^{\alpha}_{V-}\bar{g}, \phi > .
$$

Then setting $f_\alpha = \mathscr{P}^{\alpha}_{V-}\bar{g}$, we conclude the proof.

ii) follows from Proposition 2–2 and Proposition 2–4. Q. E. D.

§ 3. Theorem of the Paley-Wiener type

As is well known, the Paley-Wiener theorem expresses the relation between entire functions behaving like $(1+|z|)^n \times \exp A|\mathrm{Im}\, z|$ ($n>0$ an integer, $A>0$) near the infinity and the distribution with compact support by means of the Fourier $=$Laplace transform.

In this section using the Fourier-Laplace transform, we shall consider a relation between entire functions increasing in a particular polynomial near the infinity and the distributions of V-order finite. We call this result the Paley-Wiener type theorem associated with the cone V.

First of all we shall introduce some function space:

DEFINITION 3–1. The function space $\mathscr{E}^m_V [a, b]$ is the set of all continuous functions f such that for $\{-\alpha^*\}<m$, $\mathscr{P}^{\alpha}_{V+} f$ is continuous in $[a, b]$.

The expression $[a, b]$ denotes a set of all $x \in \mathbf{R}^n$ satisfying $x-a \in \overline{V}$ and $b-x \in \overline{V}$.

REMARK. The space $\mathscr{E}^m_V [a, b]$ is a Fréchet space with a seminorm $P_k(\phi) = \sum_{\{-\alpha^*\}<m} \sup_{V_k \cap [a,b]} |\mathscr{P}^{\alpha}_{V+}\phi|$ where $V_k = \{x;\ |x| \leqq k\}$.

Henceforth we assume that the set $[a, b]$ is compact.

We begin with proving a lemma

LEMMA 3–2. *If a distribution* T *in* $\mathscr{E}'\,[a, b]$ *is of V-order* m *in* $\mathscr{E}'[a, b]$, *we have the expression*

$$(3\text{–}1)\qquad < T, \phi > = \sum_{\{-\alpha^*\}<m} \int_{[a,b]} \mathscr{P}^{\alpha}_{V+}\phi(x)\mu_{\alpha}(dx) \qquad \phi \in \mathscr{E}[a, b],$$

where $\mu_\alpha(dx)$ are some complex Baire measure on $[a, b]$.

PROOF. Since the space $\mathscr{E}[a, b]$ is dense in $\mathscr{E}^m_V[a, b]$, we can extend T as a continuous linear form on $\mathscr{E}^m_V[a, b]$. Therefore the distribution T is a continuous linear form on the product space $\prod_{\{-\alpha^*\}<m} \mathscr{E}^\circ[a, b]$, and then from the Riesz theorem we get the lemma (cf. K. Yosida [11] p. 119).

Now we arrive at a main result, that is, the theorem of the Paley-Wiener type associated with the cone V. We divide it into two parts.

THEOREM 3–3A. *Let T be a distribution of V-order m in $\mathscr{E}'[a, b]$. Then the Fourier-Laplace transform $\mathscr{L}(T)(\omega)=\mathscr{L}(T)(\xi+i\eta)$ is an entire function which satisfies for some positive constant C*

$$(3\text{–}2)\qquad |\mathscr{L}(T)(\omega)| \leqq \begin{cases} C \sum_{\{-\alpha^*\}<m} |(-i\omega)^{-\alpha^*}_*|\, e^{(b,\eta)} & \eta \in V^*, \\ C \sum_{\{-\alpha^*\}<m} |(-i\omega)^{-\alpha^*}_*|\, e^{(a,\eta)} & -\eta \in V^*. \end{cases}$$

PROOF. Since the distribution T is in $\mathscr{E}'[a, b]$, the Fourier-Laplace transform $\mathscr{L}(T)(\omega)$ equals $<e^{-i(x,\varpi)}, T>$, and from (1–5) and Lemma 3–2, we obtain

$$(3\text{–}3)\qquad \mathscr{L}(T)(\omega) = \sum_{\{-\alpha^*\}<m} \int_{[a,b]} \mathscr{P}^{\alpha}_{V+} e^{-i(x,\varpi)}\, \mu_\alpha(dx)$$
$$= \sum_{\{-\alpha^*\}<m} (-i\omega)^{-\alpha^*}_* \int_{[a,b]} e^{-i(x,\varpi)}\, \mu_\alpha(dx) \qquad -\eta \in V^*.$$

Since α is a V-integral vector, the function $(-i\omega)^{-\alpha^*}_*$ is a polynomial of each component of ω, and so the Fourier-Laplace transform $\mathscr{L}(T)(\omega)$ is an entire function satisfying (3–3) for any η in $\mathbf{R}^n$. If the vector η is in V^*, we have

$$(3\text{–}4)\qquad (x, \eta) \leqq (b, \eta) \qquad x \in [a, b].$$

Also if the vector η is in $-V^*$, we have

$$(3\text{–}5)\qquad (x, \eta) \leqq (a, \eta) \qquad x \in [a, b].$$

Therefore using (3–3), we have the estimate (3–2) from (3–4) and (3–5). We conclude the proof.

To show the converse of Theorem 3–3A, we prepare a lemma. S. G. Gindikin [3] noted that every point x in $\mathbf{R}^n$ can be represented in the form

$$(3\text{–}6)\qquad x = {}^\delta x(e_\delta) \qquad ({}^\delta x \in G(V)),$$

where e_δ is a diagonal form $(e_{\delta_1}, \dots, e_{\delta_l})$ with elements ± 1 or 0, and if e_δ is a diagonal form with only elements 1 or -1, the representation (3–6) is unique. Then we define the function x_δ^ρ by $({}^\delta x)^\rho$ when each e_{δ_i} is not zero, and by 0 when some e_{δ_i} is zero.

LEMMA 3–4. *For any y in $\mathbf{R}^n \backslash \overline{V}$, where $\overline{V}$ is the closure of V, there exists a vector η in the dual cone V^* such that*

$$(y, \eta) \lneqq 0. \tag{3–7}$$

PROOF. When the function $y_\delta^{[1]}$ ($[1]=(1,\dots,1)$) is not zero, there exists a vector η_1 in V^* such that the vector y is transformed into the diagonal form $\eta_1^* \cdot y=(y_{\delta_1}e_{\delta_1},\dots, y_{\delta_l}e_{\delta_l})$, where η_1^* is the dual vector of η_1 in V with respect to the inner product in $\mathbf{R}^n$. From the hypothesis, there exists an i_0 such that $e_{\delta_{i_0}}=-1$. Therefore we can choose an element η which satisfies $(\eta^* y, e) \lneqq 0$. If the function $y_\delta^{[1]}$ equals zero, there exists some small positive real number ε and δ' such that $(y+\varepsilon e)_{\delta'}^{[1]} \neq 0$ and $e_{\delta'}$ is not e. Then by the above argument we can choose an element η in V^* which satisfies $(y, \eta)=(y+\varepsilon e, \eta)-(\varepsilon e, \eta) \lneqq 0$. We conclude the proof.

By using the lemma, we have the following theorem.

THEOREM 3–3 B. *Suppose an entire function $F(\omega)$ on $\mathbf{C}^n$ satisfies for some positive number C*

$$|F(\omega)| \leqq \begin{cases} C \sum_{\{-\alpha^*\}<m} |(-i\omega)_*^{-\alpha^*}| e^{(b,\eta)} & \eta \in V^*, \\ C \sum_{\{-\alpha^*\}<m} |(-i\omega)_*^{-\alpha^*}| e^{(a,\eta)} & \eta \in -V^*. \end{cases} \tag{3–8}$$

Then there exist a positive integer m_0 and a unique distribution T of V-order $m+m_0$ in a sense of Definition 1–1, *whose support is contained in* $[a, b]$, *such that Fourier-Laplace transform $\mathscr{L}(T)(\omega)$ equals $F(\omega)$.*

PROOF. For any function ϕ in $C_0^\infty(\mathbf{R}^n)$, we write $\check{\phi}(x)=\phi(-x)$, and then $(2\pi)^{-n}\int_{\mathbf{R}^n+i\eta} F(\xi+i\eta)(\mathscr{L}\check{\phi})(\xi+i\eta)d\xi$ is independent of the choice of the vector η. There we define a linear operator T on $C_0^\infty(\mathbf{R}^n)$ by

$$<T, \phi> = (2\pi)^{-n}\int_{\mathbf{R}^n+i\eta} F(\omega)(\mathscr{L}\check{\phi})(\omega)d\xi. \tag{3–9}$$

From (3–8), we see that for $-\eta \in V^*$

$$|<T, \phi>| \leqq C\, e^{(b,\eta)} \sum_{\{-\alpha^*\}<m} \int_{\mathbf{R}^n+i\eta} |(-i\omega)_*^{-\alpha^*}|\, |(\mathscr{L}\phi)(\omega)|\, d\omega. \tag{3–10}$$

Since the function ϕ is in $C_0^\infty(\mathbf{R}^n)$, for any V-integral vector $-\beta$ there exists a positive constant C_ϕ depending on ϕ to satisfy

(3–11) $$|(\mathscr{L}\phi)(\omega)| \leqq C_\phi |(-i\omega)_*^{-\beta^*}| \exp\{-\min_{x \in \operatorname{supp}\phi}(x, \eta)\}.$$

Then if we substitute (3–11) into (3–10), we obtain from (1–6)

(3–12) $$\begin{aligned} |<T, \phi>| &\leqq C_\phi \exp\{(b, \eta) - \min_{x \in \operatorname{supp}\phi}(x, \eta)\} \\ &\quad \times \int_{\mathbf{R}^n + i\eta} |(-i\omega)_*^{-\alpha_0^*}|^2 \, d\omega \\ &= C_\phi \exp\{(b, \eta) - \min_{x \in \operatorname{supp}\phi}(x, \eta)\} \int_V e^{-2(x,\eta)} x^{2(\alpha_0 + d)} \, dx \\ &= C_\phi \exp\{(b, \eta) - \min_{x \in \operatorname{supp}\phi}(x, \eta)\} \Gamma_V(2\alpha_0 + d)(2\eta)_*^{-(2\alpha_0 + d)^*}, \end{aligned}$$

where $|\alpha_0| = \sum_{i=1}^{l} \alpha_{0i}$ is large enough. Further, there exists a vector x_0 satisfying

(3–13) $$\min_{x \in \operatorname{supp}\phi}(x, \eta) = (x_0, \eta).$$

If the vector $b - x_0$ is not in $\bar{V}$, in virtue of Lemma 3–2, there exists a vector η in V^* satisfying $(b - x_0, \eta) \lneqq 0$. Therefore the left term of (3–12) converges to zero, since the vector η tends to infinity along some direction. Hence $\operatorname{supp}(T) \subset (-\infty, b]$. Also we can prove that $\operatorname{supp}(T) \subset [a, \infty)$. These prove that $\operatorname{supp}(T) \subset [a, b]$. From (1–6) and the Hölder inequality, we see that (3–10) becomes

(3–13) $$\begin{aligned} &|<T, \phi>| \\ &\leqq C e^{(b,\eta)} \sum_{\{-\alpha^*\} < m} \left(\int_{\mathbf{R}^n + i\eta} |\mathscr{L}(\mathscr{P}_{V-}^{\alpha + \alpha_0} \phi)(\omega)|^2 d\omega \right)^{1/2} \\ &\quad \times \left(\int_{\mathbf{R}^n + i\eta} |(-i\omega)_*^{\alpha_0^*}|^2 \, d\omega \right)^{1/2} \\ &\leqq C(\phi, \eta) \sum_{\{-\alpha^*\} < m} \left(\int e^{-2(x,\eta)} |\mathscr{P}_{V-}^{\alpha + \alpha_0} \phi(x)|^2 dx \right)^{1/2} \\ &\leqq C(\phi, \eta) \sum_{\{-\alpha^*\} < m} \sup |\mathscr{P}_{V+}^{\alpha + \alpha_0} \phi| \end{aligned}$$

for $\eta \in V^*$ and some suitable vector α_0, where the constant $C(\phi, \eta)$ depends only on the support of ϕ and η. We finish the proof.

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