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On *Dedekind Domains

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In this paper a ring is always a commutative ring with a unit and a graded ring is a ring with a grading of Z-type. Let A be a graded domain. Then the set of non-zero graded fractional ideals of A becomes a monoid naturally. We shall say that A is a Dedekind domain in the category of graded rings when the monoid is a group; it is not so difficult to show that A is a Dedekind domain in the category of graded rings if and only if the global dimension of A in the category of graded A-modules is less than or equal to 1.

The main purpose of this paper is to determine the structure of graded domains of global dimension 1, or equivalently to determine the structure of Dedekind domains in the category of graded rings. To do this we shall introduce a notion of exceptional primes, which plays an important role in this paper. As a main result we shall show that, in case exceptional primes do not appear, there is a 1:1 correspondence between the set of isomorphism classes and the class group of the Dedekind domain consisting of homogeneous elements of degree zero.

1. Graded rings

Let $A=\oplus A_n$, $n \in Z$, be a graded ring. We let $h(A)$ denote the set of homogeneous elements of A. Given a homogeneous element x, $\deg(x)$ stands for the degree of x. Let $\mathrm{Gr}(A)$ be the category of graded modules over A. A morphism f of M to N is an A-homomorphism of M to N such that $f(M_n) \subset N_n$ for every n, where $M=\oplus M_n$, $n \in Z$, and $N=\oplus N_n$, $n \in Z$. An asterisk (*) means 'graded', 'homogeneous' or 'in $\mathrm{Gr}(A)$'. For example, a *module is a graded module, and a *ideal is a homogeneous ideal, and so on. If every *ideal of A is finitely generated, then we say A is *noetherian. When A is *noetherian, A_0 is noetherian and moreover the underlying ring of A is noetherian ([3], p. 306). For an ideal $\mathfrak{a}$ of a *ring, we denote by $\mathfrak{a}^*$ the *ideal generated by all the *elements of $\mathfrak{a}$. If $\mathfrak{a}$ is a prime ideal, then $\mathfrak{a}^*$ is also a prime *ideal.

PROPOSITION 1.1. *Let A be a *domain. If the Jacobson radical is not zero, then $A=A_0$.*

PROOF. Suppose that there exists a non-zero *element b with $\deg(b)\neq 0$.

We take a non-zero element a of the Jacobson radical. Then $1+b^r a$ is a unit for every $r>0$. Since A is a *domain, a unit must be homogeneous. Therefore $1+b^r a$ is homogeneous for every r and this leads to a contradiction.

COROLLARY 1.2. *Let A be a *ring and let $\mathfrak{p}$ be a prime *ideal. If $\mathfrak{p}$ does not contain A_n for some $n \neq 0$, then $\mathfrak{p} = \cap \mathfrak{m}^*$, where $\mathfrak{m}$ runs over all maximal ideals containing $\mathfrak{p}$.*

REMARK. Any local domain has not a non-trivial grading by Prop. 1.1. Moreover we can show that any local reduced ring also has not a non-trivial grading.

Let A be a *ring and let $S \subset h(A)$ be a *multiplicative set not containing 0. Then $S^{-1}A$ is also a *ring in a natural way. If $S = h(A) - \mathfrak{p}$, where $\mathfrak{p}$ is a prime *ideal, we use the symbol $A_{(p)}$ instead of $S^{-1}A$. If A has only one *maximal *ideal, then we say A is *local. It is clear that $A_{(p)}$ is a *local *ring and the unique *maximal *ideal is $\mathfrak{p}A_{(p)}$. We can readily see that A is *local if and only if A_0 is local. In particular we say A is a *field when the *maximal *ideal is trivial. If A is a *field, then A is a field or $A \simeq A_0[T, T^{-1}]$, where A_0 is a field ([2], p. 291). This implies that a *maximal *ideal is a prime *ideal. Similarly to the ungraded case any *module over a *field is *free. If A is a *domain and $S = h(A) - \{0\}$, then $S^{-1}A$ is a *field, which we denote by $^*Q(A)$. Let $\mathfrak{a}$ be an A-*submodule of $^*Q(A)$. $\mathfrak{a}$ is said to be a *fractional *ideal of A if there is a *element d $(\neq 0)$ of A such that $d\mathfrak{a} \subset A$. Let $^*G(A)$ be the set of non-zero *fractional *ideal of A. Given $\mathfrak{a}, \mathfrak{b} \in {}^*G(A)$, $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$ and $\mathfrak{a} : \mathfrak{b}$ belong to $^*G(A)$. $^*G(A)$ becomes a monoid by the operation: $(\mathfrak{a}, \mathfrak{b}) \to \mathfrak{a}\mathfrak{b}$. We say that $\mathfrak{a}$ is *invertible if $\mathfrak{a}$ is an invertible element in the monoid $^*G(A)$. If $\mathfrak{a}$ is *invertible, then $\mathfrak{a}$ is finitely generated.

PROPOSITION 1.3. *Let $\mathfrak{a}$ be an element of $^*G(A)$. Then, $\mathfrak{a}$ is *invertible if and only if it is *projective.*

PROOF. Since $\mathfrak{a}$ has a *element of $^*Q(A)$, $\{\alpha \in {}^*Q(A);\ \alpha\mathfrak{a} \subset A\} = \{\beta \in Q(A);\ \beta\mathfrak{a} \subset A\}$. Hence, $\mathfrak{a}$ is a *invertible *ideal of A if and only if $\mathfrak{a}$ is an invertible ideal of A. It is well known that $\mathfrak{a}$ is invertible if and only if $\mathfrak{a}$ is projective over A, and for a *module it is *projective if and only if it is projective. Thus the proof is obtained.

LEMMA 1.4. *Let A be a *local domain. If a *ideal $\mathfrak{a}$ is *invertible, then $\mathfrak{a}$ is generated by one *element.*

PROOF. The proof is the same as in the ungraded case.

2. *Dedekind domains

In this section A is a *domain. We say that A is *normal if any *element of $*Q(A)$, which is integral over A, belongs to A. Since $*Q(A)$ is normal, the derived normal ring of A is contained in $*Q(A)$. By [1] (§ 1 Prop. 20), the derived normal ring is a *subring of $*Q(A)$. Hence, A is normal if and only if A is *normal. It is reasonable to say that A is a *Dedekind domain if $*G(A)$ is a group. If A is a *Dedekind domain and S is a *multiplicative set, then $S^{-1}A$ is also a *Dedekind domain, because of the surjectivity of the canonical monoid-homomorphism of $*G(A)$ to $*G(S^{-1}A)$. Similarly to the ungraded case, we have:

PROPOSITION 2.1. *Let A be a *domain. Then, A is a *Dedekind domain if and only if A satisfies the following conditions*: (1) *A is *noetherian*; (2) $*\dim(A)\leqq 1$; (3) *A is *normal, where* $*\dim(A)$ *means the maximal length of chains of prime *ideals.*

PROOF. Suppose that A is a *Dedekind domain. Then any *ideal is *invertible, and hence finitely generated. Thus A is *noetherian. Let $\mathfrak{p}$ be any non-zero prime *ideal. Then $A_{(\mathfrak{p})}$ is also a *Dedekind domain. By the *locality the *maximal *ideal of $A_{(\mathfrak{p})}$ is principal. It implies that $*\dim(A)\leqq 1$. Let $x \in h(*Q(A))$ be integral over A. Then, there is a *fractional *ideal $\mathfrak{a}$ ($\neq 0$) such that $x\mathfrak{a}\subset\mathfrak{a}$. $\mathfrak{a}:\mathfrak{a}$, which we denote by $\mathfrak{b}$, is contained in $*G(A)$. Since $\mathfrak{b}^2\subset\mathfrak{b}$ and $\mathfrak{b}$ contains 1, $\mathfrak{b}^2=\mathfrak{b}$. Hence $\mathfrak{b}=A$, for $*G(A)$ is a group. Thus x is an element of A, and A is *normal. Now suppose that A satisfies the conditions (1), (2) and (3). If A is *local and the *maximal *ideal is $\mathfrak{p}$, then $\mathfrak{p}$ is principal. For, let a be a non-zero *element of $\mathfrak{p}$. Then the radical ideal of aA is $\mathfrak{p}$. Since A is *noetherian, $\mathfrak{p}^n$ is contained in aA for some integer n. Hence $A\subsetneqq(A: aA)\subset(A: \mathfrak{p}^n)$. It follows that $A\neq(A: \mathfrak{p})$. Suppose that $\mathfrak{p}(A: \mathfrak{p})=\mathfrak{p}$; then each element of $(A: \mathfrak{p})$ is integral over A. It contradicts the *normality of A. Thus $\mathfrak{p}(A: \mathfrak{p})=A$. Therefore $\mathfrak{p}$ is principal. Since $\bigcap_n \mathfrak{p}^n=0$, any *ideal is principal, and hence $*G(A)$ is a group. Thus A is a *Dedekind domain. The proof of the general case can easily be reduced to the *local case.

Let A be a *Dedekind domain. A is said to be of *the first type* if there is a *ideal which is maximal. A is said to be of *the second type* if it is not of the first type.

PROPOSITION 2.2. *Let A be a *domain such that the underlying ring of A is a Dedekind domain. Then, there are only three cases*: (1) $A=A_0$; (2) $A\simeq A_0[X]$, *where A_0 is a field and X is an indeterminate*; (3) *A is a *field.*

PROOF. Suppose that A is of the second type. Then the zero ideal is a *maximal *ideal of A, and A is a *field. Let A be of the first type and let $\mathfrak{p}$ be a maximal *ideal. $\mathfrak{p}$ contains $A_+ \ (=\bigoplus_{n>0} A_n)$ and $A_- \ (=\bigoplus_{n<0} A_n)$ since $A/\mathfrak{p}$ is a field. If $S=A_0-\mathfrak{p}$, then $S^{-1}A$ is a *local *Dedekind domain. Hence $\mathfrak{p}S^{-1}A=xS^{-1}A$ for some *element x of $\mathfrak{p}$. If $\deg(x)=0$, then $S^{-1}A_+\subset xS^{-1}A_+$ and $S^{-1}A_- \subset xS^{-1}A_-$. By Krull's intersection theorem $A=A_0$. If $\deg(x)>0$, then $S^{-1}A_- \subset xS^{-1}A_-$. Hence $A_n=0$ for every $n<0$, $\mathfrak{p}=A_+$ and A_0 is a field. Thus $A \simeq A_0[x]$. If $\deg(x)<0$, we get a similar result.

COROLLARY 2.3. *Let A be a *Dedekind domain of the first type. Then $A=A_0$ or $A\simeq A_0[X]$, in which case A_0 is a field and X is an indeterminate.*

PROPOSITION 2.4. *Let A be a *Dedekind domain with non-trivial grading. Then, A_0 is a field if and only if A is of the first type or a *field.*

PROOF. The 'if' part is trivial. Suppose that A_0 is a field. If A_+ or A_- is zero, then A is of the first type. Assume that A_+ and A_- are not zero. Given a *element a $(\neq 0)$ with $\deg(a)>0$, we see that $a^{-\deg(b)}\,b^{\deg(a)}$ is a non-zero element of A_0 for any *element b $(\neq 0)$ with $\deg(b)<0$. This implies that A is a *field.

Let A be a *ring and m be an integer. If $m=0$, then we set $A^{(m)}=A_0$. If $m\neq 0$, then we set $A^{(m)}=\bigoplus_{n\in Z} A_{mn}$. It is clear that $A^{(m)}$ is a *subring of A.

PROPOSITION 2.5. *Let A be a *Dedekind domain. Then, $A^{(m)}$ is also a *Dedekind domain for every m. Especially A_0 is a Dedekind domain.*

PROOF. Let $\mathfrak{a}$ be an *ideal of $A^{(m)}$. There are *elements $w_1,\dots, w_r$ of $\mathfrak{a}$ such that $\mathfrak{a}A=w_1A+\cdots+w_rA$. For a *element $x\in\mathfrak{a}$, there are *elements $s_1,\dots, s_r$ in A such that $x=s_1w_1+\cdots+s_rw_r$. Thus the s_i are contained in $A^{(m)}$, and $\mathfrak{a}$ is a finitely generated *ideal of $A^{(m)}$. This shows that $A^{(m)}$ is *noetherian. Let $\mathfrak{p}_0$ be a prime ideal of A_0. By a *localization we see that there exists a prime *ideal $\mathfrak{P}$ of A such that $\mathfrak{P}\cap A_0=\mathfrak{p}_0$. If $m\neq 0$, A is integral over $A^{(m)}$. Hence there exists a prime *ideal $\mathfrak{P}$ of A such that $\mathfrak{P}\cap A^{(m)}=\mathfrak{p}$, for any prime *ideal $\mathfrak{p}$ of $A^{(m)}$. Let $\mathfrak{p}$ $(\neq 0)$ be a prime *ideal of $A^{(m)}$ and let $\mathfrak{P}$ be a prime *ideal of A such that $\mathfrak{P}\cap A^{(m)}=\mathfrak{p}$. Then $[A/\mathfrak{P}]^{(m)}=A^{(m)}/\mathfrak{p}$. Since $A/\mathfrak{P}$ is a *field, $A^{(m)}/\mathfrak{p}$ is also a *field. It follows that *$\dim(A)\leqq 1$. The *normality of $A^{(m)}$ is trivial. The proof is completed by Prop. 2.1.

REMARK 1. Let A be a *Dedekind domain. Then, A is of the second type if and only if $A_+\neq 0$ and $A_-\neq 0$. When A is of the second type, $\{n;\ A_n\neq 0\}$ is a subgroup of the group of integers Z.

2. Let $S=A_0-\{0\}$ and let A be of the second type. Then $S^{-1}A={}^*Q(A)$.

3. *Local *Dedekind domains of the second type

In Section 2 we determined the form of *Dedekind domains of the first type. From now on we consider *Dedekind domains of the second type. In this section A is always a *Dedekind domain of the second type. We may suppose, without loss of generality, that $A_1 \neq 0$ from Remark 1 in § 2. Let $\mathfrak{p}$ be a prime *ideal of A. $\mathfrak{p}$ is said to be *exceptional* if $\mathfrak{p}$ is not generated by all the elements of $\mathfrak{p}_0$ $(=\mathfrak{p} \cap A_0)$, i.e., $\mathfrak{p} \supsetneqq \mathfrak{p}_0 A$.

PROPOSITION 3.1. *The following statements concerning a non-zero prime *ideal $\mathfrak{p}$ of A are equivalent:*

(1) *$\mathfrak{p}$ is exceptional.*

(2) *$\mathfrak{p}$ contains A_1.*

(3) *$\mathfrak{p}$ contains A_{-1}.*

PROOF. Since $A/\mathfrak{p}$ is a *field, which is of the form $k[X, X^{-1}]$, the assertions of (2) and (3) are equivalent to each other. Suppose that $\mathfrak{p}$ does not contain both A_1 and A_{-1}. Let S be $A_0 - \mathfrak{p}$; then $S^{-1}A$ is *local. There is a *element $y \in \mathfrak{p}$ such that $\mathfrak{p}S^{-1}A = yS^{-1}A$. Localizing A by *maximal *ideals, we can see that, if y is contained in $\mathfrak{p}_0$, then we obtain $\mathfrak{p} = \mathfrak{p}_0 A$. Set $\deg(y) = s$. We take out a *element x from $A_t - \mathfrak{p}$, where $t = -s/|s|$. Set $y' = yx^{|s|}$. Then y' is contained in p_0 and $y'S^{-1}A = \mathfrak{p}S^{-1}A$. Hence $\mathfrak{p} = \mathfrak{p}_0 A$, which shows that the assertion (1) implies (2) and (3). Now we suppose that $\mathfrak{p}$ is not exceptional, i.e., $\mathfrak{p} = \mathfrak{p}_0 A$. If $\mathfrak{p}$ contains A_1, then $A_1 = \mathfrak{p}_0 A_1$. Since A_1 is finitely generated over A_0, there is an element a of $\mathfrak{p}_0$ such that $(1+a)A_1 = 0$. This is a contradiction to the fact $A_1 \neq 0$. Hence $\mathfrak{p}$ does not contain A_1.

COROLLARY 3.2. *There are only a finite number of exceptional prime *ideals.*

REMARK. Let $\mathfrak{p}$ be exceptional. Any *localization of $\mathfrak{p}$ is also exceptional.

PROPOSITION 3.3. *Let A be *local with the *maximal *ideal $\mathfrak{p}$. If $\mathfrak{p}$ is not exceptional, then A is of the form $A_0[U, U^{-1}]$, where U is a unit contained in A_1.*

PROOF. We take a unit $U \in A_1 - p$. For every n, $U^{-n}A_n = A_0$. Hence $A = A_0[U, U^{-1}]$.

PROPOSITION 3.4. *Let A and $\mathfrak{p}$ be as above. If $\mathfrak{p}$ is exceptional, then A is of the form $A_0[U, U^{-1}][x]$, where U is a unit and x is a *element.*

PROOF. We put $e = \mathrm{Min}\,\{n;\ (A/p)_n \neq 0$ *and* $n > 0\}$. By Prop. 2.5 $A^{(e)}$ is a *local *Dedekind domain with the *maximal *ideal $\mathfrak{p}^{(e)}$ $(=\bigoplus_n \mathfrak{p}_{en})$. Since $\mathfrak{p}^{(e)}$ is not exceptional, $A^{(e)}=A_0[U, U^{-1}]$, where U is a unit contained in A_e. Clearly any unit of A is contained in $A^{(e)}$. Let x be a generator of $\mathfrak{p}$. Then A is of the form $A_0[U, U^{-1}][x]$.

REMARK 1. $\deg(x)$ and $\deg(U)$ are prime to each other because $A_1 \neq 0$.

2. We call $\deg(U)$ the index of ramification of A to A_0. We denote it by e and let p be the prime element of the discrete valuation ring A_0. Then $p=x^e v$, where v is a unit.

3. By Prop. 3.4 any *local *Dedekind domain is obtained as follows: Let A_0 be a discrete valuation ring and let U, X be indeterminates. We take two integers e, f in such a way that $e \geqq 1, f \geqq 0, (e, f)=1$ and if $f=0$ (resp. $e=1$), then $e=1$ (resp. $f \geqq 0$). We set $A=A_0[U, U^{-1}, X]/(U^f X^e - p)$, where p is the prime element of A_0, $\deg(U)=e$ and $\deg(X)=f$. This is the required *ring.

4. *Dedekind domains of the second type

PROPOSITION 4.1. *Let A_0 be a Dedekind domain and let $\mathfrak{a}$ be a non-zero fractional ideal of A_0. Then $A={}_n\bigoplus \mathfrak{a}^n X^n$ is a *Dedekind domain of the second type, where X is an indeterminate.*

PROOF. Let $\{a_1, \dots, a_m\}$ (resp. $\{b_1, \dots, b_s\}$) be a generators of $\mathfrak{a}$ (resp. $\mathfrak{a}^{-1}$). Then $A=A_0[a_1X, \dots, a_mX, b_1X^{-1}, \dots, b_sX^{-1}]$. Hence A is *noetherian. Let S be the complement of a prime ideal in A_0. Since $\mathfrak{a}S^{-1}A_0$ is principal, $S^{-1}A$ is of the form $S^{-1}A_0[Y, Y^{-1}]$, where Y is an indeterminate. Hence *$\dim(A) \leqq 1$. Moreover A is *normal because $A=\bigcap S^{-1}A$, where S runs over all the complements of maximal ideals of A_0, and $S^{-1}A$ is *normal for each S. By Prop. 2.1 A is a *Dedekind domain.

REMARK. In (4.1) A has no exceptional prime *ideals because $A_1A_{-1} = A_0$.

PROPOSITION 4.2. *Let A be a *Dedekind domain of the second type. If A has no exceptional prime *ideals, then A is of the form* (4.1).

PROOF. By Prop. 3.1 $A_1A_{-1}=A_0$. We can see immediately that $A_nA_{-n} = A_0$ for every n. Then given any integers m, n, $A_mA_n=A_{m+n}$. Let x be a non-zero element of A_1. Then the ideal $\mathfrak{a}=x^{-1}A_1$ is a fractional ideal of A_0. Since $A_1=\mathfrak{a}x$ and $A_{-1}=\mathfrak{a}^{-1}x^{-1}$, $A=\bigoplus_n \mathfrak{a}^n x^n$.

Let A be a *ring. We denote the A_0-*isomorphism class of A by $[A]$.

THEOREM 4.3. *Let A_0 be a Dedekind domain, and let $D=\{[R]; R$ is a*

*Dedekind domain with no exceptional prime *ideals and $R_0 = A_0\}$. Then, there is a bijection of D onto $cl(A_0)$, where $cl(A_0)$ is the class group of A_0.*

PROOF. The proof follows by Prop. 4.1 and Prop. 4.2.

Let $\mathfrak{p}$ be a non-zero prime *ideal of A, and let $e_{\mathfrak{p}}$ be the index of ramification of $A_{(\mathfrak{p})}$ to $[A_{(\mathfrak{p})}]_0$. We say the index of $\mathfrak{p}$ to $\mathfrak{p}_0$ is $e_{\mathfrak{p}}$, where $\mathfrak{p}_0 = \mathfrak{p} \cap A_0$. Then $\mathfrak{p}_0 A = \mathfrak{p}^{e_{\mathfrak{p}}}$.

PROPOSITION 4.4. *Let A be a *Dedekind domain of the second type. Then there is a *Dedekind *subdomain of A such that it has no exceptional prime *ideals and A is finite over it.*

PROOF. If A has no exceptional prime *ideals, then we have nothing to do. Suppose that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are all the exceptional prime *ideals. Let $e_{\mathfrak{p}_i}$ be the index of $\mathfrak{p}_i$ to $[\mathfrak{p}_i]_0$ for every i. Then $A^{(m)}$ is a *Dedekind *subdomain of A with no exceptional prime *ideals, where m is the least common multiple of $e_{\mathfrak{p}_1}, \dots, e_{\mathfrak{p}_r}$. On the other hand A is integral over $A^{(m)}$. $A_k A_m = A_{k+m}$ and $A_k A_{-m} = A_{k-m}$ for every integer k. Since $A = A^{(m)}[A_{m+1}, \dots, A_{-1}, A_1, \dots, A_{m-1}]$, A is finite over $A^{(m)}$.

5. *Class groups of *Dedekind domains

In this section we shall define 'class group' in the same way as the ungraded case. In the ungraded case, the class group of a semi-local Dedekind domain is trivial, but it is false for the graded case as follows (see Prop. 5.2).

PROPOSITION 5.1. *Let A be a *Dedekind domain. Then $*G(A)$ is a free abelian group generated by all the prime *ideals as a basis.*

PROOF. The proof is the same as in the ungraded case.

We define an equivalence relation on $*G(A)$ as follows: Given $\mathfrak{a}, \mathfrak{b} \in *G(A)$, $\mathfrak{a} \sim \mathfrak{b}$ if and only if $\mathfrak{a}\mathfrak{b}^{-1}$ is principal. $*G(A)/\sim$ is said to be the *class group of A, and denoted by $*cl(A)$.

PROPOSITION 5.2. *Let A be a *semi-local *Dedekind domain, where 'semi-local' means that there are only a finite number of *maximal *ideals. Then the order of the *class group of A is equal to $\prod e_{\mathfrak{p}}/L.C.M.\ (e_{\mathfrak{p}})$, where $\mathfrak{p}$ runs over all the *maximal *ideals.*

PROOF. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all the *maximal *ideals. By Prop. 5.1 there is the canonical isomorphism f of $*G(A)$ onto Z^n, where Z is the set of integers. We identify a *element x with the principal *ideal xA for the convenience. Let H be the image of the set of non-zero *elements of $*Q(A)$ by the isomorphism f.

Then $*cl(A) \simeq Z^n/H$. By the hypothesis A_0 is semi-local and it is a PID. Hence the image of $[Q(A_0)-\{0\}]$ by f, which we denote by I, is $e_{\mathfrak{p}_1}Z+\cdots+e_{\mathfrak{p}_n}Z$. H/I is a cyclic group because the *field $*Q(A)$ is of the form $Q(A_0)[X, X^{-1}]$. Let t be a non-zero element of A_1. $f(t)+I$ is a generator of H/I. For an integer q, $qf(t) \in I$ if and only if there are $a \in A_0$ and a unit u such that $t^q = au$. This implies that the order of H/I is $\mathrm{Min}\{\deg(v); v \in A_+$ and v is a unit$\}$. On the other hand every $e_{\mathfrak{p}_i}$ divides $\deg(v)$ for any unit v. Each $\mathfrak{p}_i$ does not contain A_k for any multiple k of $e_{\mathfrak{p}_i}$. Since A_s is a cyclic A_0-module for every s, there is a unit with the degree equal to L.C.M. $(e_{\mathfrak{p}_1},\dots, e_{\mathfrak{p}_n})$. Thus the order of H/I is L.C.M. $(e_{\mathfrak{p}_1},\dots, e_{\mathfrak{p}_n})$. Now the order of Z^n/I is $e_{\mathfrak{p}_1}\cdots e_{\mathfrak{p}_n}$. The proof is completed.

REMARK. By Cor. 3.2 and the proof of Prop. 5.2, in Prop. 5.2 we may assume only that A_0 is a PID. In this case L.C.M. $(e_{\mathfrak{p}})$ is well-defined by Cor. 3.2.

EXAMPLE. Let $A = Z[\sqrt{p_1\cdots p_r}\,T, T, T^{-1}]$, where the p_i are prime numbers of the ring of integers Z, different to each other, and T is an indeterminate with the degree 2. Then A is a *Dedekind domain. There are r exceptional prime *ideals and the *class group has the order 2^{r-1}.

References

[1] N. Bourbaki, Algèbre commutative, Chap. 5, Hermann, Paris.
[2] R. Fossum and H. B. Foxby, The category of graded modules, Math. Scand., **35** (1974), 288–300.
[3] R. Fossum, The structure of indecomposable injective modules, Math. Scand., **36** (1975), 291–312.

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