HIROSHIMA MATH. J.
7 (1977), 411–425

Meromorphic Mappings into a Compact Complex Space

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(Received December 20, 1976)

Introduction. Let M be an m-dimensional smooth complex projective variety, $\Delta_n=\{(z_j)\in \boldsymbol{C}^n;\ |z_j|<1\}$ the unit polydisc in the complex affine space $\boldsymbol{C}^n$ of dimension n and $\Delta_n^*=\Delta_n-\{z_1=0\}$. Kobayashi-Ochiai ([5]) proved that if a holomorphic mapping $f: \Delta_m^*\to M$ is of rank m, i.e., the differential df is non-singular at some point, and if the canonical bundle K_M over M is positive, then f has a meromorphic extension from Δ_m into M. Kodaira showed that this extension theorem remains valid in the case where M is of general type (see Kobayashi-Ochiai [5, Addendum]). The condition that M is of general type is birationally invariant, whereas the positivity of K_M is not. For holomorphic mappings $f: \Delta_n^*\to M$ with $n<m$, Carlson ([1]) proved the analogous extension theorem under the condition that the vector bundle $\Omega(n)$ of holomorphic n-forms over M is positive.

In the present paper we shall establish such an extension theorem for algebraically non-degenerate holomorphic mappings $f: \Delta_n^*\to M$ with $n\leqq m$ under an assumption for $\Omega(n)$ which is birationally, moreover, bimeromorphically invariant and coincides, in case $n=m$, with that M is of general type (see (1.1), Corollary 1.2 and Theorem 3.1). Furthermore we shall deal with the case where M is a Moišezon space.

In the proof of that theorem, the key is a lemma of the Schwarz type (Lemma 2.2). In the last section we shall apply this lemma to study the family $\mathscr{M}$ of meromorphic mappings from an n-dimensional compact complex manifold N into M of rank n. We shall prove that if the analytic set B in M defined in section 4 is empty, then $\mathscr{M}$ is m-normal and the limits belong to $\mathscr{M}$, i.e., the space $\mathscr{M}$ endowed with m-convergence is compact (see Definitions 5.1, 5.2 and Theorem 5.3). In general, however, it seems that the m-convergence does not determine a topology in the precise sense. To make clear this fact we shall prove that $\mathscr{M}$ can be embedded into some complex affine and projective spaces in such a way that $\mathscr{M}$ is compact in each of them (Theorem 5.4).

1. Preliminaries

Let M be a compact complex manifold of dimension m and $\Omega(n)$ the vector bundle of holomorphic n-forms over M. Suppose that there exists an effective divisor D on M with $\kappa(D, M)=m$ (Iitaka [4]), that is,

$$\overline{\lim_{k\to\infty}} \dim \Gamma(M, [kD])/k^m > 0,$$

where $\Gamma(M, [kD])$ denotes the vector space of global holomorphic sections of the line bundle $[kD]$ determined by the divisor kD with integral coefficient $k\in \boldsymbol{Z}$. By Iitaka [4] there is a positive integer k_0 such that the image of the meromorphic mapping

$$T: M \ni x \longmapsto (\tau_1(x), \dots, \tau_N(x)) \in \boldsymbol{P}^{N-1}$$

is m-dimensional, where $\{\tau_j\}$ is a basis of $\Gamma(M, [k_0D])$, $N=\dim \Gamma(M, [k_0D])$ and $\boldsymbol{P}^{N-1}$ denotes the $(N-1)$-dimensional complex projective space. Pulling back rational functions on $\boldsymbol{P}^{N-1}$ through T we see that the meromorphic function field of M is of transcendental degree m, i.e., M is a Moišezon manifold. We consider the following condition for M:

(1.1) *For some effective divisor D on M with $\kappa(D, M)=m$, there are a positive integer l and a point $x_0\in M$ such that for any $\xi \in \Omega^*_{x_0}(n)$ with $\xi\neq 0$ there is a section $\sigma\in\Gamma(M, S^l(\Omega(n))\otimes[-D])$ such that $\sigma_{x_0}(S^l\xi)\neq 0$,*

where $\Omega^*(n)$ denotes the dual bundle of $\Omega(n)$ and $S^l(\cdot)$ the l-th symmetric tensor power. When E is a line bundle, we shall simply write $S^l(E)=E^l$.

REMARK. In the proof of Theorem 3.1 in section 3 and in section 4, we shall see that condition (1.1) is independent of the choice of such a D. In case $n=m$, (1.1) is equivalent to that M is of general type (cf. [7]).

Let O denote the zero section of the bundle $\Omega^*(n)$ and $P\Omega^*(n)$ the quotient of $\Omega^*(n)-O$ by the multiplicative group $\boldsymbol{C}^*$. Then $\Omega^*(n)-O\to P\Omega^*(n)$ is a principal bundle with group $\boldsymbol{C}^*$. Let L be the dual of the associated line bundle over $P\Omega^*(n)$. Letting $\pi: P\Omega^*(n)\to M$ denote the projection, we have

$$\Gamma(M, S^l(\Omega(n))\otimes[-D]) = \Gamma(P\Omega^*(n), L^l\otimes\pi^*[-D]) \quad \text{(cf. [3])}.$$

Let A be the analytic set of the common zeros of global holomorphic sections of $L^l\otimes\pi^*[-D]$ and $B=\pi(A)$. Then (1.1) is equivalent to

$$B \neq M \tag{1.1'}$$

and the set of points at which (1.1) does not hold is the analytic set B.

PROPOSITION 1.1. *Let M_1 and M_2 be compact complex manifolds of dimension m and $f: M_1\to M_2$ a surjective meromorphic mapping. If M_2 satisfies (1.1), then so does M_1.*

PROOF. Let S be the singular locus (indeterminant points) of f. Then $f|_{M_1-S}: M_1-S \to M_2$ is holomorphic and $d(f|_{M_1-S})$ is non-singular in a non-empty open set. Let D_2 be an effective divisor on M_2 with which (1.1) holds. By the above argument we may assume that (1.1) holds at a point $x_2=f(x_1)$ with $x_1 \in M_1-S$ at which $(df)_{x_1}$ is non-singular. Let $D_1=f^*D_2$ be the pullback of the effective divisor D_2 on M_2. Since $\dim \Gamma(M_1, [kD_1]) \geqq \dim \Gamma(M_2, [kD_2])$, $\kappa(D_1, M_1)=m$. We naturally get a homomorphism

$$f^*: \Gamma(M_2, S^l(\Omega_{M_2}(n)) \otimes [-D_2]) \longrightarrow \Gamma(M_1, S^l(\Omega_{M_1}(n)) \otimes [-D_1]).$$

Since condition (1.1) for M_2 is satisfied at x_2 and $(df)_{x_1}$ is non-singular, M_1 satisfies (1.1) at x_1.

COROLLARY 1.2. *Condition* (1.1) *is bimeromorphically invariant.*

DEFINITION 1.1. We say that a Moišezon space X*) satisfies condition (1.1) if a non-singular model $\tilde{X}$ of X satisfies (1.1).

By Corollary 1.2 this condition for X is independent of the choice of $\tilde{X}$.

In general, a meromorphic mapping f into a complex space X is said to be algebraically degenerate if the image of f is contained in a proper subvariety of X. If it is not the case, f is said to be algebraically non-degenerate.

Let Y be a complex space and $f: Y \to X$ a holomorphic mapping. We define the rank of f by

$$\text{rank of } f = \max_{y \in Y} \{\dim Y - \dim_y f^{-1}(f(y))\} \quad (\text{see [9, Chap. VII]}).$$

In the case where f is meromorphic, there is a modification $\tilde{Y} \to Y$ and a holomorphic mapping $\tilde{f}: \tilde{Y} \to X$ such that the diagram

$$\begin{array}{ccc} \tilde{Y} & & \\ \downarrow & \searrow^{\tilde{f}} & \\ Y & \xrightarrow[f]{} & X \end{array}$$

is commutative. We set

$$\text{rank of } f = \text{rank of } \tilde{f}.$$

2. Schwarz lemma

In this section we let M be a smooth complex projective variety, D an ample

*) Throughout the present paper, complex spaces are assumed to be reduced and irreducible.

divisor*) on M, and assume that M satisfies (1.1) with D. Let $\{\tau_0,\dots,\tau_N\}$ be a basis of $\Gamma(M,[D])$. Then $\rho=\sum|\tau_j|^2\in\Gamma(M,[D]\otimes\overline{[D]})$ is a positive section which naturally determines a metric in $[D]\to M$, where the bar denotes the complex conjugate. We denote by ω the curvature form of the metric, which is positive definite. By using a local coordinate system (x_α), we set

$$\omega=\sum h_{\alpha\bar\beta}\frac{i}{2\pi}dx_\alpha\wedge d\bar x_\beta.$$

The Kähler metric h associated with ω is locally given by

$$h=\sum h_{\alpha\bar\beta}\frac{1}{\pi}dx_\alpha\otimes d\bar x_\beta.$$

The metric naturally induces a metric $h^{(n)}$ in $\Omega^*(n)$ in the following manner: For decomposable vectors $\xi=\xi_1\wedge\cdots\wedge\xi_n$ and $\eta=\eta_1\wedge\cdots\wedge\eta_n$ in $\Omega_x^*(n)$,

$$h_x^{(n)}(\xi,\eta)=\det(h_x(\xi_i,\eta_j)) \tag{2.1}$$

and $h^{(n)}$ is defined for general ξ and η by linearity. Let $\{\sigma_1,\dots,\sigma_s\}$ be a basis of $\Gamma(M,S^l(\Omega(n))\otimes[-D])$ and set

$$\psi=(\sigma_1\otimes\bar\sigma_1+\cdots+\sigma_s\otimes\bar\sigma_s)\otimes\rho\in\Gamma(M,S^l\Omega(n)\otimes S^l\overline{\Omega(n)}).$$

Let Σ be the unit sphere bundle of $\Omega^*(n)$ with respect to $h^{(n)}$. For $\xi\in\Sigma$,

$$\psi(S^l\xi,S^l\xi)=\sum_i|\sigma_i(S^l\xi)|^2\otimes\rho$$

is a smooth function. Since Σ is compact, we can take the above $\{\sigma_i\}$ so that $\psi(S^l\xi,S^l\xi)\leqq 1$ for $\xi\in\Sigma$. This implies

$$\psi(S^l\xi,S^l\xi)\leqq(h^{(n)}(\xi,\xi))^l \tag{2.2}$$

for $\xi\in\Omega^*(n)$. Let W be an n-dimensional complex submanifold in a domain of M and assume that the restriction $\psi|_W\in\Gamma(W,K_W^l\otimes\bar K_W^l)$ does not vanish identically. Then $\psi|_W$ is locally written as

$$\psi|_W=\rho|_W(x)(\sum|a_i(x)|^2)|dx_1\wedge\cdots\wedge dx_n|^{2l},$$

where $x=(x_1,\dots,x_n)$ is a local coordinate system in W and a_i are holomorphic functions. We define the curvature form $\Theta(\psi,W)$ of ψ relative to W by

$$\Theta(\psi,W)=\frac{i}{2\pi}\partial\bar\partial\log((\rho|_W)(\sum|a_i|^2)),$$

*) We call D ample if the associated line bundle $[D]$ is ample in the sense of Griffiths [3], i.e., $\Gamma(M,[D])$ gives an immersion of M into some complex projective space.

which may be singular on a subvariety S of W. Since a_i are holomorphic,

$$\Theta(\psi, W) \geqq \omega|_W \quad \text{out of } S. \tag{2.3}$$

Therefore $\wedge_1^n \Theta(\psi, W) \geqq \wedge_1^n \omega|_W$. Now we let $v(x)=b(x)(i/2)dx_1 \wedge d\bar{x}_1 \wedge \cdots \wedge (i/2)dx_n \wedge d\bar{x}_n$ be a volume form. Then $v(x)$ can be written as $v(x)=b(x)|dx_1 \wedge \cdots \wedge dx_n|^2$. We shall freely use this identification. Combining (2.3) with (2.2) we have

LEMMA 2.1. *For any n-dimensional complex submanifold W in a domain of M,*

$$(\bigwedge_1^n \Theta(\psi, W))^l \geqq \psi|_W.$$

We set

$$\Delta(r) = \{z \in \boldsymbol{C};\ |z| < r\},$$

$$\Delta^*(r) = \{z \in \boldsymbol{C};\ 0 < |z| < r\},$$

$$\Delta_n(r) = \Delta(r) \times \cdots \times \Delta(r) \qquad (n\text{-times}),$$

$$\Delta_n^*(r) = \Delta^*(r) \times \Delta_{n-1}(r).$$

In case $r=1$, we simply write $\Delta_n(r)=\Delta_n$ and $\Delta_n^*(r)=\Delta_n^*$. Let $(z_1,\dots, z_n)$ be the natural coordinate system in $\Delta_n(r)$ and set

$$v_r = \prod_1^n \frac{r^2}{(r^2-|z_j|^2)^2}\left(\frac{1}{\pi}\right)^n |dz_1 \wedge \cdots \wedge dz_n|^2,$$

$$v = v_1.$$

LEMMA 2.2. *Let $f\colon \Delta_n \to M$ be a meromorphic mapping. Then*

$$f^*\psi \leqq c_0 v^l,$$

where $c_0 = l^{ln}$.

PROOF. We may suppose that $f^*\psi \not\equiv 0$. Set

$$f^*\psi = a(z)|dz_1 \wedge \cdots \wedge dz_n|^{2l},$$

$$v_r(z) = b_r(z)|dz_1 \wedge \cdots \wedge dz_n|^2,$$

$$c_r(z) = \log((b_r(z))^l/a(z)),$$

where $0<r<1$. First one notes that $a(z)$ is a smooth function. If some $|z_j| \to r$, then $b_r(z) \to +\infty$ and if $a(z)=0$ at $z \in \Delta_n(r)$, then $c_r(z)=+\infty$. The infimum of

$c_r(z)$ in $\Delta_n(r)$ is attained at some point $z_0 \in \Delta_n(r)$ at which

$$a(z_0) \neq 0. \tag{2.4}$$

We shall see that f is holomorphic at z_0. Let $\{\tau_0, \dots, \tau_N\}$ be the basis of $\Gamma(M, [D])$ taken above and set

$$T = (\tau_0, \dots, \tau_N): M \longrightarrow \boldsymbol{P}^N,$$

which is an immersion. Then $f^*\psi = \sum_{i,j} |f^*(\sigma_i \otimes \tau_j)|^2$ and (2.4) implies that there is a section $f^*(\sigma_i \otimes \tau_j)$, say, $f^*(\sigma_1 \otimes \tau_0)$ such that $f^*(\sigma_1 \otimes \tau_0)(z_0) \neq 0$. The meromorphic mapping $T \circ f$ is represented by

$$T \circ f = (f^*(\sigma_1 \otimes \tau_0), \dots, f^*(\sigma_1 \otimes \tau_N)).$$

Since $f^*(\sigma_1 \otimes \tau_0)(z_0) \neq 0$, $T \circ f$ is holomorphic at z_0 and so is f.

Since $i(2\pi)^{-1}\partial\bar\partial \log c_r(z_0)$ is semi-positive definite,

$$l \frac{i}{2\pi} \partial\bar\partial \log b_r(z_0) \geqq \frac{i}{2\pi} \partial\bar\partial \log a(z_0),$$

so that

$$l^n \bigwedge_1^n \frac{i}{2\pi} \partial\bar\partial \log b_r(z_0) \geqq \bigwedge_1^n \frac{i}{2\pi} \partial\bar\partial \log a(z_0). \tag{2.5}$$

It follows from (2.4) that $(df)_{z_0}$ is of maximal rank. There is a neighborhood W of z_0 which is biholomorphically embedded into a domain of M by f. We regard W as a submanifold in the domain. The right hand side of (2.5) is equal to $\wedge_1^n \Theta(\psi, W)$. From Lemma 2.1 and the identity, $\wedge_1^n \mathrm{Ric}\, v_r = v_r$, it follows that

$$l^{nl}(v_r(z_0))^l \geqq f^*\psi(z_0).$$

Hence $c_r(z_0) \geqq -nl \log l$ and so $f^*\psi \leqq c_0 v_r^l$ in $\Delta_n(r)$. Letting $r \to 1$, we deduce that $f^*\psi \leqq c_0 v^l$ in Δ_n.

3. Extension theorem

THEOREM 3.1. *Let X be a Moišezon space of dimension m satisfying condition* (1.1) *and $f: \Delta_n^* \to X$ an algebraically non-degenerate meromorphic mapping of rank n. Then f can be meromorphically extended over Δ_n.*

REMARK. Since (1.1) is bimeromorphically invariant (Corollary 1.2), X may contain $\boldsymbol{P}^{m-1}$. Therefore the algebraic non-degeneracy of f can not be dropped.

As immediate consequences of this theorem we get

COROLLARY 3.2. *Let N be an n-dimensional complex manifold and S a thin analytic set in N. Then any algebraically non-degenerate holomorphic mapping of $N-S$ into X of rank n has a meromorphic extension of N into X.*

COROLLARY 3.3. *Let $f: \boldsymbol{C}^n \to X$ be a meromorphic mapping. Then f is algebraically degenerate or the rank of f is less than n.*

PROOF OF THEOREM 3.1. By Moišezon's theorem [8], there is a modification $\lambda: (\tilde{X}, \tilde{S}) \to (X, S)$, where $\tilde{X}$ is a smooth projective variety in some complex projective space $\boldsymbol{P}^N$. By Proposition 1.1 $\tilde{X}$ satisfies (1.1) with a divisor D such that $\kappa(D, \tilde{X}) = m$. Let $\tilde{D}$ be a general hyperplane section of $\tilde{X}$. Then by Kodaira [7] there is an exact sequence

$$0 \longrightarrow \Gamma(X, [kD-\tilde{D}]) \longrightarrow \Gamma(\tilde{X}, [kD]) \longrightarrow \Gamma(\tilde{D}, [kD]|_{\tilde{D}}) \longrightarrow \cdots.$$

Since $\overline{\lim\limits_{k\to\infty}} \dim \Gamma(\tilde{X}, [kD])/k^m > 0$ and $\dim \Gamma(\tilde{D}, [kD]|_{\tilde{D}}) = O(k^{m-1})$ as $k \to \infty$, $\dim \Gamma(\tilde{X}, [kD-\tilde{D}]) > 0$ for a large k. Replacing l in (1.1) by kl we easily see that (1.1) is valid for the divisor kD. Using a section $\alpha \in \Gamma(\tilde{X}, [kD-\tilde{D}])$ with $\alpha \not\equiv 0$, we get an into-isomorphism

$$\Gamma(\tilde{X}, S^{kl}(\Omega(n)) \otimes [-kD]) \ni \sigma \longmapsto \sigma \otimes \alpha \in \Gamma(\tilde{X}, S^{kl}(\Omega(n)) \otimes [-\tilde{D}]).$$

Hence X satisfies (1.1) with the very ample divisor $\tilde{D}$.

Since f is algebraically non-degenerate, f can be lifted to a meromorphic mapping $\tilde{f}: \Delta_n^* \to \tilde{X}$, which is algebraically non-degenerate and of rank n. Now assume that f has a meromorphic extension over Δ_n. We denote it by $\hat{f}$. Let $\hat{\Gamma} \subset \Delta_n \times \tilde{X}$ be the graph of $\hat{f}$ and Γ that of f. Then we have

$$\begin{array}{ccc} \hat{\Gamma} & \subset\!\!=\!\!=\!\!=\!\!= & \Delta_n \times \tilde{X} \\ & & \downarrow{\scriptstyle \Lambda} \\ \Gamma \subset \Delta_n^* \times X & \subset & \Delta_n \times X, \end{array}$$

where $\Lambda = (\text{identity}) \times \lambda$. Since Λ is proper, $\Lambda(\hat{\Gamma})$ is an analytic set in $\Delta_n \times X$ and $\Lambda(\hat{\Gamma}) \supset \Gamma$. From the construction it is easily seen that $\bar{\Gamma}$ (closure of Γ) $= \Lambda(\hat{\Gamma})$. Thus f has a meromorphic extension from Δ_n into X.

Therefore it is sufficient to prove Theorem 3.1 in the case where X is a smooth complex projective variety M and the divisor D on M in condition (1.1) is very ample (i.e., global holomorphic sections of $[D]$ give an embedding into some $\boldsymbol{P}^N$). Let B be the analytic set in (1.1′). Since $f: \Delta_n^* \to M$ is algebraically non-degenerate, $f^{-1}(B)$ is a proper subvariety in Δ_n^*. Since df is of maximal rank in a non-empty open set in Δ_n^*, there is a section $\sigma \in \Gamma(M, S^l(\Omega(n)) \otimes [-D])$ such that $f^*\sigma \in \Gamma(\Delta_n^*, K_{\Delta_n^*}^l \otimes f^*[-D])$ does not vanish identically. Let $\{\tau_0, \ldots, \tau_N\}$ be a basis of $\Gamma(M, [D])$ and set $T=(\tau_0,\dots,\tau_N)\colon M\to \boldsymbol{P}^N$, which is an embedding. We set

$$\alpha_i = f^*\sigma \otimes f^*\tau_i \in \Gamma(\Delta_n^*, K_{\Delta_n^*}^l),$$

$$F = (\alpha_0,\dots,\alpha_N)\colon \Delta_n^* \longrightarrow \boldsymbol{P}^N. \tag{3.1}$$

Then (3.1) gives a representation of $T\circ f$. It is enough to show that each α_i can be meromorphically extended over Δ_n. Letting α denote one of $\{\alpha_i\}$, we may assume that

$$|\alpha|^2 \leqq f^*\psi \quad (\text{see section 2 for } \psi).$$

Setting

$$\alpha(z) = a(z)(dz_1 \wedge \cdots \wedge dz_n)^l,$$

we have by Lemma 2.2

$$|a(z)|^2 \leqq c_0(b(z))^l,$$

where

$$b(z_1,\dots,z_n) = \pi^{-n}|z_1|^{-2}(\log|z_1|^2)^{-2}\prod_{j=2}^{n}(1-|z_j|^2)^{-2}.$$

We expand $a(z)$ as a Laurent series

$$a(z) = \sum_{\mu_1=-\infty}^{+\infty} z_1^{\mu_1} \sum_{\substack{\mu_j\geqq 0\\ j\geqq 2}} a_{\mu_2\cdots\mu_n}^{(\mu_1)} z_2^{\mu_2}\cdots z_n^{\mu_n}$$

and set each $z_j=r_je^{i\theta_j}$ with $0<r_j<1$. Then

$$\int_0^{2\pi}\frac{d\theta_1}{2\pi}\cdots\int_0^{2\pi}\frac{d\theta_n}{2\pi}\,|a(r_1e^{i\theta_1},\dots,r_ne^{i\theta_n})|^2$$
$$\leqq c_0\pi^{-ln}r_1^{-2l}(\log r_1^2)^{-2l}\prod_{j=2}^{n}(1-r_j^2)^{-2l}.$$

Hence

$$\sum_{\mu_1=-\infty}^{+\infty} r_1^{2\mu_1}\sum_{\substack{\mu_j\geqq 0\\ j\geqq 2}} |a_{\mu_2\cdots\mu_n}^{(\mu_1)}|^2 r_2^{2\mu_2}\cdots r_n^{2\mu_n}$$
$$\leqq c_0\pi^{-ln}r_1^{-2l}(\log r_1^2)^{-2l}\prod_{j=2}^{n}(1-r_j^2)^{-2l}.$$

Comparing the orders of both the sides as $r_1\to 0$, we infer that $a_{\mu_2\cdots\mu_n}^{(\mu_1)}=0$ for $\mu_1\leqq -l$. Thus $\alpha(z)$ has singularities which are at most poles of order $l-1$ on $\{z_1=0\}$.

REMARK 1. It should be noted that the poles of all α_i in the representation of $T\circ f$ (see (3.1)) are at most of order $l-1$ which is independent of each f.

REMARK 2. As proved above, the theorem remains valid without the assumption that f is algebraically non-degenerate, unless $f(\Delta_n^*)$ is contained in B.

In the case where $\Omega(n)$ is positive, this theorem was proved by Carlson [1] without algebraic non-degeneracy. In this case, we can take l in condition (1.1) so that $B=\emptyset$.

4. The analytic set B

Let M be a smooth complex projective variety of dimension m and D an ample divisor on M. The purpose of the present section is to show that the analytic set B in (1.1′) can be defined independently of each D, provided D is ample.

Let $B_{l,k}(D)$ be the analytic set of all points $x\in M$ at each of which there is an element $\xi\in\Omega_x^*(n)$ with $\xi\neq 0$ such that $\sigma_x(S^l\xi)=0$ for all $\sigma\in\Gamma(M, S^l(\Omega(n))\otimes[-kD])$. We set

$$B(D)=\bigcap_{\substack{l>0\\k>0}} B_{l,k}(D). \tag{4.1}$$

PROPOSITION 4.1. *Let D_i $(i=1, 2)$ be ample divisors on M. Then*
(i) $B(D_1)=B(D_2)$,
(ii) $B(D_1)=B_{l,1}(D_1)$
for some $l\in\mathbf{Z}$ $(l>0)$.

PROOF. To prove (i), it is enough to show $B(D_1)\subset B(D_2)$. Let x be any point of $B(D_1)$. Since D_2 is ample, there is a positive integer k_0 such that there is a section $\phi\in\Gamma(M, [k_0D_2-D_1])$ with $\phi(x)\neq 0$. For an arbitrary $\sigma\in\Gamma(M, S^l(\Omega(n))\otimes[-kD_2])$, $S^{k_0}\sigma\in\Gamma(M, S^{k_0l}(\Omega(n))\otimes[-k_0kD_2])$, so that $S^{k_0}\sigma\otimes\phi^k\in\Gamma(M, S^{k_0l}(\Omega(n))\otimes[-kD_1])$. Since $x\in B(D_1)$, there is an element $\xi\in\Omega_x^*(n)$ with $\xi\neq 0$ such that $\sigma'_x(S^{k_0l}\xi)=0$ for all $\sigma'\in\Gamma(M, S^{k_0l}(\Omega(n))\otimes[-kD_1])$. Therefore we have $(S^{k_0}\sigma\otimes\phi^k)_x(S^{k_0l}\xi)=0$. Since $\phi(x)\neq 0$, $\sigma_x(S^l\xi)=0$. Hence $x\in B(D_2)$.

For the proof of (ii) we simply write $D_1=D$. We first prove

$$B(D)=\bigcap_{l} B_{l,1}(D). \tag{4.2}$$

If it is proved that $B_{l,k}(D)\supset B_{l,1}(D)$, then (4.2) immediately follows. Let x be an arbitrary point of $B_{l,1}(D)$. Since D is ample, there is a section $\tau\in\Gamma(M, [D])$ with $\tau(x)\neq 0$. For any $\sigma\in\Gamma(M, S^l(\Omega(n))\otimes[-kD])$, $\sigma\otimes\tau^{k-1}$ belongs to $\Gamma(M, S^l(\Omega(n))\otimes[-D])$. Since $x\in B_{l,1}(D)$, there is an element $\xi\in\Omega^*(n)$ with $\xi\neq 0$ such that $\sigma'_x(S^l\xi)=0$ for all $\sigma'\in\Gamma(M, S^l(\Omega(n))\otimes[-D])$, so that $(\sigma\otimes\tau^{k-1})_x(S^l\xi)=$

0. Since $\tau(x)\neq 0$, $\sigma_x(S^l\xi)=0$. This proves (4.2).

Since M is compact, $B(D)=\bigcap_{l=1}^{s}B_{l,1}(D)$ for a positive integer s. In the same manner as above, we see that $B_{l,1}(D)\supset B_{ll',1}(D)$ for any $l'\in \boldsymbol{Z}$, $l'>0$. Let l_0 be the least common multiple of $\{2,\dots, s\}$. Then $B(D)=B_{l_0,1}(D)$. This completes the proof.

In the rest of this paper we shall denote by B the analytic set $B(D)$. One should note that this does not depend on the choice of D but essentially on the vector bundle $\Omega(n)$ of holomorphic n-forms over M.

5. Meromorphic mappings of N into M

Let M be a smooth complex projective variety of dimension m and N a complex manifold.

DEFINITION 5.1. A sequence $\{f_\nu\}_{\nu=1,2,\dots}$ of meromorphic mappings of N into M is said to be meromorphically convergent (simply, m-convergent) to a meromorphic mapping f of N into M if there are an embedding $T\colon M\to \boldsymbol{P}^N$ and a neighborhood U of each point of N in which $T\circ f_\nu$ and $T\circ f$ have representations

$$(5.1)\qquad \begin{aligned} T\circ f_\nu &= (\alpha_{\nu 0},\dots, \alpha_{\nu N}),\\ T\circ f &= (\alpha_0,\dots, \alpha_N), \end{aligned}$$

where $(w_0,\dots, w_N)$ is a homogeneous coordinate system in $\boldsymbol{P}^N$ and $\alpha_{\nu j}$, α_j are holomorphic functions in U such that each $\{\alpha_{\nu j}\}_\nu$ converges uniformly on any compact set in U to α_j.

DEFINITION 5.2. A family $\mathscr{M}$ of meromorphic mappings of N into M is said to be m-normal if any sequence of $\mathscr{M}$ has a subsequence which is m-convergent.

REMARK. Fujimoto ([2]) first introduced the notion of m-convergence. In his definition the representation of each $T\circ f_\nu$ in (5.1) is assumed to be reduced, i.e., $\operatorname{codim}\{f_{\nu 0}=\cdots=f_{\nu N}=0\}\geqq 2$, while ours is not. By using Stoll's theorem [10] we easily see that if $\{f_\nu\}$ is m-convergent to f in the present sense, a subsequence of $\{f_\nu\}$ is m-convergent to f in that of Fujimoto. Hence, so far as the m-normality is concerned, the present definition coincides with that of Fujimoto.

In the rest of this paper we assume that N is a compact complex manifold of dimension n and restrict ourselves in the special case where the analytic set B in M defined in section 5 is empty. Let $\mathscr{M}$ denote the family of meromorphic mappings from N into M of rank n.

Let D be a very ample divisor on M, $\{\tau_0,\dots, \tau_N\}$ a basis of $\Gamma(M, [D])$ and $\{\sigma_1,\dots, \sigma_s\}$ that of $\Gamma(M, S^l(\Omega(n))\otimes[-D])$ where l is a positive integer such that

$B_{l,1}(D)=\emptyset$ (see Proposition 4.1). Set $\vartheta_{ij}=\sigma_i\otimes\tau_j\in\Gamma(M, S^l\Omega(n))$ and

$$\psi = \sum_{i,j}\vartheta_{ij}\otimes\bar{\vartheta}_{ij}.$$

By the assumption $B=\emptyset$, $\psi_x(S^l\xi, S^l\xi)=\sum_{i,i}|(\vartheta_{ij})_x(S^l\xi)|^2>0$ for $\xi\in\Omega_x^*(n)$ with $\xi\neq 0$. Let T: $M\to \boldsymbol{P}^N$ be the embedding defined by

$$X\ni x\longmapsto(\tau_0(x), .., \tau_N(x))\in\boldsymbol{P}^N,$$

and $\omega=i(2\pi)^{-1}\partial\bar{\partial}\log(\sum_{j=0}^N|\tau_j|^2)$ the positive (1, 1)-form belonging to the first Chern class $c_1([D])$ of $[D]$. Let χ be the Kähler form associated with the standard Fubini-Study metric on $\boldsymbol{P}^N$. Then $\omega=T^*\chi$. We may assume that ψ satisfies (2.2). Let Σ be the unit sphere bundle of $\Omega^*(n)$ with respect to the metric defined by (2.1). Then

$$\inf\{\psi(S^l\xi, S^l\xi);\ \xi\in\Sigma\}>0,$$

since Σ is compact. Thus there is a positive constant c_0 such that for any n-dimensional complex submanifold W in a domain of M

$$c_0(\overset{n}{\underset{1}{\wedge}}\omega|_W)^l\leqq\psi|_W\leqq(\overset{n}{\underset{1}{\wedge}}\omega|_W)^l. \tag{5.2}$$

By Lemma 2.2 we have

LEMMA 5.1. *There is a smooth volume form υ on N satisfying*

$$f^*\psi\leqq\upsilon^l$$

for every $f\in\mathcal{M}$.

LEMMA 5.2. *For every $f\in\mathcal{M}$*

$$C_0\leqq\int_N(f^*\psi)^{1/l}\leqq C_1,$$

where $C_0=c_0^{1/l}$ with the constant c_0 in (5.2) and $C_1=\int_N\upsilon$.

PROOF. The second inequality immediately follows from Lemma 5.1.

Let $W=f(N)$. Then W is a complex n-dimensional subvariety in M and

$$\int_N(f^*\psi)^{1/l}=\deg(f)\int_W(\psi|_W)^{1/l},$$

where $\deg(f)$ denotes the degree of the meromorphic mapping f: $N\to W$ (cf. Kobayashi-Ochiai [6, Lemma 4]). By (5.2)

$$\int_W (\psi|_W)^{1/l} \geqq C_0 \int_W T^*(\bigwedge_1^n \chi) = C_0 \int_{T(W)} \bigwedge_1^n \chi = C_0 \deg(T(W)),$$

where $\deg(T(W))$ denotes the degree of the subvariety $T(W)$ in $\boldsymbol{P}^N$. Hence

$$\int_N (f^*\psi)^{1/l} \geqq C_0 \deg(f) \deg(T(W)) \geqq C_0.$$

THEOREM 5.3. *The family $\mathscr{M}$ of meromorphic mappings from N into M of rank n is m-normal. Moreover the limits belong to $\mathscr{M}$.*

PROOF. Let $\{f_\nu\}_{\nu=1,2,\ldots}$ be a sequence of $\mathscr{M}$. By Lemma 5.1

$$|f_\nu^*\vartheta_{ij}|^2 \leqq f^*\psi \leqq v^l.$$

This implies that $f_\nu^*\vartheta_{ij} \in \Gamma(N, K_N^l)$ are uniformly bounded. There is a subsequence $\{f_{\nu_k}\}$ such that each $f_{\nu_k}^*\vartheta_{ij}$ converges uniformly to $\alpha_{ij} \in \Gamma(N, K_N^l)$. By Lemma 5.2

$$\int_N (\sum_{i,j} |f_{\nu_k}^*\vartheta_{ij}|^2)^{1/l} \geqq C_0.$$

We have

$$\int_N (\sum_{i,j} |\alpha_{ij}|^2)^{1/l} \geqq C_0.$$

Therefore there is a section $\alpha_{ij} \not\equiv 0$, say, $\alpha_{10} \not\equiv 0$. We define a meromorphic mapping F by

$$F = (\alpha_{10}, \ldots, \alpha_{1N}) \colon N \longrightarrow \boldsymbol{P}^N.$$

We may assume that all $f_{\nu_k}^*\vartheta_{10} \not\equiv 0$. Then $T \circ f_{\nu_k}$ are represented by

$$T \circ f_{\nu_k} = (f_{\nu_k}^*\vartheta_{10}, \ldots, f_{\nu_k}^*\vartheta_{1N}) \colon N \longrightarrow T(M) \subset \boldsymbol{P}^N.$$

Hence $F(N) \subset T(M)$. Setting $f = T^{-1} \circ F$ we infer that $\{f_{\nu_k}\}$ is m-convergent to f and

$$\alpha_{ij} = f^*\vartheta_{ij} \qquad \text{for all} \quad i, j. \tag{5.3}$$

Since $f^*\vartheta_{10} \not\equiv 0$, f belongs to $\mathscr{M}$.

Theorem 5.3 means that $\mathscr{M}$ is compact in the sense of the m-convergence. But it seems that, in general, the m-convergence does not define a topology in the precise sense. In the following we shall make clear this point.

Let Γ_0 be the vector subspace in $\Gamma(M, S^l\Omega(n))$ generated by $\{\vartheta_{ij}\}$ and $\Gamma_1 = \Gamma(N, K_N^l)$. Then a meromorphic mapping $f \in \mathscr{M}$ induces a homomorphism

$$f^*\colon \Gamma_0 \ni \vartheta \longmapsto f^*\vartheta \in \Gamma_1.$$

We set

$$\iota\colon \mathscr{M} \ni f \longmapsto f^* \in \mathrm{Hom}(\Gamma_0, \Gamma_1) - \{O\}.$$

Composing ι with the natural mapping

$$\mathrm{Hom}(\Gamma_0, \Gamma_1) - \{O\} \longrightarrow (\mathrm{Hom}(\Gamma_0, \Gamma_1) - \{O\})/\boldsymbol{C}^* = P\mathrm{Hom}(\Gamma_0, \Gamma_1),$$

We get

$$\tilde{\iota}\colon \mathscr{M} \longrightarrow P\mathrm{Hom}(\Gamma_0, \Gamma_1).$$

We shall show that $\tilde{\iota}$ and ι are injective. Let $f_i \in \mathscr{M}$, $i=1, 2$ and assume that $\tilde{\iota}(f_1)=\tilde{\iota}(f_2)$. Then $f_1^*=cf_2^*$ with some $c \in \boldsymbol{C}^*$. There is a section ϑ_{ij}, say, ϑ_{10} such that $f_1^*\vartheta_{10}=cf_2^*\vartheta_{10}\not\equiv 0$. The meromorphic mappings $T\circ f_1$ and $T\circ f_2$ are represented by

$$\text{(5.4)}\qquad \begin{aligned} T\circ f_1 &= (f_1^*\vartheta_{10},\dots, f_1^*\vartheta_{1N}),\\ T\circ f_2 &= (f_2^*\vartheta_{10},\dots, f_2^*\vartheta_{1N}). \end{aligned}$$

Since $f_1^*\vartheta_{1j}=cf_2^*\vartheta_{1j}$ for all j, $T\circ f_1=T\circ f_2$ and so $f_1=f_2$.

Next we show that the image $\iota(\mathscr{M})$ is compact in $\mathrm{Hom}(\Gamma_0, \Gamma_1)-\{O\}$ endowed with the usual topology. Let $\{\iota(f_\nu)\}$ be any sequence of $\iota(\mathscr{M})$. Taking a suitable subsequence, we may assume by Theorem 5.3 that $\{f_\nu\}$ is m-convergent to a meromorphic mapping $f \in \mathscr{M}$. Then each $\{f_\nu^*\vartheta_{ij}\}_\nu \subset \Gamma(N, K_N^l)$ converges uniformly on any compact set in N minus a thin analytic set to $f^*\vartheta_{ij} \in \Gamma(N, K_N^l)$. By the maximal principle of holomorphic functions, each $\{f_\nu^*\vartheta_{ij}\}_\nu$ converges uniformly to $f^*\vartheta_{ij}$. Therefore $\{\iota(f_\nu)\}$ converges to $\iota(f)$ in $\mathrm{Hom}(\Gamma_0, \Gamma_1)-\{O\}$ and so $\iota(\mathscr{M})$ and $\tilde{\iota}(\mathscr{M})$ are compact sets.

Let $\{\tilde{\iota}(f_\nu)\}_\nu$ be a sequence of $\tilde{\iota}(\mathscr{M})$ converging to $\tilde{\iota}(f)$ in $P\mathrm{Hom}(\Gamma_0, \Gamma_1)$. Then, using the representations of $T\circ f_\nu$ and $T\circ f$ of the type (5.4), we deduce that $\{f_\nu\}$ is m-convergent to f. Thus we have

THEOREM 5.4. (i) *The mappings* $\iota\colon \mathscr{M}\to\mathrm{Hom}(\Gamma_0, \Gamma_1)-\{O\}$ *and* $\tilde{\iota}\colon \mathscr{M}\to P\mathrm{Hom}(\Gamma_0, \Gamma_1)$ *are injective.*

(ii) $\iota(\mathscr{M})$ *and* $\tilde{\iota}(\mathscr{M})$ *are compact sets in each space.*

(iii) *Let* $f_\nu \in \mathscr{M}$, $\nu=1, 2,\dots$, *and* $f \in \mathscr{M}$. *Then the following convergences are equivalent*:

(a) $\{f_\nu\}$ *is m-convergent to* f,

(b) $\iota(f_\nu) \longrightarrow \iota(f)$ *in* $\mathrm{Hom}(\Gamma_0, \Gamma_1)$,

(c) $\tilde{\iota}(f_\nu) \longrightarrow \tilde{\iota}(f)$ *in* $P\mathrm{Hom}(\Gamma_0, \Gamma_1)$.

Therefore $\tilde{\iota}(\mathscr{M}) \ni \tilde{\iota}(f) \longmapsto \iota(f) \in \iota(\mathscr{M})$ *is continuous.*

REMARK 1. In the case where $n=m$ and M is of general type, Kobayashi-Ochiai ([6]) recently proved that $\mathscr{M}$ is a finite set. They also dealt with the case $n\geqq m$ and obtained the same result.

In case $n<m$, the finiteness of $\mathscr{M}$ does not hold in general. In fact, let N be a closed Riemann surface with genus greater than one, $M=N\times N$ and denote by G the holomorphic automorphism group of N. Then the vector bundle $\Omega(1)$ over M is positive, so that $B=\emptyset$. In this case we have

$$\mathscr{M} = \{(a, f);\ a\in N, f\in G\} \cup \{(f, b); f\in G,\ b\in N\} \cup \{(f, g); f, g\in G\},$$

which is infinite.

REMARK 2. Theorem 5.4 (iii) implies that $\tilde{\iota}(\mathscr{M})$ can not be an analytic set of positive dimension, since the $\boldsymbol{C}^*$-bundle $\mathrm{Hom}(\Gamma_0, \Gamma_1)-\{O\}\to P\mathrm{Hom}(\Gamma_0, \Gamma_1)$, restricted to any subvariety of positive dimension in $P\mathrm{Hom}(\Gamma_0, \Gamma_1)$ is topologically non-trivial.

REMARK 3. Let N' be an n-dimensional compact complex manifold N minus a thin analytic set. Let $\mathscr{M}'$ be the family of meromorphic mappings from N' into M of rank n. Then, by the remark in section 3, Theorems 5.3 and 5.4 are still valid for $\mathscr{M}'$.

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