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On Stable Homotopy Types of Stunted Lens Spaces, II

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§1. Introduction

Let $L^n(p^r)=S^{2n+1}/Z_{p^r}$ be the standard lens space mod p^r, where p is a prime and r is a positive integer. In the earlier paper [6] we have obtained some necessary conditions that two stunted lens spaces are stably homotopy equivalent (briefly: S-equivalent), and some sufficient conditions in case $r=2$.

The main purpose of this paper is to improve the sufficient conditions using the results on the structure of $J(L^n(p^r))$ obtained in the previous note [7].

Using [2, Prop. (2.6) and Lemma (2.4)], [3, Th. 1], [4, Th. 1.1] and [8, Th. 1.4], we prove

THEOREM 1. *Let n, m and r be integers such that $n>m>0$ and $p^r>2$. Assume that t is a positive integer which is a multiple of*

$$\begin{array}{ll} p^{r+n-m-2}, & \textit{if}\quad p=2 \quad \textit{and} \quad n-m\equiv 1 \bmod 2,\\ p^{r+n-m-1}, & \textit{if}\quad p=2 \quad \textit{and} \quad n-m\equiv 0 \bmod 2,\\ p^{r+[(n-m-2)/(p-1)]}, & \textit{if}\quad p \quad \textit{is an odd prime.} \end{array}$$

Then $L^n(p^r)/L^{m-1}(p^r)$ is S-equivalent to $L^{n+t}(p^r)/L^{m-1+t}(p^r)$. The same is true for $L^n(p^r)/L_0^m(p^r)$, $L_0^n(p^r)/L^{m-1}(p^r)$ and $L_0^n(p^r)/L_0^m(p^r)$, where $L_0^n(p^r)$ denotes the $2n$-skeleton of the natural CW-decomposition of $L^n(p^r)$.

For a prime p and integers n, m and r such that $n>m>0$ and $r\geqq 1$, we define the integers a_s and b_s $(0\leqq s<r)$ by

$$n-m-p^s+1=a_sp^s(p-1)+b_s, \qquad 0\leqq b_s<p^s(p-1).$$

In case $p\geqq 3$, and $r=3$ or 4, define $\varepsilon(r)$ as follows:

$$\varepsilon(3)=\begin{cases} 2 & \text{if}\quad a_0=(a_2+1)p^2,\\ 1 & \text{if}\quad a_0=(a_2+1)p^2+p,\ (a_2+1)p^2+1, \text{ or}\\ & \quad\ \ a_0=(a_1+1)p<(a_2+1)p^2,\\ 0 & \text{otherwise.} \end{cases}$$

$$\varepsilon(4) = \begin{cases} 3 & \text{if} \quad a_0 = (a_3+1)p^3, \\ 2 & \text{if} \quad a_0 = (a_3+1)p^3+1,\ (a_3+1)p^3+p^2,\ (a_2+1)p^2, \\ 1 & \text{if} \quad a_0 = (a_3+1)p^3+2,\ (a_3+1)p^3+qp\ (1 \leqq q < p), \\ & (a_3+1)p^3+p^2+1,\ (a_3+1)p^3+p^2+p,\ (a_2+1)p^2+1, \\ & (a_2+1)p^2+p,\ (a_1+1)p, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following results by using [7, Prop. 7.5, 7.7].

THEOREM 2. *Let p be an odd prime and $r=3$ or 4. If $t \equiv 0 \bmod p^{a_0+\varepsilon(r)}$, then $L^n(p^r)/L^{m-1}(p^r)$ is S-equivalent to $L^{n+t}(p^r)/L^{m-1+t}(p^r)$. The same is true for $L^n(p^r)/L_0^m(p^r)$, $L_0^n(p^r)/L^{m-1}(p^r)$ and $L_0^n(p^r)/L_0^m(p^r)$.*

The results of Theorem 2 for small values of $\varepsilon(r)$ $(r=3, 4)$ are better than those of Theorem 1, although Theorem 2 for large values of $\varepsilon(r)$ gives the same results as Theorem 1.

THEOREM 3. *Let p be an odd prime, and $r=3$ or 4. $L^n(p^r)/L^{m-1}(p^r)$ (resp. $L_0^n(p^r)/L^{m-1}(p^r)$) is S-equivalent to $L^{n-m}(p^r)^+$ (resp. $L_0^{n-m}(p^r)^+$) if and only if $m \equiv 0 \bmod p^{a_0+\varepsilon(r)}$. Here X^+ denotes the disjoint union of X and a point.*

In case $p=2$ and $r=3$, we have

THEOREM 4. *Assume that $t \equiv 0 \bmod 2^{n-m+\varepsilon}$, where*

$$\varepsilon = \begin{cases} 2 & \text{if} \quad n-m \equiv 0 \bmod 4, \\ 1 & \text{if} \quad n-m \equiv 1 \text{ or } 2 \bmod 4, \quad \text{and} \quad n-m > 1, \\ 0 & \text{if} \quad n-m \equiv 3 \bmod 4, \quad \text{or} \quad n-m = 1. \end{cases}$$

Then $L^n(8)/L^{m-1}(8)$ is S-equivalent to $L^{n+t}(8)/L^{m-1+t}(8)$. The same is true for $L^n(8)/L_0^m(8)$.

Assume that $t \equiv 0 \bmod 2^{n-m+\varepsilon'}$, where

$$\varepsilon' = \begin{cases} 1 & \text{if} \quad n-m \not\equiv 3 \bmod 4, \quad \text{and} \quad n-m > 1, \\ 0 & \text{if} \quad n-m \equiv 3 \bmod 4, \quad \text{or} \quad n-m = 1. \end{cases}$$

Then $L_0^n(8)/L^{m-1}(8)$ is S-equivalent to $L_0^{n+t}(8)/L^{m-1+t}(8)$. The same is true for $L_0^n(8)/L_0^m(8)$.

If $n-m \not\equiv 0 \bmod 4$, Theorem 4 gives better results than Theorem 1.

Corresponding to Theorem 3, we obtain

THEOREM 5. *$L^n(8)/L^{m-1}(8)$ is S-equivalent to $L^{n-m}(8)^+$ if and only if* $m \equiv 0 \bmod 2^{n-m+\varepsilon}$. *$L_0^n(8)/L^{m-1}(8)$ is S-equivalent to $L_0^{n-m}(8)^+$ if and only if* $m \equiv 0 \bmod 2^{n-m+\varepsilon'}$. *Here ε and ε' are defined in Theorem* 4.

We prove Theorems 1–3 in §2 evaluating the orders of the J-images of the canonical bundles according to [4], [7] and [8]. In §3 and §4, we determine the K- and KO-rings of $L^n(8)$ and $L_0^n(8)$ in the line of [5]. In §5, we determine the J-groups of $L^n(8)$ and $L_0^n(8)$ in the line of [6], and give proofs of Theorems 4 and 5. It is proved in [6] that the J-homomorphism $\widetilde{KO}(L^n(4)) \to \tilde{J}(L^n(4))$ is isomorphic, but, in this note, it will be shown that the J-homomorphism $\widetilde{KO}(L^n(8)) \to \tilde{J}(L^n(8))$ is not isomorphic for $n>1$.

§2. Proofs of Theorems 1–3

First, we recall some notations. Let X be a finite CW-complex. Denote by

$$r\colon K(X) \longrightarrow KO(X),\ c\colon KO(X) \longrightarrow K(X) \quad \text{and} \quad J\colon KO(X) \longrightarrow J(X)$$

the real restriction, the complexification and the J-homomorphism, respectively.

Let G be a finite group and a be an element of G. Then, by $\# G$ and $\# a$ we mean the order of G and the order of a, respectively.

Let η be the canonical complex line bundle over the standard lens space $L^l(p^r) = S^{2l+1}/Z_{p^r}$, where p is a prime and r is a positive integer, and let $\sigma = \eta - 1$ $(\in \tilde{K}(L^l(p^r)))$ be the stable class of η. Then the following is due to [4, Th. 1.1].

$$\# r\sigma = p^{r+[(l-2)/(p-1)]}, \qquad \text{for an odd prime} \quad p. \tag{2.1}$$

For $p=2$ and $r \geqq 2$, we obtain in [8, Th. 1.4] the following result, by completing the partial result of M. Yasuo [10, Prop. (3.5)].

$$\# r\sigma = \begin{cases} 2^{r+l-1}, & \text{if} \quad l \equiv 0 \bmod 2, \\ 2^{r+l-2}, & \text{if} \quad l \equiv 1 \bmod 2. \end{cases} \tag{2.2}$$

PROOF OF THEOREM 1. Let $\sigma = \eta - 1$ be the stable class of the canonical complex line bundle η over $L^{n-m}(p^r)$. If t is an integer given in the theorem, it follows from (2.1) and (2.2) that $J(tr\sigma) = 0$ in $J(L^{n-m}(p^r))$. Therefore we have

$$J(mr\eta \oplus 2t) = J(mr\eta) + J(2t) + J(tr\sigma) = J((m+t)r\eta).$$

According to [2, Prop. (2.6)], this means that $(L^{n-m}(p^r))^{mr\eta\oplus 2t}$ and $(L^{n-m}(p^r))^{(m+t)r\eta}$ have the same S-type. But by [2, Lemma (2.4)] and [3, Th. 1] (or [9, Th. 2.1]), we have the following homeomorphisms.

$$(L^{n-m}(p^r))^{mr\eta\oplus 2t} \approx S^{2t}(L^{n-m}(p^r))^{mr\eta} \approx S^{2t}(L^n(p^r)/L^{m-1}(p^r)),$$

$$(L^{n-m}(p^r))^{(m+t)r\eta} \approx L^{n+t}(p^r)/L^{m-1+t}(p^r).$$

Thus we see that $L^n(p^r)/L^{m-1}(p^r)$ and $L^{n+t}(p^r)/L^{m-1+t}(p^r)$ are S-equivalent. Other cases are proved similarly. q. e. d.

We determined $\# Jr\sigma$ for $r=2$ in [6], and for $r=3$ and 4 in [7]. If we use $\# Jr\sigma$ instead of $\#\sigma$ or $\# r\sigma$, we may obtain sharper results than Theorem 1.

PROOF OF THEOREM 2. In [7, Prop. 7.5, 7.7], the group $\tilde{J}(L_0^n(p^r))$ $(r=3, 4)$ is determined and the order of $Jr\sigma$ is given as follows: $\# Jr\sigma = p^{a_0+\varepsilon(r)}$. Then the proof is carried out in the same way as the above proof. q. e. d.

PROOF OF THEOREM 3. Since $L^n(p^r)/L^{m-1}(p^r) \approx (L^{n-m}(p^r))^{mr\eta}$ by [3, Th. 1] (or [9, Th. 2.1]), and since $L^{n-m}(p^r)^+$ $(=(L^{n-m}(p^r))^0)$ is S-equivalent to $(L^{n-m}(p^r))^{2m}$ by [2, Lemma (2.4)], we see from [2, Prop. (2.9)] and [7, Prop. 7.5, 7.7] that $L^n(p^r)/L^{m-1}(p^r)$ and $L^{n-m}(p^r)^+$ have the same S-type if and only if $m \equiv 0 \bmod p^{a_0+\varepsilon(r)}$. Another case is proved similarly. q. e. d.

REMARK. In Theorem 3, we have immediately the following result: $L^n(p^r)/L_0^m(p^r)$ (resp. $L_0^n(p^r)/L_0^m(p^r)$) is S-equivalent to $L^{n-m}(p^r)$ (resp. $L_0^{n-m}(p^r)$) if $m \equiv 0 \bmod p^{a_0+\varepsilon(r)}$. The similar result holds also for Theorem 5. But the converses cannot be proved by the same way, and we should omit in [6, Th. 1.6] the statements 'and only if' for the spaces $L^n(p^2)/L_0^m(p^2)$ and $L_0^n(p^2)/L_0^m(p^2)$.

§3. The structure of $\tilde{K}(L^n(8))$

Let η be the canonical complex line bundle over the lens space $L^n(8)=S^{2n+1}/Z_8$, and let $\sigma=\eta-1$ $(\in \tilde{K}(L^n(8)))$ be the stable class of η. Then the relations

$$(3.1) \qquad (\sigma+1)^8=1, \qquad \sigma^{n+1}=0$$

hold, and the elements $\sigma, \sigma^2, \dots, \sigma^7$ generate $\tilde{K}(L^n(8))$ (cf. [5, Lemma 3.6]). Furthermore, we have

$$(3.2) \qquad \#\tilde{K}(L^n(8)) = 8^n = 2^{3n} \qquad \text{(cf. [5, Lemma 3.8])}.$$

Let us define

$$(3.3) \qquad \sigma(1)=\eta^2-1=\sigma^2+2\sigma, \ \sigma(2)=\eta^4-1=\sigma(1)^2+2\sigma(1).$$

Then, from (3.1) and (3.3), we obtain

$$(3.4) \qquad \begin{aligned} &(\sigma(1)+1)^4=1, \quad (\sigma(2)+1)^2=1, \\ &\sigma(2)^i=(-1)^{i-1}2^{i-1}\sigma(2), \qquad \text{for} \quad i \geqq 1. \end{aligned}$$

By routine calculations using (3.1), we have

LEMMA (3.5). *The following relations hold.*

(3.5.1) $2^{i+3}\sigma^{n-i}=0$ *for* $0\leqq i\leqq n-1$, *if* $n\geqq 1$.

(3.5.2) $2^{i+2}\sigma^{n-i}\pm 2^{i+3}\sigma^{n-i-1}=0$ *for* $0\leqq i\leqq n-2$, *if* $n\geqq 2$.

(3.5.3) $2^2\sigma^{n-1}-2^3\sigma^{n-2}=0$ *if* $n\geqq 3$.

(3.5.4) $2^{i+1}\sigma^{n-i}+2^{i+2}\sigma^{n-i-1}=0$ *for* $2\leqq i\leqq n-2$, *if* $n\geqq 4$.

(3.5.5) $2\sigma^n-3\cdot 2^2\sigma^{n-2}-2^3\sigma^{n-3}=0$ *if* $n\geqq 4$.

(3.5.6) $2^3\sigma^{n-3}-3\cdot 2^4\sigma^{n-4}=0$ *if* $n\geqq 5$.

(3.5.7) $2^i\sigma^{n-i}+2^{i+1}\sigma^{n-i-1}=0$ *for* $4\leqq i\leqq n-2$, *if* $n\geqq 6$.

PROOF. Multiplying each side of the first equality $(\sigma+1)^8=1$ in (3.1) by σ^{n-1}, and using the second equality $\sigma^{n+1}=0$ in (3.1), we have $2^3\sigma^n=0$. Assume inductively $2^{i+2}\sigma^{n-i+1}=0$ for any i with $0<i\leqq n-1$. Multiplying each side of the first equality in (3.1) by $2^i\sigma^{n-i-1}$, and using the second equality in (3.1) and the inductive assumption, we get $2^{i+3}\sigma^{n-i}=0$. This completes the proof of (3.5.1). The proofs of the other equalities are similar. (Cf. [5, Lemmas 4.1 and 4.3].) q.e.d.

By routine calculations using (3.1), (3.3) and (3.4), we obtain

LEMMA (3.6). *Let* $m=[n/2]\geqq 0$. *Then the following relations hold.*

(3.6.1) $\sigma(1)^{m+2}=0$.

(3.6.2) $\sigma(1)^{m+1}\sigma=0$ *if* $n=2m$.

(3.6.3) $2^{i+2}\sigma(1)^{m+1-i}=0$ *for* $0\leqq i\leqq m$.

(3.6.4) $2^{i+1}\sigma(1)^{m+1-i}\pm 2^{i+2}\sigma(1)^{m-i}=0$ *for* $0\leqq i\leqq m-1$, *if* $m\geqq 1$.

(3.6.5) $2\sigma(1)^m-2^2\sigma(1)^{m-1}=0$ *if* $m\geqq 2$.

(3.6.6) $2^i\sigma(1)^{m+1-i}+2^{i+1}\sigma(1)^{m-i}=0$ *for* $2\leqq i\leqq m-1$, *if* $m\geqq 3$.

(3.6.7) $\sigma(1)^{m+1}-2\sigma(1)^{m-1}+2^2\sigma(1)^{m-2}=0$ *if* $m\geqq 3$.

(3.6.8) $2^2\sigma(1)^{m-2}-3\cdot 2^3\sigma(1)^{m-3}=0$ *if* $m\geqq 4$.

(3.6.9) $2^{i-1}\sigma(1)^{m+1-i}+2^i\sigma(1)^{m-i}=0$ *for* $4\leqq i\leqq m-1$, *if* $m\geqq 5$.

PROOF. Since $\sigma(1)=\sigma^2+2\sigma$ by the first equality in (3.3), we have

$$\sigma(1)^{m+2}=(\sigma^2+2\sigma)^{m+2}=\sum_{i=0}^{m+2}\binom{m+2}{i}2^i\sigma^{2m+4-i},$$

which, in turn, is equal to zero by (3.5.1) and the second equality in (3.1). Thus we have completed the proof of (3.6.1). In the same way we obtain (3.6.2).

Multiplying each side of the first equality in (3.4) by $\sigma(1)^m$ and using (3.6.1), we have $2^2\sigma(1)^{m+1}=0$. Assume inductively $2^{i+1}\sigma(1)^{m+2-i}=0$ for any i with $0<i\leqq m$. Multiplying each side of the first equality in (3.4) by $2^i\sigma(1)^{m-i}$, and using the inductive assumption, we have $2^{i+2}\sigma(1)^{m+1-i}=0$. This completes the proof of (3.6.3). The others are proved similarly. (Cf. [5, Lemmas 4.1 and 4.3].)

q. e. d.

We determine the structure of $\tilde{K}(L^n(8))$ using (3.1)–(3.6). The method of the proof is quite similar to that of [5, Th. A].

PROPOSITION (3.7). *If* $n\geqq 3$,

$$\tilde{K}(L^n(8))\cong Z_{2^{n+2}}\oplus Z_{2^{[n/2]+1}}\oplus Z_{2^{[(n+1)/2]}}\oplus Z_{2^{[n/4]}}$$
$$\oplus Z_{2^{[(n-1)/4]}}\oplus Z_{2^{[(n-2)/4]}}\oplus Z_{2^{[(n-3)/4]}}.$$

Each direct summand is generated by

$$\sigma,\ \sigma(1)\pm 2^{2t+2}\sigma,\ \sigma(1)\sigma,\ \sigma(2)\pm(2^{3t+3}\sigma+2^{t+1}\sigma(1)),\ \sigma(2)\sigma\pm 2^{3t+4}\sigma$$
$$\pm 2^{t+1}\sigma(1)\sigma,\ \sigma(2)\sigma(1)\pm 2^{t+2}\sigma(1),\ \sigma(2)\sigma(1)\sigma \qquad \textit{for} \quad n=4t+3;$$

$$\sigma,\ \sigma(1),\ \sigma(1)\sigma\pm 2^{2t+2}\sigma,\ \sigma(2)\pm 2^{3t+3}\sigma\pm 2^{t+1}\sigma(1),\ \sigma(2)\sigma\pm 2^{3t+3}\sigma,$$
$$\sigma(2)\sigma(1),\ \sigma(2)\sigma(1)\sigma\pm(2^{3t+3}\sigma-2^{t+1}\sigma(1)\sigma)\pm 2^{t+2}\sigma(1)$$
$$\textit{for} \quad n=4t+2;$$

$$\sigma,\ \sigma(1)\pm 2^{2t+1}\sigma,\ \sigma(1)\sigma,\ \sigma(2)\pm 2^{t+1}\sigma(1),\ \sigma(2)\sigma,\ \sigma(2)\sigma(1)\pm(2^{3t+2}\sigma$$
$$-2^{t+1}\sigma(1))\pm 2^{t+1}\sigma(1)\sigma,\ \sigma(2)\sigma(1)\sigma\pm 2^{3t+3}\sigma\pm 2^{t+2}\sigma(1)\pm 2^{t+1}\sigma(1)\sigma$$
$$\textit{for} \quad n=4t+1;$$

$$\sigma,\ \sigma(1),\ \sigma(1)\sigma\pm 2^{2t+1}\sigma,\ \sigma(2),\ \sigma(2)\sigma\pm(2^{3t+1}\sigma-2^t\sigma(1)\sigma),$$
$$\sigma(2)\sigma(1)\pm 2^{t+1}\sigma(1),\ \sigma(2)\sigma(1)\sigma\pm 2^{3t+2}\sigma \qquad \textit{for} \quad n=4t.$$

If $n=2$, $\tilde{K}(L^2(8))\cong Z_{2^4}\oplus Z_{2^2}$, *where the first summand is generated by* σ *and the second is generated by* $\sigma(1)$.

If $n=1$, $\tilde{K}(L^1(8))\cong Z_{2^3}$, *generated by* σ.

The multiplicative structure is given by

$$\sigma^8 = -\sum_{i=1}^{7}\binom{8}{i}\sigma^i, \quad \sigma^{n+1}=0.$$

PROOF. Let $n=4t$. Since the elements $\sigma, \sigma^2, \ldots, \sigma^7$ generate $\tilde{K}(L^n(8))$, the elements σ, $\sigma(1)=\sigma^2+2\sigma$, $\sigma(1)\sigma \pm 2^{2t+1}\sigma = \sigma^3+\cdots$, $\sigma(2)=\sigma^4+\cdots$, $\sigma(2)\sigma \pm (2^{3t+1}\sigma - 2^t\sigma(1)\sigma) = \sigma^5+\cdots$, $\sigma(2)\sigma(1) \pm 2^{t+1}\sigma(1) = \sigma^6+\cdots$, $\sigma(2)\sigma(1)\sigma \pm 2^{3t+2}\sigma = \sigma^7+\cdots$ generate $\tilde{K}(L^n(8))$. In order to get the additive structure of $\tilde{K}(L^n(8))$, it is sufficient to prove the following equalities:

$$2^{n+2}\sigma = 0,\ 2^{2t+1}\sigma(1) = 0,\ 2^{2t}(\sigma(1)\sigma \pm 2^{2t+1}\sigma) = 0,$$

$$2^t\sigma(2) = 0,\ 2^{t-1}(\sigma(2)\sigma \pm (2^{3t+1}\sigma - 2^t\sigma(1)\sigma)) = 0,$$

$$2^{t-1}(\sigma(2)\sigma(1) \pm 2^{t+1}\sigma(1)) = 0,\ 2^{t-1}(\sigma(2)\sigma(1)\sigma \pm 2^{3t+2}\sigma) = 0,$$

because $2^{n+2}\cdot 2^{2t+1}\cdot 2^{2t}\cdot 2^t\cdot(2^{t-1})^3\ (=2^{3n})$ is equal to the order of $\tilde{K}(L^n(8))$ by (3.2).

The first equality follows immediately from (3.5.1).

To prove the second and the third, put $n=2m$. Then we have

$$\begin{aligned}
2^{m+1}\sigma(1) &= \pm 2\sigma(1)^{m+1} && \text{by (3.6.4)}\\
&= \pm 2(\sigma^2+2\sigma)^{m+1} && \text{by (3.3)}\\
&= \pm \sum_{i=0}^{m+1}\binom{m+1}{i}2^{i+1}\sigma^{n+2-i} = 0 && \text{by (3.1) and (3.5.1)},\\
2^m\sigma(1)\sigma &= (-1)^{m-2}2^2\sigma(1)^{m-1}\sigma \quad \text{for } m\geqq 2 && \text{by (3.6.6)}\\
&= (-1)^m 2\sigma(1)^m\sigma && \text{by (3.6.5)}\\
&= (-1)^m\sum_{i=0}^{m}\binom{m}{i}2^{i+1}\sigma^{n+1-i} && \text{by (3.3)}\\
&= (-1)^m\sum_{i=1}^{m}\binom{m}{i}2^{n+1}\sigma && \text{by (3.1) and (3.5.2)}\\
&= (-1)^m(2^m-1)2^{n+1}\sigma = \pm 2^{n+1}\sigma && \text{by (3.1) and (3.5.1)},
\end{aligned}$$

and hence $2^m(\sigma(1)\sigma \pm 2^{m+1}\sigma)=0$. This holds also for $m=1$ by (3.5.1) and (3.5.2).

The fourth equality is proved as follows:

$$\begin{aligned}
2^t\sigma(2) &= (-1)^t\sigma(2)^{t+1} = (-1)^t(\sigma(1)^2+2\sigma(1))^{t+1} && \text{by (3.4)}\\
&= (-1)^t\sum_{i=0}^{t+1}\binom{t+1}{i}2^i\sigma(1)^{2t+2-i}\\
&= (-1)^t\left\{\sigma(1)^{m+2} + \sum_{i=1}^{t+1}\binom{t+1}{i}(-1)^{2t+1-i}2^{2t+1}\sigma(1)\right\}
\end{aligned}$$

by (3.6.4)

$$= (-1)^t 2^{2t+1}\sigma(1) \qquad \text{by (3.6.1)}$$

$$= 0 \qquad \text{by the second equality.}$$

Before proving the fifth equality, we notice that the following relations hold by Lemma (3.6).

$$2\sigma(1)^{m-1}\sigma - 2^2\sigma(1)^{m-2}\sigma = 0 \quad \text{if} \quad n = 2m \geqq 6. \tag{3.7.1}$$

$$2^2\sigma(1)^{m-2}\sigma + 2^3\sigma(1)^{m-3}\sigma = 0 \quad \text{if} \quad n = 2m \geqq 8. \tag{3.7.2}$$

Now, we have, by (3.4) and (3.3),

$$\begin{aligned} 2^{t-1}\sigma(2)\sigma &= (-1)^{t-1}\sigma(2)^t\sigma = (-1)^{t-1}(\sigma(1)^2 + 2\sigma(1))^t\sigma \\ &= (-1)^{t-1}\Big\{\sigma(1)^m\sigma + t\cdot 2\sigma(1)^{m-1}\sigma + \binom{t}{2}2^2\sigma(1)^{m-2}\sigma \\ &\quad + \textstyle\sum_{i=3}^{t}\binom{t}{i}2^i\sigma(1)^{m-i}\sigma\Big\}, \end{aligned}$$

where $n=2m=4t$. On the other hand,

$$\begin{aligned} \sigma(1)^m\sigma &= (\sigma^2 + 2\sigma)^m\sigma && \text{by (3.3)} \\ &= t\cdot 2^2\sigma^n + \binom{m}{2}2^2\sigma^{n-1} + \textstyle\sum_{i=3}^{m}\binom{m}{i}2^i\sigma^{n+1-i} && \text{by (3.1)} \\ &= \pm t\cdot 2^{n+1}\sigma + \binom{m}{2}2^3\sigma^{n-2} + \textstyle\sum_{i=3}^{m}\binom{m}{i}(-1)^{n-i}2^n\sigma \\ & && \text{by (3.5.2)–(3.5.4)} \\ &= \pm t\cdot 2^{n+1}\sigma - 2^n\sigma && \text{by (3.5.4) and (3.1)}, \end{aligned}$$

$$\begin{aligned} t\cdot 2\sigma(1)^{m-1}\sigma &= t\cdot 2^2\sigma(1)^{m-2}\sigma && \text{for} \quad m \geqq 3 \quad \text{by} \quad (3.7.1) \\ &= -t\cdot 2^3\sigma(1)^{m-3}\sigma && \text{for} \quad m \geqq 4 \quad \text{by} \quad (3.7.2) \\ &= -t\cdot 2^{m-1}\sigma(1)\sigma && \text{for} \quad m \geqq 5 \quad \text{by} \quad (3.6.9), \end{aligned}$$

$$\begin{aligned} \binom{t}{2}2^2\sigma(1)^{m-2}\sigma &= -\binom{t}{2}2^3\sigma(1)^{m-3}\sigma && \text{for} \quad m \geqq 4 \quad \text{by} \quad (3.7.2) \\ &= -\binom{t}{2}2^{m-1}\sigma(1)\sigma && \text{for} \quad m \geqq 5 \quad \text{by} \quad (3.6.9), \end{aligned}$$

$$\textstyle\sum_{i=3}^{t}\binom{t}{i}2^i\sigma(1)^{m-i}\sigma = \Big\{1 - t + \binom{t}{2}\Big\}2^{m-1}\sigma(1)\sigma \qquad \text{for} \quad m \geqq 5$$

by (3.6.9).

Therefore, we obtain, by the third equality,

$$2^{t-1}\sigma(2)\sigma = \pm(2^n\sigma - 2^{m-1}\sigma(1)\sigma), \qquad \text{for} \quad t>2,$$

and so $2^{t-1}(\sigma(2)\sigma \pm (2^{3t+1}\sigma - 2^t\sigma(1)\sigma)) = 0$. This holds also for $t=2$.

The remaining relations follow easily from the above relation.

The proofs in the other cases are similar, and hence we omit the details here.

The multiplicative structure is given by (3.1). q.e.d.

§4. The structure of $\widetilde{KO}(L^n(8))$ and $\widetilde{KO}(L_0^n(8))$

The following is obtained by considering the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}(L^n(8))$.

LEMMA (4.1). *$\widetilde{KO}(L^n(8))$ has only 2-component, and*

$$\# \widetilde{KO}(L^n(8)) \leqq \begin{cases} 2^{8t+1} & \textit{for} \quad n=4t, \\ 2^{8t+2} & \textit{for} \quad n=4t+1, \\ 2^{8t+5} & \textit{for} \quad n=4t+2, \\ 2^{8t+5} & \textit{for} \quad n=4t+3. \end{cases}$$

Let ρ be the non-trivial real line bundle over $L^n(8)$, and let $\kappa = \rho - 1$ ($\in \widetilde{KO}(L^n(8))$) be the stable class of ρ. Then we have

LEMMA (4.2). *For the complexification c and the real restriction r,*

(4.2.1) $\quad c\kappa = \sigma(2)$,

(4.2.2) $\quad cr\sigma = \sigma^2/(\sigma+1)$

$$= 2\sigma + \sigma(1) + \sigma(1)\sigma + \sigma(2) + \sigma(2)\sigma + \sigma(2)\sigma(1) + \sigma(2)\sigma(1)\sigma,$$

(4.2.3) $\quad cr\sigma(1) = \sigma(1)^2/(\sigma(1)+1) = 2\sigma(1) + \sigma(2) + \sigma(2)\sigma(1)$,

(4.2.4) $\quad c(\kappa \cdot r\sigma) = -2\sigma(2) - \sigma(2)\sigma(1) - \sigma(2)\sigma(1)\sigma$.

PROOF. According to [5, Prop. 3.3], $c\rho = \eta^4$. Therefore

$$c\kappa = c\rho - 1 = \eta^4 - 1 = \sigma(2),$$

by the second equality of (3.3). Thus we obtain (4.2.1).

To prove (4.2.2), let t be the conjugation. Then $cr = 1 + t$ and $t\sigma = t(\eta - 1)$

$=\eta^{-1}-1=(\sigma+1)^{-1}-1$. Hence

$$\begin{aligned} cr\sigma &= (1+t)\sigma = \sigma + (\sigma+1)^{-1} - 1 \quad (=\sigma^2/(\sigma+1)) \\ &= \sigma + (\sigma+1)^7 - 1 \qquad \text{by the first equality of (3.1)} \\ &= \sigma + (\sigma+1)(\sigma(1)+1)(\sigma(2)+1) - 1 \qquad \text{by (3.3)} \\ &= 2\sigma + \sigma(2)\sigma(1)\sigma + \sigma(1)\sigma + \sigma(2)\sigma + \sigma(2)\sigma(1) + \sigma(2) + \sigma(1). \end{aligned}$$

Thus we complete the proof of (4.2.2). Other equalities are obtained similarly.

q. e. d.

Using Proposition (3.7), Lemmas (4.1) and (4.2), we determine the structure of $\widetilde{KO}(L^n(8))$ as follows. The method of the proof is similar to that of [5, Th. B].

PROPOSITION (4.3). *In case* $n=4t+3$ *or* $4t+2$ $(t>0)$,

$$\widetilde{KO}(L^n(8)) \cong Z_{2^{4t+4}} \oplus Z_{2^{2t+1}} \oplus Z_{2^t} \oplus Z_{2^t}.$$

Each direct summand is generated by $r\sigma$, $r\sigma(1)+2^{2t+2}r\sigma$, $\kappa+2^t(r\sigma(1)+2^{2t+2}r\sigma)$, $\kappa\cdot r\sigma-2^{3t+3}r\sigma$, *respectively.*

In case $n=4t+\varepsilon$, $\varepsilon=0$ *or* 1 $(t>0)$,

$$\widetilde{KO}(L^n(8)) \cong Z_{2^{4t+2}} \oplus Z_{2^{2t}} \oplus Z_{2^t} \oplus Z_{2^{t-1+\varepsilon}}.$$

Each direct summand is generated by $r\sigma$, $r\sigma(1)+2^{2t+1}r\sigma$, $\kappa+2^t r\sigma(1)$, $\kappa\cdot r\sigma+2^t(r\sigma(1)+2^{2t+1}r\sigma)$, *respectively, where the last generator may be replaced by* $\kappa\cdot r\sigma$ *if* $n=4t+1$.

In case $n=3$, 2 *or* 1,

$$\widetilde{KO}(L^3(8)) \cong \widetilde{KO}(L^2(8)) \cong Z_{2^4} \oplus Z_2, \quad \widetilde{KO}(L^1(8)) \cong Z_2 \oplus Z_2,$$

where, in each group, the first summand is generated by $r\sigma$ *and the second is generated by* κ.

PROOF. First we consider the case $n=4t+3$ $(t>0)$. Put the generators of $\tilde{K}(L^n(8))$ as follows (cf. Prop. (3.7)):

$$\sigma=\sigma_1,\ \sigma(1)+2^{2t+2}\sigma=\sigma_2,\ \sigma(1)\sigma=\sigma_3,\ \sigma(2)+2^{3t+3}\sigma+2^{t+1}\sigma(1)=\sigma_4,$$

$$\sigma(2)\sigma-2^{3t+4}\sigma-2^{t+1}\sigma(1)\sigma=\sigma_5,\ \sigma(2)\sigma(1)-2^{t+2}\sigma(1)=\sigma_6,$$

$$\sigma(2)\sigma(1)\sigma=\sigma_7.$$

Define $A=r\sigma$, $B=r\sigma(1)+2^{2t+2}r\sigma$, $C=\kappa+2^t(r\sigma(1)+2^{2t+2}r\sigma)$, $D=\kappa\cdot r\sigma-2^{3t+3}r\sigma$; and $A'=cA$, $B'=cB$, $C'=cC$, $D'=cD$. Then

$$A' = cr\sigma = \sigma_7 + \sigma_6 + \sigma_5 + \sigma_4 + (1 + 2^{t+1})\sigma_3 + (1 + 2^{t+1})\sigma_2$$
$$+ 2(1 - 2^{2t+1})\sigma_1,$$
$$B' = c(r\sigma(1) + 2^{2t+2}r\sigma) = \sigma_6 + \sigma_4 + 2(1 + 2^t)\sigma_2 + 2^{3t+4}(2^t - 1)\sigma_1,$$
$$C' = c(\kappa + 2^t(r\sigma(1) + 2^{2t+2}r\sigma)) = \sigma_4 + 2^{2t+1}\sigma_2 + 2^{4t+4}\sigma_1,$$
$$D' = c(\kappa\cdot r\sigma - 2^{3t+3}r\sigma) = -\sigma_7 - \sigma_6 - 2\sigma_4.$$

It can be seen without difficulty that the elements A', B', C' and D' generate the subgroup

$$Z_{2^{4t+4}} \oplus Z_{2^{2t+1}} \oplus Z_{2^t} \oplus Z_{2^t}$$

of $\tilde{K}(L^n(8))$. Since the order of this subgroup is $2^{4t+4}\cdot 2^{2t+1}\cdot(2^t)^2 = 2^{8t+5}$, the order of the subgroup (of $\widetilde{KO}(L^n(8))$) generated by A, B, C and D is larger than or equal to 2^{8t+5}. On the other hand, $\#\widetilde{KO}(L^{4t+3}(8)) \leqq 2^{8t+5}$ by Lemma (4.1). It follows that A, B, C and D generate $\widetilde{KO}(L^{4t+3}(8))$ and that $\#\widetilde{KO}(L^{4t+3}(8)) = 2^{8t+5}$.

In case $n=4t+2$ $(t>0)$, the inclusion map $j\colon L^n(k)\to L^{n+1}(k)$ induces the isomorphism $j^!\colon \widetilde{KO}(L^{n+1}(k))\to\widetilde{KO}(L^n(k))$ for any integer $k>1$. Therefore we obtain the result for the case $n=4t+2$ from the result for the case $n=4t+3$. It follows that $\widetilde{KO}(L^n(8))$ is additively generated by $r\sigma$, $r\sigma(1)$, κ and $\kappa\cdot r\sigma$ for the cases $n=4t+3$ and $n=4t+2$.

As in the proofs of Lemmas 5.9–5.12 in [5], we see that the inclusion map $L^{4t+1}(8)\to L^{4t+2}(8)$ induces the epimorphism $\widetilde{KO}(L^{4t+2}(8))\to\widetilde{KO}(L^{4t+1}(8))$ and that $\#\widetilde{KO}(L^{4t+1}(8)) = 2^{8t+2}$. Thus the result for the case $n=4t+1$ $(t>0)$ follows from the relations:

$$2^{4t+2}r\sigma = 0,\quad 2^{2t}(r\sigma(1) + 2^{2t+1}r\sigma) = 0,$$
$$2^t(\kappa + 2^t r\sigma(1)) = 0,\quad 2^t\kappa\cdot r\sigma = 0,$$

which are proved as follows. In $\widetilde{KO}(L^{4t+2}(8))$, we have

$$2^{4t+4}r\sigma = 0, \tag{1}$$
$$2^{2t+1}(r\sigma(1) + 2^{2t+2}r\sigma) = 0, \tag{2}$$
$$2^t(\kappa + 2^t(r\sigma(1) + 2^{2t+2}r\sigma)) = 0, \tag{3}$$
$$2^t(\kappa\cdot r\sigma - 2^{3t+3}r\sigma) = 0. \tag{4}$$

On the other hand, by (4.2.2) and (4.2.3),

$$c(r\sigma(1)+2^{2t+1}r\sigma)=(\sigma(2)\sigma(1)+2^{3t+2}\sigma-2^{t+1}\sigma(1)+2^{t+1}\sigma(1)\sigma)$$
$$+(\sigma(2)+2^{t+1}\sigma(1))-2^{t+1}\sigma(1)\sigma+2(\sigma(1)+2^{2t+1}\sigma)-2^{3t+2}(2^t+1)\sigma.$$

Thus we see from Proposition (3.7) that, in $\tilde{K}(L^{4t+1}(8))$, $2^{2t}c(r\sigma(1)+2^{2t+1}r\sigma)=0$. Since $rc=2$, we have

$$2^{2t+1}(r\sigma(1)+2^{2t+1}r\sigma)=0. \tag{5}$$

It follows from (2) and (5) that $2^{4t+2}r\sigma=0$. This equality combined with (3) and (4) gives $2^t(\kappa+2^t r\sigma(1))=0$ and $2^t\kappa\cdot r\sigma=0$, respectively. Now, by (4.2.2)–(4.2.4),

$$c(\kappa\cdot r\sigma+2^t(r\sigma(1)+2^{2t+1}r\sigma)+2^{4t+1}r\sigma)$$
$$=-(\sigma(2)\sigma(1)\sigma-2^{3t+3}\sigma-2^{t+2}\sigma(1)-2^{t+1}\sigma(1)\sigma)-(\sigma(2)\sigma(1)+2^{3t+2}\sigma$$
$$-2^{t+1}\sigma(1)+2^{t+1}\sigma(1)\sigma)-2(\sigma(2)+2^{t+1}\sigma(1)).$$

Hence we see from Proposition (3.7) that, in $\tilde{K}(L^{4t+1}(8))$, $2^{t-1}c(\kappa\cdot r\sigma+2^t(r\sigma(1)+2^{2t+1}r\sigma)+2^{4t+1}r\sigma)=0$. Since $rc=2$,

$$2^t(\kappa\cdot r\sigma+2^t(r\sigma(1)+2^{2t+1}r\sigma)+2^{4t+1}r\sigma)=0.$$

Therefore, we obtain $2^{2t}(r\sigma(1)+2^{2t+1}r\sigma)=0$.

Finally, we consider the case $n=4t$ $(t>0)$. As in the previous case, we notice that the inclusion $i\colon L^{4t}(8)\to L^{4t+1}(8)$ induces the epimorphism $i^!\colon \widetilde{KO}(L^{4t+1}(8))\to\widetilde{KO}(L^{4t}(8))$, that $\#\widetilde{KO}(L^{4t}(8))=2^{8t+1}$, and that $\operatorname{Ker} i^!$ is equal to the kernel of the complexification $c\colon\widetilde{KO}(L^{4t+1}(8))\to\tilde{K}(L^{4t+1}(8))$, which is isomorphic to Z_2 generated by $2^{t-1}(\kappa\cdot r\sigma+2^t(r\sigma(1)+2^{2t+1}r\sigma))$. For the proofs of these facts, refer to [5, Lemmas 5.14–5.17]. Combining the result for the case $n=4t+1$ with the above facts, we obtain the result for the case $n=4t$.

The proofs in the remaining cases are easy. q.e.d.

COROLLARY (4.4). *In case $n=4t+3$, the complexification $c\colon\widetilde{KO}(L^n(8))\to\tilde{K}(L^n(8))$ is monomorphic.*

Next, we determine the structure of $\widetilde{KO}(L_0^n(8))$, where $L_0^n(8)$ is the $2n$-skeleton of the standard CW-decomposition of $L^n(8)$.

PROPOSITION (4.5). *In case $n\not\equiv 0 \bmod 4$, $\widetilde{KO}(L_0^n(8))\cong\widetilde{KO}(L^n(8))$. In case $n=4t$,*

$$\widetilde{KO}(L_0^n(8))\cong Z_{2^{4t+1}}\oplus Z_{2^{2t}}\oplus Z_{2^t}\oplus Z_{2^{t-1}}.$$

Each direct summand is generated by $r\sigma$, $r\sigma(1)$, κ, $\kappa\cdot r\sigma-2^{3t+1}r\sigma-2^t r\sigma(1)$, res-

pectively.

PROOF. The first part follows easily from the Puppe exact sequence.

To prove the second part, we put the generators of $\tilde{K}(L_0^n(8))$ $(\cong \tilde{K}(L^n(8)))$ as follows (cf. Prop. (3.7)):

$$\sigma = \sigma_1,\ \sigma(1) = \sigma_2,\ \sigma(1)\sigma + 2^{2t+1}\sigma = \sigma_3,\ \sigma(2) = \sigma_4,$$

$$\sigma(2)\sigma - (2^{3t+1}\sigma - 2^t\sigma(1)\sigma) = \sigma_5,\ \sigma(2)\sigma(1) + 2^{t+1}\sigma(1) = \sigma_6,$$

$$\sigma(2)\sigma(1)\sigma + 2^{3t+2}\sigma = \sigma_7.$$

Define $A=r\sigma$, $B=r\sigma(1)$, $C=\kappa$, $D=\kappa\cdot r\sigma-2^{3t+1}r\sigma-2^t r\sigma(1)$; and $A'=cA$, $B'=cB$, $C'=cC$, $D'=cD$. Then

$$\begin{aligned} A' &= cr\sigma = \sigma_7 + \sigma_6 + \sigma_5 + \sigma_4 + (1-2^t)\sigma_3 + (1-2^{t+1})\sigma_2 + (2-2^{2t+1})\sigma_1, \\ B' &= cr\sigma(1) = \sigma_6 + \sigma_4 + (2-2^{t+1})\sigma_2, \\ C' &= c\kappa = \sigma_4, \\ D' &= c(\kappa\cdot r\sigma - 2^{3t+1}r\sigma - 2^t r\sigma(1)) = -\sigma_7 - \sigma_6 - 2\sigma_4. \end{aligned}$$

It can be easily shown that the elements A', B', C' and D' generate the subgroup

$$Z_{2^{4t+1}} \oplus Z_{2^{2t}} \oplus Z_{2^t} \oplus Z_{2^{t-1}}$$

of $\tilde{K}(L_0^{4t}(8))$. Now, we see $\#\widetilde{KO}(L_0^{4t}(8)) \leqq 2^{8t}$ in the way similar to Lemma (4.1). Therefore, A, B, C and D generate $\widetilde{KO}(L_0^n(8))$, as desired. q.e.d.

The multiplicative structure of $\widetilde{KO}(L^n(8))$ and $\widetilde{KO}(L_0^n(8))$ is given by the next proposition.

PROPOSITION (4.6). *The following relations hold.*

$$(r\sigma)^2 = -4r\sigma + r\sigma(1),\ (r\sigma(1))^2 = -4r\sigma(1) + 2\kappa,$$

$$\kappa^2 = -2\kappa = \kappa\cdot r\sigma(1),\ r\sigma\cdot r\sigma(1) = \kappa\cdot r\sigma - 2r\sigma(1) + 2\kappa.$$

PROOF. It is sufficient to prove the relations in the case $n=4t+3$. According to Corollary (4.4), the complexification c is monomorphic in this case. Hence, the first equality is seen from the equalities.

$$\begin{aligned} c(r\sigma)^2 &= \sigma^4/(1+\sigma)^2 && \text{by (4.2.2)} \\ &= \{-4\sigma^2(1+\sigma) + (2\sigma+\sigma^2)^2\}/(1+\sigma)^2 \\ &= -4\sigma^2/(1+\sigma) + \sigma(1)^2/(1+\sigma(1)) && \text{by (3.3)} \end{aligned}$$

$$= c(-4r\sigma + r\sigma(1)) \qquad \text{by (4.2.2)–(4.2.3)}.$$

Other equalities are obtained similarly. q. e. d.

§5. The structure of $J(L^n(8))$ and $J(L_0^n(8))$

It is known that the J-homomorphism $J\colon \widetilde{KO}(X)\to\tilde{J}(X)$ is isomorphic if X is the real projective space RP^n [1, (6.3)] or the mod 4 lens space $L^n(4)$ [6, Th. 4.5]. In this section we see that J is not isomorphic for the mod 8 lens space $L^n(8)$ $(n>1)$.

In order to study the groups $J(L^n(8))$ and $J(L_0^n(8))$ we determine the Adams operations.

LEMMA (5.1). *The Adams operation Ψ^j on $\widetilde{KO}(L^n(8))$ (or $\widetilde{KO}(L_0^n(8))$) is given by*

$$\Psi^j(\kappa) = \begin{cases} \kappa & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

$$\Psi^j(r\sigma) = \begin{cases} r\sigma & \text{if } j \equiv \pm 1 \bmod 8, \\ r\sigma(1) & \text{if } j \equiv \pm 2 \bmod 8, \\ r\sigma + \kappa\cdot r\sigma + 2\kappa & \text{if } j \equiv \pm 3 \bmod 8, \\ 2k & \text{if } j \equiv 4 \bmod 8, \\ 0 & \text{if } j \equiv 0 \bmod 8. \end{cases}$$

$$\Psi^j(r\sigma(1)) = \begin{cases} r\sigma(1) & \text{if } j \text{ is odd,} \\ 2\kappa & \text{if } j \equiv 2 \bmod 4, \\ 0 & \text{if } j \equiv 0 \bmod 4. \end{cases}$$

PROOF. As in the proof of Proposition (4.6), it is sufficient to prove the equalities in the case $n=4t+3$. Let us prove $\Psi^j(r\sigma)=r\sigma+\kappa\cdot r\sigma+2\kappa$ if $j\equiv \pm 3 \bmod 8$. In fact,

$$\begin{aligned} c\Psi^j(r\sigma) &= \Psi^j(cr\sigma) = \Psi^j(\sigma^2/(\sigma+1)) && \text{by (4.2.2)} \\ &= \Psi^j\sigma^2/\Psi^j(\sigma+1) = ((\sigma+1)^j-1)^2/(\sigma+1)^j \\ &= (\sigma+1)^3 - 2 + (\sigma+1)^5 && \text{by the first equality of (3.1)} \\ &= \sigma(2)\sigma + \sigma(2) + \sigma(1)\sigma + \sigma(1) + 2\sigma && \text{by (3.3)} \end{aligned}$$

$$= cr\sigma + c(\kappa\cdot r\sigma) + 2c\kappa \qquad \text{by (4.2.1), (4.2.2), (4.2.4)}$$

$$= c(r\sigma + \kappa\cdot r\sigma + 2\kappa).$$

Other equalities are proved similarly. q.e.d.

Now, we prove

LEMMA (5.2). *The kernel of the J-homomorphism*

$$J\colon \widetilde{KO}(L^n(8)) \longrightarrow \tilde{J}^n(L(8)) \quad (or\ J\colon \widetilde{KO}(L_0^n(8)) \longrightarrow \tilde{J}(L_0^n(8)))$$

is the submodule of $\widetilde{KO}(L^n(8))$ *generated by* $\kappa\cdot r\sigma + 2\kappa$, *whose order is* $2^{[(n+2)/4]}$.

PROOF. Notice that

$$\operatorname{Ker} J = \textstyle\sum_k \bigcap_e k^e(\Psi^k - 1)\widetilde{KO}(L^n(8)) \quad (\text{cf. } [7, (1.1)]).$$

If $k=2l$ $(l\geqq 1)$, $k^e\widetilde{KO}(L^n(8)) = l^e 2^e \widetilde{KO}(L^n(8)) = 0$ for a sufficiently large integer e by Proposition (4.3).

If $k\equiv \pm 1 \bmod 8$, $\Psi^k - 1 = 0$ by Lemma (5.1).

If $k\equiv \pm 3 \bmod 8$, $k^e(\Psi^k-1)\widetilde{KO}(L^n(8)) = (\Psi^3-1)\widetilde{KO}(L^n(8))$ for a sufficiently large integer e by Proposition (4.3) and the proof of Lemma (5.1). On the other hand, we have

$$(\Psi^3-1)(\kappa) = (\Psi^3-1)(r\sigma(1)) = 0,\ (\Psi^3-1)(r\sigma) = \kappa\cdot r\sigma + 2\kappa,$$

$$(\Psi^3-1)(r\sigma\cdot r\sigma(1)) = (\Psi^3-1)(\kappa\cdot r\sigma - 2r\sigma(1) + 2\kappa) = (\Psi^3-1)(\kappa\cdot r\sigma)$$

$$= \kappa(r\sigma + \kappa\cdot r\sigma + 2\kappa) - \kappa\cdot r\sigma = \kappa^2\cdot r\sigma + 2\kappa^2 = -2(\kappa\cdot r\sigma + 2\kappa)$$

by Lemma (5.1) and Proposition (4.6). Thus, $\operatorname{Ker} J$ is generated by $\kappa\cdot r\sigma + 2\kappa$. The order of $\kappa\cdot r\sigma + 2\kappa$ is easily seen by Proposition (4.3). q.e.d.

Combining Proposition (4.3) with Lemma (5.2), we obtain

PROPOSITION (5.3). *If* $n=4t+3$ *or* $n=4t+2$,

$$\tilde{J}(L^n(8)) \cong \tilde{J}(L_0^n(8)) \cong Z_{2^{4t+3}} \oplus Z_{2^{2t+1}} \oplus Z_{2^t}.$$

Each direct summand is generated by $Jr\sigma$, $Jr\sigma(1) + 2^{2t+2}Jr\sigma$, $J\kappa + 2^t(Jr\sigma(1) + 2^{2t+2}Jr\sigma)$, *respectively.*

If $n=4t+1$ $(t>0)$,

$$\tilde{J}(L^n(8)) \cong \tilde{J}(L_0^n(8)) \cong Z_{2^{4t+2}} \oplus Z_{2^{2t}} \oplus Z_{2^t}.$$

Each direct summand is generated by $Jr\sigma$, $Jr\sigma(1) + 2^{2t+1}Jr\sigma$, $J\kappa + 2^tJr\sigma(1)$, *respectively.*

If $n=4t$,

$$\tilde{J}(L^n(8)) \cong Z_{2^{4t+2}} \oplus Z_{2^{2t-1}} \oplus Z_{2^t},$$

where each direct summand is generated by $J r\sigma$, $J r\sigma(1)-2^{2t+1}J r\sigma$, $J\kappa+2^t J r\sigma(1)$, *respectively, and*

$$\tilde{J}(L_0^n(8)) \cong Z_{2^{4t+1}} \oplus Z_{2^{2t-1}} \oplus Z_{2^t},$$

where each direct summand is generated by $J r\sigma$, $J r\sigma(1)+2^{2t+1}J r\sigma$, $J\kappa$, *respectively.*

If $n=1$,

$$\tilde{J}(L^1(8)) \cong \tilde{J}(L_0^1(8)) \cong Z_2 \oplus Z_2 \quad (\cong \widetilde{KO}(L^1(8))),$$

where the first summand is generated by $J r\sigma$ *and the second is generated by* $J\kappa$.

PROOF. Consider the case $n=4t+3$. Since $J(\kappa\cdot r\sigma)=-2J\kappa$ and J is epimorphic, the elements $J r\sigma$, $J r\sigma(1)+2^{2t+2}J r\sigma$ and $J\kappa+2^t(J r\sigma(1)+2^{2t+2}J r\sigma)$ generate $\tilde{J}(L^n(8))$. By Proposition (4.3), the equality

$$\begin{aligned}\kappa\cdot r\sigma + 2\kappa &= (\kappa\cdot r\sigma - 2^{3t+3}r\sigma) + 2(\kappa + 2^t(r\sigma(1) + 2^{2t+2}r\sigma)) \\ &\quad - 2^{t+1}(r\sigma(1) + 2^{2t+2}r\sigma) + 2^{3t+3}r\sigma\end{aligned}$$

implies that $2^{4t+3}J r\sigma=0$. On the other hand, we have evidently

$$2^{2t+1}(J r\sigma(1) + 2^{2t+2}J r\sigma) = 0,\ 2^t(J\kappa + 2^t(J r\sigma(1) + 2^{2t+2}J r\sigma)) = 0,$$

by Proposition (4.3). But, $2^{4t+3}\cdot 2^{2t+1}\cdot 2^t = 2^{7t+4} = \#\widetilde{KO}(L^n(8))/\#\operatorname{Ker} J = \#\tilde{J}(L^n(8))$. Hence we get the desired result.

The other cases are similar. q.e.d.

As a corollary we obtain

COROLLARY (5.4). *The order of the element* $J r\sigma$ *is equal to*

2^{n+2} *in* $J(L^n(8))$ *for* $n\equiv 0 \bmod 4$,

2^{n+1} *in* $J(L^n(8))$ *for* $n\equiv 1, 2 \bmod 4$ *and* $n>1$,

or in $J(L_0^n(8))$ *for* $n\not\equiv 3 \bmod 4$ *and* $n>1$,

2^n *in* $J(L^n(8))$ *or in* $J(L_0^n(8))$ *for* $n\equiv 3 \bmod 4$ *or* $n=1$.

Now we are in a position to prove Theorems 4 and 5.

PROOF OF THEOREM 4. According to Corollary (5.4), the element $Jr\sigma$ ($\in J(L^{n-m}(8))$) is of order $2^{n-m+\varepsilon}$ and the element $Jr\sigma$ ($\in J(L_0^{n-m}(8))$) is of order $2^{n-m+\varepsilon'}$. Then the theorem is proved in the same way as Theorem 1.

q.e.d.

PROOF OF THEOREM 5. Using Corollary (5.4) we prove the theorem in the same way as Theorem 3. q.e.d.

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