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The Minimal Condition for Ascendant Subalgebras of Lie Algebras

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1.

Let L be a Lie algebra over an arbitrary field Φ which is not necessarily of finite dimension. We write $H \leq L$ when H is a subalgebra of L and $H \lhd L$ when H is an ideal of L. For an integer $n \geq 0$, $H \leq L$ is an n-step subideal of L if there is a series of not necessarily distinct subalgebras

$$H = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_{n-1} \lhd H_n = L.$$

In this case we write $H \lhd^n L$. H is a subideal of L if $H \lhd^n L$ for some n. We then write H si L.

There is a transfinite generalization: For an ordinal σ, $H \leq L$ is a σ-step ascendant subalgebra of L if there is a series $\{H_\alpha\}_{\alpha \leq \sigma}$ of not necessarily distinct subalgebras of L such that

(i) $H_0 = H$, $H_\sigma = L$,
(ii) $H_\alpha \lhd H_{\alpha+1}$ for any $\alpha < \sigma$,
(iii) $H_\lambda = \bigcup_{\alpha < \lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

In this case we write $H \lhd^\sigma L$. H is an ascendant subalgebra of L if $H \lhd^\sigma L$ for some ordinal σ. We then write H asc L. When σ is finite, the σ-step ascendant subalgebras are of course the σ-step subideals.

We denote by Min-si (resp. Min-$\lhd^\sigma$) the class of Lie algebras over Φ satisfying the minimal condition for subideals (resp. σ-step ascendant subalgebras).

The classes Min-si and Min-$\lhd^n$ ($n \in \mathbf{N}$) are related by the series of inclusions

$$\text{Min-}\lhd \supseteq \text{Min-}\lhd^2 \supseteq \text{Min-}\lhd^3 \supseteq \cdots \supseteq \text{Min-si}. \tag{1}$$

In [2] Stewart showed that

$$\text{Min-}\lhd^3 = \text{Min-si}. \tag{2}$$

The purpose of this paper is to investigate the minimal condition for ascendant subalgebras and to show a transfinite analogue of the result (2).

We introduce the class Min-(asc of step $< \sigma$). By this we mean the class of Lie algebras over Φ satisfying the minimal condition for ascendant subalgebras of step$<\sigma$. Then, for the first infinite ordinal ω, Min-(asc of step$<\omega$) is nothing but Min-si. Corresponding to (1), we have the following inclusions:

$$\begin{array}{ccccccc}
\text{Min-}\lhd^{\omega} & \supseteq \text{Min-}\lhd^{\omega+1} & \supseteq\cdots\supseteq & \text{Min-}\lhd^{\omega n_1+n_2} & \supseteq\cdots\supseteq & \text{Min-(asc of step}<\omega^2) \\
\text{Min-}\lhd^{\omega^2} & \supseteq \text{Min-}\lhd^{\omega^2+1} & \supseteq\cdots\supseteq & \text{Min-}\lhd^{\omega^2 n_1+\omega n_2+n_3} & \supseteq\cdots\supseteq & \text{Min-(asc of step}<\omega^3) \\
\vdots & \vdots & & \vdots & & \ddots \\
\text{Min-}\lhd^{\omega^\alpha} & \supseteq \text{Min-}\lhd^{\omega^\alpha+1} & \supseteq\cdots & \cdots & \cdots\supseteq & \text{Min-(asc of step}<\omega^{\alpha+1}) \\
\vdots & \vdots & & & & \ddots
\end{array}$$

We shall show that for any ordinal $\alpha\geq 1$

$$\text{Min-}\lhd^{\omega^\alpha+1} = \text{Min-(asc of step} < \omega^{\alpha+1}).$$

2.

For a Lie algebra L, $L^\omega=\bigcap_{n=1}^{\infty} L^n$ and $L^{(\omega)}=\bigcap_{n=1}^{\infty} L^{(n)}$. The following lemma is well known (see [3]).

Lemma 1. *If L is a Lie algebra and H* si *L, then $H^\omega \lhd L$ and $H^{(\omega)}\lhd L$.*

L is perfect if $L=L^2$. By using Lemma 1 and transfinite induction, we can easily show the following lemma ([1, p. 11]).

Lemma 2. *Every perfect ascendant subalgebra of a Lie algebra L is an ideal of L.*

Furthermore we need the following two lemmas.

Lemma 3. Min-(asc of step$<\sigma$) *is* E*-closed.*

Proof. Let L be a Lie algebra and assume that $N\lhd L$ and N, $L/N\in$ Min-(asc of step$<\sigma$). If

$$H_1\geq H_2\geq\cdots,\quad H_i\lhd^{\sigma_i} L \quad\text{with}\quad \sigma_i<\sigma \qquad (i=1,2,\dots),$$

then

$$H_1\cap N\geq H_2\cap N\geq\cdots,\quad H_i\cap N\lhd^{\sigma_i} N,$$

$$(H_1+N)/N\geq (H_2+N)/N\geq\cdots,\quad (H_i+N)/N\lhd^{\sigma_i} L/N.$$

There exists $n\in\mathbf{N}$ such that

$$H_n\cap N=H_{n+1}\cap N=\cdots,$$

$$(H_n + N)/N = (H_{n+1} + N)/N = \cdots.$$

Therefore for any $m \geq n$

$$H_m = H_{m+1} + (H_m \cap N) = H_{m+1} + (H_{m+1} \cap N) = H_{m+1}.$$

Thus $L \in$ Min-(asc of step$<\sigma$).

LEMMA 4. *Let α and β be any ordinals such that $\omega^\alpha < \beta < \omega^{\alpha+1}$. Then there exist ordinals ρ and σ such that*

$$\beta = \rho + \sigma \quad \textit{and} \quad 1 \leq \sigma \leq \omega^\alpha.$$

PROOF. β can be written in the form

$$\beta = \omega^\alpha \gamma + \delta \quad \text{with} \quad 1 \leq \gamma < \omega \quad \text{and} \quad \delta < \omega^\alpha.$$

If $\delta = 0$, $\gamma \geq 2$ and we can take

$$\rho = \omega^\alpha(\gamma - 1) \quad \text{and} \quad \sigma = \omega^\alpha.$$

If $\delta > 0$, it suffices to take

$$\rho = \omega^\alpha \gamma \quad \text{and} \quad \sigma = \delta.$$

3.

We shall now prove the following

THEOREM. *Let α and β be any ordinals such that $\alpha \geq 1$ and $\omega^\alpha < \beta < \omega^{\alpha+1}$. Then*

$$\text{Min-}\lhd^{\omega^\alpha+1} = \text{Min-}\lhd^{\beta} = \text{Min-(asc of step} < \omega^{\alpha+1}).$$

PROOF. Put $\gamma = \omega^\alpha$ and $\delta = \omega^{\alpha+1}$. Then

$$\text{Min-}\lhd^{\gamma+1} \supseteq \text{Min-}\lhd^{\beta} \supseteq \text{Min-(asc of step} < \delta).$$

Assume that there exists a Lie algebra L such that

$$L \in \text{Min-}\lhd^{\gamma+1} \quad \text{and} \quad L \notin \text{Min-(asc of step} < \delta).$$

(a) There exists M minimal with respect to

$$M \lhd L \quad \text{and} \quad M \notin \text{Min-(asc of step} < \delta).$$

This follows immediately from the fact that $L \in$ Min-$\lhd$.

(b) Any proper ideal N of M belongs to Min-(asc of step$<\delta$). In fact, we have

$$N^i \text{ ch } N \lhd M \lhd L.$$

Hence $N^i \lhd^2 L$. Since $L \in \text{Min-}\lhd^2$,

$$N^\omega = N^c \qquad \text{for some } c \in \mathbf{N}.$$

By Lemma 1 $N^c \lhd L$ and therefore by minimality of M

$$N^c \in \text{Min-(asc of step} < \delta).$$

Since $L \in \text{Min-}\lhd^3$, $N^i \in \text{Min-}\lhd$ and therefore

$$N^i/N^{i+1} \in \mathfrak{A} \cap \text{Min-}\lhd \subseteq \mathfrak{F},$$

where $\mathfrak{A}$ (resp. $\mathfrak{F}$) is the class of all abelian (resp. finite-dimensional) Lie algebras. It follows that

$$N/N^c \in \mathfrak{F}.$$

We now use Lemma 3 to conclude that $N \in \text{Min-(asc of step} < \delta)$.

(c) $M \in \text{Min-}\lhd^\gamma$. This follows from the fact that $M \lhd L \in \text{Min-}\lhd^{\gamma+1}$.

Now, since $M \notin \text{Min-(asc of step} < \delta)$, there exists an infinite series $\{I_n\}$ of distinct subalgebras of M such that

$$I_1 \supset I_2 \supset \cdots, \quad I_i \lhd^{\beta_i} M \quad \text{with} \quad \beta_i < \delta \quad (i = 1, 2, \ldots).$$

We may assume that $I_i \lhd^\varepsilon M$ is false for any $\varepsilon < \beta_i$. By (c) $\beta_i \geq \gamma + 1$ for almost all i. Therefore we may furthermore assume that $\beta_i \geq \gamma + 1$ for all i.

Case 1. β_i is not a limit ordinal for some i. In this case, $\beta_i = \varepsilon + 1$ and

$$I_i \lhd^\varepsilon N \lhd M.$$

By (b), $N \in \text{Min-(asc of step} < \delta)$. But $I_i \lhd^\varepsilon N$ and $I_{i+k} \lhd^{\beta_{i+k}} N$ $(k = 1, 2, \ldots)$ with $\varepsilon, \beta_{i+k} < \delta$, which is a contradiction.

Case 2. β_i is a limit ordinal for some i. In this case, by Lemma 4 we have

$$\beta_i = \rho + \sigma \quad \text{with} \quad 1 \leq \sigma \leq \gamma.$$

Here σ is a limit ordinal. We can obviously write

$$I_i \lhd^\rho N \lhd^\sigma M.$$

But $N^{(i)} \lhd^\sigma M$ and therefore $N^{(i)} \lhd^{\sigma+1} L$. Since $\sigma + 1 \leq \gamma + 1$, $L \in \text{Min-}\lhd^{\gamma+1} \subseteq \text{Min-}\lhd^{\sigma+1}$. Consequently

$$N^{(\omega)} = N^{(c)} \qquad \text{for some } c \in \mathbf{N}.$$

Hence $N^{(c)}$ is a perfect ascendant subalgebra of L. It follows from Lemma 2 that $N^{(c)} \lhd L$. By (b), we obtain

$$N^{(c)} \in \text{Min-(asc of step} < \delta).$$

On the other hand, observing the facts that $N \lhd^{\sigma+1} L \in \text{Min-}\lhd^{\sigma+1}$ and that σ is a limit ordinal, we obtain $N^{(j)} \in \text{Min-}\lhd$ $(j=1, 2,\dots)$. Hence

$$N^{(j)}/N^{(j+1)} \in \mathfrak{A} \cap \text{Min-}\lhd \subseteq \mathfrak{F}.$$

Consequently

$$N/N^{(c)} \in \mathfrak{F}.$$

Therefore we can use Lemma 3 to see that

$$N \in \text{Min-(asc of step} < \delta).$$

However, $I_i \lhd^{\rho} N$ and $I_{i+k} \lhd^{\beta_{i+k}} N$ $(k=1, 2,\dots)$ with ρ, $\beta_{i+k} < \delta$, which is a contradiction. This completes the proof.

REMARK. Stewart's result (2) stated in Section 1 is shown in the above proof of Theorem where we replace ω^{α} and $\omega^{\alpha+1}$ by 2 and ω respectively.

References

[1] R. K. Amayo and I. Stewart, Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.

[2] I. Stewart, The minimal condition for subideals of Lie algebras, Math. Z. **111** (1969), 301–310.

[3] S. Tôgô, Radicals of infinite-dimensional Lie algebras, Hiroshima Math. J. **2** (1972), 179–203.

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