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# *Nonoscillation Criteria for Differential Equations of the Second Order*

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## 1. Introduction

In this paper we consider the perturbed second order nonlinear differential equation

$$(E) \qquad (a(t)x')' + q(t)f(x) = e(t, x, x').$$

In the last twenty years many authors have studied the oscillatory behavior of equations of this type especially when $e(t, x, x') \equiv 0$. Fortunately, several surveys of known results have been done, the most recent of which are by Wong [8, 9]. While many sufficient conditions for oscillation are known, there are relatively few theorems which guarantee that $(E)$ has a nonoscillatory solution (see [1–9] and the references contained therein). Far fewer results guaranteeing that all solutions of $(E)$ be nonoscillatory are known, and in fact, when $e(t, x, x') \not\equiv 0$, only the results of Graef [1] and Graef and Spikes [2–7] apply.

In this paper we obtain sufficient conditions for all solutions of $(E)$ to be nonoscillatory. This is accomplished by comparing $(E)$ to an unperturbed nonlinear equation in Theorems 3 and 4 and to an unperturbed linear equation in Theorem 5. Use is made of a nonlinear Picone type identity introduced by the authors in [7].

## 2. Nonoscillation Criteria

Consider the equations

$$(1) \qquad (a(t)x')' + q(t)f(x) = e(t, x, x')$$

and

$$(2) \qquad (a_1(t)x')' + q_1(t)f_1(x) = 0,$$

where $a, a_1, q, q_1 \colon [t_0, \infty) \to R, f, f_1 \colon R \to R$, and $e \colon [t_0, \infty) \times R^2 \to R$ are continuous, $a(t) > 0$, and $a_1(t) > 0$. It will be convenient to use the same classifica-

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tion of solutions used in [2–7]. That is, a solution $x(t)$ of (1) or (2) will be called nonoscillatory if there exists $t_1 \geq t_0$ such that $x(t) \neq 0$ for $t \geq t_1$; the solution will be called oscillatory if for any given $t_1 \geq t_0$ there exist $t_2$ and $t_3$ satisfying $t_1 < t_2 < t_3$, $x(t_2) > 0$ and $x(t_3) < 0$; and it will be called a $Z$-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive. We will say that an equation is nonoscillatory if all its solutions are nonoscillatory.

The following two lemmas will be needed in order to prove our first two nonoscillation results. In the statements of these lemmas we adopt the notation: $h'(t)_+ = \max\{h'(t), 0\}$, $h'(t)_- = \max\{-h'(t), 0\}$, and $F(x) = \int_0^x f(s)ds$.

LEMMA 1. *Suppose that* $xf_1(x) > 0$ *for* $x \neq 0$, $f_1'(x) \geq 0$ *for all* $x$, $q_1(t) > 0$, $\int_{t_0}^{\infty} [q_1'(s)_+ / q_1(s)]ds < \infty$, *there are positive constants* $m$ *and* $M$ *such that* $m \leq a_1(t) \leq M$, *and* $k > 0$ *is given. If*

$$\int_{t_0}^{\infty} q_1(s)f_1(ks)ds < \infty \left( \int_{t_0}^{\infty} q_1(s)f_1(-ks)ds > -\infty \right),$$

*then there exists a solution* $x(t)$ *of* (2) *and* $T \geq t_0$ *such that* $x(t) \geq mk(t-T)/2M$ $(x(t) \leq -mk(t-T)/2M)$ *for* $t \geq T$.

The above lemma was proved in [2; Theorem 1]. A simple modification of the proof of Lemma 2 in [6] yields the following result.

LEMMA 2. *Assume that there is a positive constant* $D$ *such that* $F(x) > -D$, $q(t) > 0$, $\int_{t_0}^{\infty} [q'(s)_+ / q(s)]ds < \infty$, $\int_{t_0}^{\infty} [a'(s)_- / a(s)]ds < \infty$, *and there is a continuous function* $r$: $[t_0, \infty) \to R$ *such that* $|e(t, x, x')| \leq r(t)$ *and* $\int_{t_0}^{\infty} [r(s)/a(s)]ds < \infty$. *If* $x(t)$ *is a solution of* (1), *then* $x'(t)$ *is bounded and there exist* $A > 0$ *and* $T \geq t_0$ *such that* $|x(t)| \leq At$ *for* $t \geq T$.

We are now ready to prove our first nonoscillation result.

THEOREM 3. *Assume that* $q_1(t) > q(t)$, $a(t) \geq a_1(t)$, $f(0) = 0$, *and the hypotheses of Lemmas* 1 *and* 2 *hold for all* $k > 0$.

(i) *Suppose that* $f'(x) \geq 0$ *for* $x \geq 0$, $e(t, x, x') \geq 0$, *and there exists* $K > 0$ *such that* $v \geq K$ *and* $v \geq u \geq 0$ *implies* $f_1'(v) \geq f'(u)$. *Then no solution of* (1) *is oscillatory or nonnegative Z-type. If, in addition,* $f(x) \leq 0$ *for* $x \leq 0$, *then equation* (1) *is nonoscillatory.*

(ii) *Suppose that* $f'(x) \geq 0$ *for* $x \leq 0$, $e(t, x, x') \leq 0$, *and there exists* $K > 0$ *such that* $v \leq -K$ *and* $v \leq u \leq 0$ *implies* $f_1'(v) \geq f'(u)$. *Then no solution of* (1) *is oscillatory or nonpositive Z-type. If, in addition,* $f(x) \geq 0$ *for* $x \geq 0$, *then equation* (1) *is nonoscillatory.*

PROOF. To prove (i), assume that $x(t)$ is an oscillatory or nonnegative $Z$-type solution of (1). By Lemma 2, there exist $A>0$ and $T\geq t_0$ such that $|x(t)|\leq At$ for $t\geq T$. From Lemma 1 it follows that there is a solution $y(t)$ of (2) and $T_1\geq T$ such that $y(t)\geq 2At\geq K$ for $t\geq T_1$. Now let $t_1$ and $t_2$ be consecutive zeros of $x(t)$ with $t_2>t_1\geq T_1$ and $x(t)>0$ for $t_1<t<t_2$. Define $S$: $[T_1, \infty)\to R$ by

$$S(t) = f(x(t))\,[f_1(y(t))a(t)x'(t) - f(x(t))a_1(t)y'(t)]/f_1(y(t)).$$

Then

$$\begin{aligned} S'(t) &= f(x(t))\,(a(t)x'(t))' + a(t)f'(x(t))\,[x'(t)]^2 \\ &\quad - f^2(x(t))\,(a_1(t)y'(t))'/f_1(y(t)) \\ &\quad - 2a_1(t)y'(t)f(x(t))f'(x(t))x'(t)/f_1(y(t)) \\ &\quad + a_1(t)\,[y'(t)]^2 f^2(x(t))f_1'(y(t))/f_1^2(y(t)) \\ &= f(x(t))e(t, x(t), x'(t)) + [q_1(t) - q(t)]f^2(x(t)) \\ &\quad + a_1(t)f_1'(y(t))\,\{f^2(x(t))\,[y'(t)]^2/f_1^2(y(t)) \\ &\quad - 2f(x(t))f'(x(t))y'(t)x'(t)/f_1(y(t))f_1'(y(t)) \\ &\quad + [f'(x(t))]^2[x'(t)]^2/[f_1'(y(t))]^2\} \\ &\quad - a_1(t)\,[f'(x(t))]^2[x'(t)]^2/f_1'(y(t)) \\ &\quad + a(t)f'(x(t))\,[x'(t)]^2. \end{aligned}$$

Hence,

$$\begin{aligned} S'(t) &= f(x(t))e(t, x(t), x'(t)) + [q_1(t) - q(t)]f^2(x(t)) \\ &\quad + a_1(t)f_1'(y(t))\,\{f(x(t))y'(t)/f_1(y(t)) \\ &\quad - f'(x(t))x'(t)/f_1'(y(t))\}^2 + f'(x(t))\,[x'(t)]^2\{a(t) \\ &\quad - a_1(t)f'(x(t))/f_1'(y(t))\}. \end{aligned}$$

By our choice of $T_1$, we have $f'(x(t))/f_1'(y(t))\leq 1$, so we now integrate from $t_1$ to $t_2$. While the integral of the left hand side of the above equation is zero, the integral of the right hand side is positive, and we have a contradiction.

To complete the proof of (i) suppose that $f(x)\leq 0$ for $x\leq 0$ and let $x(t)$ be a nonpositive $Z$-type solution of (1), say $x(t)\leq 0$ for $t\geq t_3\geq t_0$. From (1) we have

$$(a(t)x'(t))' = e(t, x(t), x'(t)) - q(t)f(x(t)) \geq 0$$

for $t \geq t_3$. Choosing $t_4 \geq t_3$ to be a zero of $x'(t)$ and integrating we have

$$a(t)x'(t) \geq a(t_4)x'(t_4) = 0$$

for $t \geq t_4$. Thus $x'(t) \geq 0$ for $t \geq t_4$ which is impossible for a $Z$-type solution.

The proof of part (ii) proceeds in a similar fashion by taking a negative loop of a solution. The details will be omitted.

It is interesting to note that the hypotheses on $f$ and $f_1$ in the above theorem (and Theorem 4 below) are satisfied if $f(x)=x^n$ and $f_1(x)=x^N$ where $n$ and $N$ are odd positive integers with $N \geq n$. For example, we can conclude from part (i) of Theorem 3 that the equation

$$(t^3x')' + x^3/t^8 = tx^2(\operatorname{sech} x')/(x^2+1), \qquad t \geq 1 \tag{3}$$

is nonoscillatory by comparing it to the equation

$$x'' + x^5/t^7 = 0, \qquad t \geq 1.$$

The nonoscillation of equation (3) cannot be deduced from other known nonoscillation criteria for perturbed nonlinear equations (see [1–7]).

REMARK 1. The nonlinear Picone type identity obtained by differentiating $S(t)$ in the proof of the above theorem was first introduced by the authors in [7].

REMARK 2. If $f(x) \equiv f_1(x)$, $a(t) \equiv a_1(t)$, and $e(t, x, x') \equiv r(t)$, then Theorem 3 includes Theorem 4 of Graef and Spikes [2] as a special case (see also the Theorem in [3]).

In our next theorem we place a different type of condition on the perturbation term $e(t, x, x')$.

THEOREM 4. *Suppose that* $q_1(t) > q(t)$, $a(t) \geq a_1(t)$, $xf(x) \geq 0$, $xe(t, x, x') \geq 0$, *and* $f'(x) \geq 0$ *for all* $x$. *Furthermore, assume that the hypotheses of Lemmas* 1 *and* 2 *hold for all* $k > 0$, *and there exists* $K > 0$ *such that for* $|v| \geq K$ *and either* $v \geq u \geq 0$ *or* $v \leq u \leq 0$ *we have* $f_1'(v) \geq f'(u)$. *Then equation* (1) *is nonoscillatory.*

PROOF. The proof of this theorem is quite similar to the proof of Theorem 3. The hypotheses here guarantee that $S'(t) > 0$ on a subinterval of $(t_1, t_2)$ regardless of whether $x(t) > 0$ or $x(t) < 0$ on $(t_1, t_2)$.

REMARK 3. The strict inequality $q_1(t) > q(t)$ in Theorems 3 and 4 can be relaxed to $q_1(t) \geq q(t)$ provided that other conditions are imposed to insure that $S'(t) > 0$ on some subinterval of $(t_1, t_2)$. Also, Lemmas 1 and 2 can be replaced by any other set of hypotheses which would yield the same growth estimates on the solutions of (2) and (1) respectively.

By using

$$x'' + x^3/t^5 = 0, \qquad t \geq 1$$

as a comparison equation, Theorem 4 shows that the equation

$$x'' + x^3/t^5 = x/t^2(x^2 + 1), \qquad t \geq 1$$

is nonoscillatory, and, moreover, this result cannot be obtained from other known nonoscillation criteria. Note also that $e(t, x, x')$ is allowed to change signs with $x$. This is somewhat unusual since it is known (c.f. [2, 3, 6]) that if $e(t, x, x') \equiv r(t)$ and $r(t)$ changes signs, then equation (1) may have oscillatory solutions even when the unperturbed equation is nonoscillatory.

In our final theorem we place a condition on $e(t, x, x')$ which is significantly different from those usually found in the literature. We will compare (1) to the linear equation

$$(a_2(t)x')' + q_2(t)x = 0, \tag{4}$$

where $a_2$, $q_2$: $[t_0, \infty) \to R$ are continuous and $a_2(t) > 0$.

THEOREM 5. *Let* $a(t) \geq a_2(t)$, $q_2(t) > q(t)$, $f(0) = 0$,

$$f(x) > 0 \qquad \textit{for} \quad x > 0,$$

*and assume that there exists a continuous function* $W$: $R \to R$ *such that* $W(x) > 0$, $W'(x) \geq 0$,

$$0 \leq f'(x) \leq W(x) \qquad \textit{for} \quad x \geq 0,$$

*and*

$$e(t, x, x')W(x) \geq q_2(t)[W(x) - 1]f(x) \qquad \textit{for} \quad x \geq 0.$$

*If equation* (4) *is nonoscillatory, then no solution of* (1) *is oscillatory or nonnegative Z-type. If, in addition,* $f(x) \leq 0$ *and* $e(t, x, x') \geq 0$ *for* $x \leq 0$, *and* $q(t) \geq 0$, *then equation* (1) *is nonoscillatory.*

PROOF. Suppose that $x(t)$ is an oscillatory or nonnegative $Z$-type solution of (1). Let $y(t)$ be a solution of (4); we can assume with no loss in generality that $y(t) > 0$ for $t \geq t_1 \geq t_0$. Then there exist $t_2$ and $t_3$ such that $t_3 > t_2 \geq t_1$, $x(t_2) = x(t_3) = 0$, and $x(t) > 0$ for $t_2 < t < t_3$. Defining $P$: $[t_1, \infty) \to R$ by

$$P(t) = f(x(t))[W(x(t))a(t)x'(t)y(t) - a_2(t)y'(t)f(x(t))]/y(t)$$

and differentiating, we have

$$\begin{aligned}
P'(t) &= f(x(t))W(x(t))e(t, x(t), x'(t)) - q(t)f^2(x(t))W(x(t)) \\
&\quad + a(t)f(x(t))W'(x(t))[x'(t)]^2 + a(t)f'(x(t))W(x(t))[x'(t)]^2 \\
&\quad + q_2(t)f^2(x(t)) - 2a_2(t)f(x(t))f'(x(t))x'(t)y'(t)/y(t) \\
&\quad + a_2(t)f^2(x(t))[y'(t)]^2/y^2(t) \\
&= f(x(t))\{W(x(t))e(t, x(t), x'(t)) - q_2(t)[W(x(t)) \\
&\quad - 1]f(x(t))\} + [q_2(t) - q(t)]W(x(t))f^2(x(t)) \\
&\quad + a(t)f(x(t))W'(x(t))[x'(t)]^2 \\
&\quad + a_2(t)\{f(x(t))y'(t)/y(t) - f'(x(t))x'(t)\}^2 \\
&\quad + a(t)f'(x(t))W(x(t))[x'(t)]^2 - a_2(t)[f'(x(t))]^2[x'(t)]^2 \\
&\geq f(x(t))\{W(x(t))e(t, x(t), x'(t)) - q_2(t)[W(x(t)) \\
&\quad - 1]f(x(t))\} + [q_2(t) - q(t)]W(x(t))f^2(x(t)) \\
&\quad + a(t)f(x(t))W'(x(t))[x'(t)]^2 + a_2(t)\{f(x(t))y'(t)/y(t) \\
&\quad - f'(x(t))x'(t)\}^2 + [a(t) - a_2(t)]f'(x(t))W(x(t))[x'(t)]^2.
\end{aligned}$$

Integrating from $t_2$ to $t_3$ again yields a contradiction. That, under the additional conditions, equation (1) has no nonpositive $Z$-type solutions follows as before.

REMARK 4. As mentioned in Remark 3, the strict inequalities in Theorem 5 can be relaxed as long as we have $P'(t)>0$ on some subinterval of $(t_2, t_3)$.

As examples of Theorem 5 we see that the equation

$$x'' + (\sin t)x^3/t^3 = x^6/t^2 + \cosh x' + 3t, \qquad t \geq 1$$

has no oscillatory or nonnegative $Z$-type solutions, and the equation

$$x'' + x^3/t^3 = x^6/t^2 + \cosh x' + 3t, \qquad t \geq 1 \tag{5}$$

is nonoscillatory. In both cases we compare to the nonoscillatory linear equation

$$x'' + x/t^3 = 0, \qquad t \geq 1.$$

That equation (5) is nonoscillatory is somewhat surprising since the present authors [3] have shown that if $e(t, x, x') \equiv r(t) \geq 0$, then equation (1) may have oscillatory solutions unless $\int_{t_0}^{\infty} r(s)ds < \infty$. Once again, the nonoscillation of (5) cannot be deduced from other known nonoscillation criteria.

In conclusion, we note that in Theorem 5, $e(t, x, x')$ may change signs for $x \geq 0$.

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