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On the Oscillatory and Asymptotic Behavior of Solutions of a Certain Fourth Order Linear Differential Equation

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1. Introduction. Recently there has been renewed interest in the asymptotic and oscillation properties of solutions of fourth order differential equations, see [1], [4]. The purpose of this paper is to consider these properties for the solutions of the nonselfadjoint differential equation

$$(L) \qquad (y''' + py)' + py' + qy = 0.$$

The following assumptions concerning p and q will be made at various stages in the paper.

H_0: p and q are continuous functions from $[0, \infty)$ to $[0, \infty)$.

H_1: p' is continuous, $q>0$, $pq \not\equiv 0$ on an interval.

H_2: $\int^{\infty}(q-p')dx=\infty$, p/q bounded.

A solution y of (L) is said to be *oscillatory* if y has an unbounded set of zeros $\{x_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_k=\infty$; otherwise, a solution is said to be *non-oscillatory*. Equation (L) is termed *oscillatory* if it has an oscillatory solution.

As a special case of (L) we have the selfadjoint equation

$$(1) \qquad y^{iv} + qy = 0,$$

which has been studied extensively by M. Svec [6, 7]. The oscillatory character of (1) is relatively simple, since either all the solutions of (1) oscillate or none of them oscillate [7]. Such is not the case for (L).

For example the equations

$$(2) \qquad y^{iv} + 40y' + 39y = 0$$

and

$$(3) \qquad y^{iv} + 4y' + 4y = 0$$

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are examples of (L) with H_0, H_1 and H_2 satisfied. Equation (2) has both oscillatory and nonoscillatory solutions, e.g., $y=e^{2x}\sin 3x$ and $y=e^{-x}$, while all the solutions of (3) are oscillatory. As a means of separating these cases we will study the connection between (i, j)-disconjugacy and oscillation. Recall that (L) is said to be *(i, j)-disconjugate* if i and j are positive integers such that $i+j=4$ and no solution of (L) has an (i, j)-distribution of zeros i.e. no nontrivial solution has a pair of zeros of multiplicities i and j, respectively. If no nontrivial solution of (L) has more than three (3) zeros, the equation is termed *disconjugate.*

2. Preliminary Results. We begin our study of (L) by considering a functional which plays a vital role in our investigation. Throughout this section we assume that H_0 and H_1 hold.

LEMMA 2.1. *Let $y(x)$ be a solution of (L), then*

$$F[y(x)] = y(x)(y''' + py)(x) - y'(x)y''(x)$$

is nonincreasing

PROOF. Differentiating $F[y(x)]$ and making substitutions from (L) we find that $F'[y(x)]=-y''^2(x)-q(x)y^2(x)$, from which the Lemma follows. An immediate consequence of the lemma is

COROLLARY 2.2. *Equation (L) is $(2, 2)$-disconjugate.*

Using the functional $F[y]$ we now classify the solutions of (L). A solution y will be called type I if $F[y(x)]\geq 0$ on $[0, \infty)$ and type II otherwise. While it is easy to construct a type II solution, it is not immediately evident that (L) has a nontrivial type I solution. We sketch a proof which shows the existence of such a solution.

THEOREM 2.3. *Equation (L) has a nontrivial type I solution.*

PROOF. Let $x=a$ be a number such that $0\leq a<1$ and suppose $y_1(x)$, $y_2(x)$ and $y_3(x)$ are three linearly independent solutions of (L) which vanish at $x=a$. For each positive integer n, let

$$u_n(x) = c_{1n}y(x) + c_{2n}y(x) + c_{3n}y_3(x)$$

be a solution of (L) satisfying

$$u_n(a) = 0, \quad u_n(n) = u_n'(n) = 0$$

and $c_{1n}^2+c_{3n}^2+c_{3n}^2=1$. Suppose further, without loss of generality that $\lim_{n\to\infty}c_{in}=c_i$, $i=1, 2, 3$. Let $u(x)=c_1y_1(x)+c_2y_2(x)+c_3y_3(x)$. Since $u_n(x)\to u(x)$ on compact subsets of $[0, \infty)$ and $F[u_n(x)]>0$ on $[0, n)$ for each n, it follows that

$F[u(x)] \geq 0$ on $[0, \infty)$. Moreover, $u(x) \not\equiv 0$, since $c_1^2 + c_2^2 + c_3^2 = 1$ and the proof is complete.

Before we proceed we need the following lemma due to Leighton and Nehari [3].

LEMMA 2.4. *Let $u(t)$ and $v(t)$ be two functions with continuous first derivatives in (α, β) and let $v(t)$ be of constant sign in this interval. If $x = a$ and $x = b$ $(\alpha < a < b < \beta)$ are two consecutive zeros of $u(t)$, then there exists a constant λ such that $u(t) - \lambda v(t)$ has a double zero in (a, b).*

THEOREM 2.5. *All type II solutions have the same oscillatory character.*

PROOF. The solution $y_3(x, a)$ defined by $y_3(a) = y_3'(a) = y_3''(a) = 0$, $y_3'''(a) = 1$ is a type II solution. We assert that if $y_3(x, a)$ is oscillatory and $b > a$, then $y_3(x, b)$ is oscillatory. Suppose the contrary, i.e., assume $y_3(x, a)$ oscillatory and $y_3(x, b)$ is positive on (x_1, ∞), for some $x_1 > b$. We can assume, without loss of generality, that $y_3(b, a) > 0$.

If $\alpha < \beta$ are consecutive zeros of $y_3(x, a)$ in (x_1, ∞) on which $y_3(x, a) > 0$ then by lemma 2.4 there is a constant $k > 0$ such that the solution $Z(x) = y(x, a) - ky(x, b)$ has a double zero in (α, β). Thus $F[Z(b)] > 0$ since $b < \alpha$. But $F[Z(b)] = F[y_3(b, a)] - ky_3(b, a) < 0$. This contradiction proves our assertion.

Now let $u(x)$ be an arbitrary type II solution of (L). Suppose, without loss of generality, that $u(x)$ is nonoscillatory and positive on some interval $[c, \infty)$, where c is chosen large enough so that $F[u(c)] < 0$. Then, since $y_3(x, c)$ is oscillatory, there is a linear combination
$w(x) = y_3(x, c) - ku(x)$, $k > 0$, having a double zero at $x = d > c$. Thus $F[w(d)] = 0$. But

$$F[w(c)] = -ku(c)[1 - ku'''(c) - kp(c)u(c)] - k^2u'(c)u''(c)$$
$$= -ku(c) + k^2F[u(c)] < 0,$$

a contradiction, since F is nonincreasing on (c, d). This contradiction shows that if $y_3(x, a)$ oscillates, then every type II solution oscillates. It can be shown in a similar manner that whenever (L) has an oscillatory type II solution then $y_3(x, a)$ oscillates. Thus our proof is complete.

A sufficient condition for (L) to be oscillatory is now given.

THEOREM 2.6. *Suppose $\int^{\infty} p(t)dt = \infty$ and $y(x)$ is a solution of (L) satisfying $F[y(c)] < 0$, for some $c \geq 0$. Then $y(x)$ is oscillatory.*

PROOF. Suppose the contrary, i.e., suppose $F[y(c)] < 0$ and that $y(x)$

nonoscillatory. Then there exists $a \geq c$ such that $y(x) \neq 0$, for $x \geq a$. We can suppose, without loss of generality, that $y(x) > 0$ on $[a, \infty)$.

Consider the function

$$H(x) = \frac{y''(x)}{y(x)} + \int_a^x p(t)\,dt.$$

By differentiating $H(x)$ we obtain

$$H'(x) = \frac{F[(x)]}{y^2(x)} < 0\,.$$

Consequently, $H(x)$ is decreasing. Since $\int_a^\infty p(x)dx = \infty$, it follows that $y''(x) < 0$ for large x and since $y(x) > 0$ it follows that $y'(x) > 0$ for large x.

The fact that $\int_a^x p(t)dt \to \infty$ as $x \to \infty$ implies $y''(x)/y(x) \to -\infty$ as $x \to \infty$. But this implies $y''(x)$ is bounded away from zero for large x since $y(x)$ is increasing. This is clearly impossible because $y''(x)$ bounded away from zero and negative will imply $y(x) \to -\infty$ as $x \to \infty$. We therefore conclude that $y(x)$ oscillates. This completes the proof.

We now investigate the effect of certain disconjugacy conditions upon the oscillation of (L). For a complete discussion of the oscillation numbers $r_{ij}(t)$ which appear below, consult Peterson [5]. For our purposes $r_{ij}(t) < \infty$ will mean (L) is not (i, j)-disconjugate on $[t, \infty)$ and $r_{ij} = \infty$ will mean (L) is (i, j)-disconjugate on $[0, \infty)$.

THEOREM 2.7. *If $r_{13}(t) < \infty$ and $r_{31}(t) < \infty$ for all t on $[0, \infty)$, then (L) is strongly oscillatory.*

PROOF. We show that type I solutions are oscillatory. Suppose $y(t) > 0$ on $[a, \infty)$ for some $a > 0$ and $F[y(x)] > 0$ on $[a, \infty)$. Let $u(x)$ be a nontrivial solution of (1) having a 1–3 distribution of zeros at $x = b$ and $x = c = r_{13}(b)$, $a < b < c$, $u(x) \neq 0$ on (b, c). We can assume $u'''(c) = 1$. Since $y(x) > 0$ on $[a, \infty)$ and $u(x) < 0$ on (b, c), there exists a positive constant k such that $v(x) = y(x) + ku(x)$ has a double zero in (b, c). Since $F[v(x)]$ is decreasing and $F[v(x)]$ vanishes in (b, c), we conclude that $F[v(c)] < 0$. But $F[v(c)] = F[y(c)] + ky(c) > 0$. This contradiction proves that type I solutions must have a zero on $(b, r_{13}(b))$ for every $b > 0$.

Fix $t_0 = b > 0$. Define $t_n = r_{13}(t_{n-1})$, for each positive integer n. Then by the above argument each type I solution vanishes in (t_{n-1}, t_n) for each n. This proves that type I solutions oscillate. The proof that type II solutions oscillate is essentially the same.

COROLLARY 2.8. *If (L) has a nonoscillatory solution, then for some* $t \in [0, \infty)$ *either* $r_{13}(t)=\infty$ *or* $r_{31}(t)=\infty$.

Using Corollary 2.8 and the fact that $z=e^{-x}$ is a type I solution of

$$z'''' + 40z' + 39z = 0 \tag{2}$$

we conclude that no nontrivial solution of (2) has a 1–3 distribution of zeros on $[0, \infty)$ and therefore $r_{13}(t)=\infty$ for each $t \geq 0$.

We now give a converse for Theorem 2.7.

THEOREM 2.9. *If (L) is strongly oscillatory, then* $r_{31}(t)<\infty$ *and* $r_{13}^{(t)}<\infty$ *for all* $t>0$.

PROOF. From the assumption (L) is strongly oscillatory, $r_{31}(t)<\infty$ follows easily. To see that $r_{13}(t)<\infty$ for each t we proceed indirectly.

Suppose there is a t such that $r_{13}(t)=\infty$. Because $r_{22}(t)=\infty$ using the techniques found in [2, Theorem 3.2] we can construct a nonoscillatory solution of (L), contradicting the assumption that all solutions are oscillatory. From this contradiction we conclude that $r_{13}(t)<\infty$ for each $t \geq 0$.

The following lemma gives sufficient conditions for (L) to be disconjugate on some interval $[a, \infty)$.

LEMMA 2.10. *If* $r_{13}=r_{31}=r_{22}=\infty$ *on* $[a, \infty)$ *for some* a, *then no nontrivial solution of (L) can have more than three zeros on* $[a, \infty)$.

A proof of lemma 2.10 can be constructed using results in [3].

3. Asymptotic Properties. We now begin a study of the asymptotic behavior of the solutions of (L). We will assume H_0, H_1 and H_2 hold throughout this section.

THEOREM 3.1. *If y is a type I solution of (L), then*

(a) $\int^{\infty} y''^2(x)dx < \infty$, *and*

(b) $\int^{\infty} q(x)y^2(x)dx < \infty$.

PROOF. Let y be a type I solution, then $F[y(x)] \geq 0$ and $\lim_{x\to\infty} F[y(x)]$ exists. Differentiating $F[y(x)]$ and then integrating from 0 to x we obtain

$$F[y(x)] = F[y(0)] - \int_0^x y''^2(t)dt - \int_0^x q(t)y^2(t)dt.$$

From which we see that

$$\int_0^x y''^2(t)dt \leq F[y(0)] \quad \text{and} \quad \int_0^x q(t)y^2(t)dt \leq F[y(0)].$$

Since x is arbitrary, our theorem follows.

THEOREM 3.2. *Let $y(x)$ be a nontrivial type I solution of (L). Then* $\lim_{x\to\infty} F[y(x)]=0$.

PROOF. Suppose not, then $\lim_{x\to\infty} F[y(x)]=k>0$ and because $F[y(x)]$ is decreasing, $F[y(x)]>k$ for all x. Writing $F[y(x)]>k$ as

$$(y(x)y''(x) - y'^2(x))' + p(x)y^2(x) > k > 0$$

and integrating from 0 to x, we conclude that

$$\lim_{x\to\infty} [y(x)y''(x) - y'^2(x) + \int_0^x p(t)y^2(t)dt] = \infty.$$

Since $p(x)\leq Mq(x)$ for some $M>0$, it follows from (b) of the preceding theorem that

$$\int^\infty p(x)y^2(x)dx < \infty.$$

But this implies that

$$\lim_{x\to\infty} [y(x)y''(x) - y'^2(x)] = \infty. \tag{4}$$

If y were oscillatory we would have an immediate contradiction and conclude that $k=0$. From (4), we see that neither y nor y'' can be oscillatory and that $\operatorname{sgn} y(x)=\operatorname{sgn} y''(x)$ on some ray $[b, \infty)$ where b is chosen large enough so that $y(x)y'(x)y''(x)\neq 0$ on $[b, \infty)$. If $y(x)>0$ and $y'(x)>0$ on $[b, \infty)$, then $y(x)>y(b)$ for all $x>b$. Integrating (L) from b to x we obtain

$$y'''(x) + 2p(x)y(x) + \int_b^x [q(t) - p'(t)]y(t)dt = y'''(b) + 2p(b)y(b). \tag{5}$$

Since $y(x)>y(b)$ for $x>b$ the following inequality is a consequence of (5)

$$y'''(x) + 2p(x)y(x) + y(b)\int_b^x [q(t) - p'(t)]dt \leqq y'''(b) + 2p(b)y(b). \tag{6}$$

Inequality (6) and H_2, however, implies $y'''(x)$ is negative and bounded away from zero. But this is impossible since $y''(x)>0$ for large x.

Now suppose $y(x)>0$, $y'(x)<0$ on $[b, \infty)$. Then $y(x)$ and $y'(x)$ are bounded. From (4), it follows that $y''(x)$ is positive and unbounded, contradicting the boundedness of $y'(x)$. Since all possibilities have been exhausted for $y'(x)$, we conclude that $\lim_{x\to\infty} F[y(x)]=0$.

A characterization of type I solutions can now be given. We begin with a result on boundedness.

THEOREM 3.3. *Let $y(x)$ be a type I solution of (L). Then $y'(x)$ is bounded.*

PROOF. From the preceding theorem, we know that $\lim_{x\to\infty} F[y(x)]=0$. If $y(x)\equiv 0$, we are done, so suppose $y(x)$ is nontrivial. Suppose $y(x)$ is oscillatory, then writing $F[y(x)]>0$ as

$$[y(x)y''(x) - y'^2(x)]' + p(x)y^2(x) > 0$$

and integrating from b to x yields

$$y(x)y''(x) - y'^2(x) + \int_b^x p(t)y^2(t)dt < y(b)y''(b) - y'^2(b).$$

As seen previously, H_2 implies

$$\int_b^\infty p(x)y^2(x)dx < \infty.$$

Let $\{x_n\}_{n-1}^\infty$ be a sequence of zeros $y''(x)$ such that $\lim_{n\to\infty} x_n=\infty$. Then for $x_n > b$

$$\int_b^{x_n} p(t)y^2(t)dt + y'^2(b) - y(b)y''(b) > y'^2(x_n)$$

and the boundedness of $y'(x)$ follows.

Now consider the case where $y(x)$ does not oscillate. Assume $y(x)>0$ on $[a, \infty)$ for some $a\geq 0$. As shown above, $F[y(x)]>0$ implies

$$y(x)y''(x) - y'^2(x) + \int_a^x p(t)y^2(t)dt + y'^2(a) - y(a)y''(a) > 0.$$

If $y''(x)$ oscillates, the bounded of $y'^2(x)$ follows immediately. So we assume $y''(x)$ is one sign on $[b, \infty)$, $b>a$. If $y''(x)<0$ on $[b, \infty)$, then $y'(x)$ must be positive for large x and is therefore bounded. Finally suppose $y''(x)>0$ and $y'(x)$ unbounded on $[b, \infty)$. Then $y(x)$ is increasing and proceeding as in theorem 3.2. We conclude that $y'''(x)$ is negative and bounded away from zero, contradicting $y''(x)>0$ on $[b, \infty)$. Hence $y(x)>0$, $y'(x)>0$, $y''(x)>0$ cannot occur on any ray. This completes the proof of the theorem.

We now turn our attention to the type II solutions. The proof of the theorem is similar to the preceding ones and therefore omitted.

THEOREM 3.4. *Let $y(x)$ be a type II solution of (L). Then $y'(x)$ is unbounded.*

From theorems 3.3 and 3.4 it follows that type I solutions are precisely those solutions of (L) with a bounded first derivative. Using this characterization it is clear that the set of type I solutions is a subspace of the solution space. In fact, using the techniques of theorem 2.3 one can construct two independent type I solutions. However, if there were three independent type I solutions, then some nontrivial linear combination of them would have a double zero and would therefore be a type II solution. Since this is impossible, the type I solutions form a two-dimensional subspace of the solution space. We record this in our final result.

THEOREM 3.5. *The set of type I solutions of (L) form a two-dimensional vector space. Moreover the zeros of any two independent type I solutions separate each other on $[0, \infty)$.*

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