

## ***Forced Nonoscillations in Second Order Functional Equations***

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### 1. Introduction

Recently a great deal of effort has been spent in obtaining criteria and asymptotic properties of oscillatory and nonoscillatory solutions of equations of the type

$$(1) \quad (r(t)y'(t))' + p(t)f(y(t)) = q(t).$$

The reader is referred to the works of Graef and Spikes [3, 4], Kusano and Onose [6], this author [9, 10, 11, 12], Skidmore and Bowers [13] and Skidmore and Leighton [14]. Most of these authors assume the nonnegative nature of  $p(t)$  to arrive at various oscillation criteria.

Very little seems to be known for retarded equations of the form

$$(2) \quad (r(t)y'(t))' + p(t)f(y(g(t))) = q(t).$$

Standard techniques that have been discovered for equation (1) simply do not work for equation (2). Our purpose in this paper is to study equation (2) and find conditions to force all solutions of equation (2) to be nonoscillatory. We shall first prove that under very general conditions, all solutions of (2) may be continued indefinitely on some positive half real line.

### 2. Definitions and assumptions

In what follows we shall assume:  $r, p, q, g \in C[[t_0, \infty), R]$ ,  $t_0 > 0$ ,  $f \in C[R, R]$ ,  $xf(x) > 0$ ,  $f(x)/x \leq m$ ,  $m > 0$ ,  $r(t) > 0$ ,  $g(t) \leq t$ ,  $g'(t) > 0$ ,  $g(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . We call a function  $h \in C[[t_0, \infty), R]$  to be oscillatory if  $h(t)$  has arbitrarily large zeros in  $[t_0, \infty)$ ; otherwise we call  $h(t)$  nonoscillatory.

### 3. Indefinite continuation of solutions

In this section, we prove that under very general conditions, all solutions of equation (2) can be continued indefinitely to the right of  $t_0 > 0$ . See [2, Theorem (2.1)].

**THEOREM (3.1).** *The continuity of  $p(t)$ ,  $f(t)$  and  $q(t)$  for  $t \in [s_0, \infty)$  is sufficient to allow any solution of equation (2) to be continued indefinitely to the right of  $t_0$ .*

**PROOF.** Suppose to the contrary that there exists a solution  $y(t)$  of (2) which exists on  $[s_0, T)$ ,  $T < \infty$  and cannot be continued to the right of  $T$ . Then we have  $g(T) = T$  and

$$(3) \quad \limsup_{t \rightarrow T^-} |y(t)| = \infty.$$

(See Burton and Grimmer [2]). There is  $s_1 \in (s_0, T)$  such that  $g(t) \geq s_0$  for  $t \geq s_1$ . Integrating equation (2) for  $t \in [s_0, T)$ , we get

$$(4) \quad y'(t) - \frac{r(s_0)y'(s_0)}{r(t)} + \frac{1}{r(t)} \int_{s_0}^t p(s)f(y(g(s)))ds = \frac{1}{r(t)} \int_{s_0}^t q(s)ds.$$

Integrating (4) again between  $s_0$  and  $g(t)$ ,  $t \in [s_1, T)$ , we get

$$y(g(t)) = y(s_0) + \int_{s_0}^{g(t)} \frac{r(s_0)y'(s_0)}{r(s)} ds - \int_{s_0}^{g(t)} \frac{1}{r(s)} \int_{s_0}^s p(x)f(y(g(x))) dx ds + \int_{s_0}^{g(t)} \frac{1}{r(s)} \int_{s_0}^s q(x) dx ds.$$

Since  $g(t) \leq t$ , there exists constants  $k_0, k_1$  such that for  $t \in [s_1, T)$  we have from above

$$(5) \quad |y(g(t))| \leq k_0 + k_1 \int_{s_0}^t \int_{s_0}^s |p(x)| |y(g(x))| \frac{|f(y(g(x)))|}{|y(g(x))|} dx ds,$$

where  $1/r(t) \leq k_1$ , and

$$\left| y(s_0) + \int_{s_0}^T \frac{r(s_0)y'(s_0)}{r(s)} ds + \int_{s_0}^T \frac{1}{r(s)} \int_{s_0}^s |q(x)| dx ds \right| \leq k_0.$$

Since  $|f(y(g(x)))|/|y(g(x))| \leq m$ , we get from (5)

$$(6) \quad |y(g(t))| \leq k_0 + mk_1 \int_{s_0}^t (t-s)|p(s)| |y(g(s))| ds$$

from which

$$\frac{|y(g(t))|}{t} \leq \frac{k_0}{t} + mk_1 \int_{s_0}^t \frac{t-s}{t} s |p(s)| \frac{|y(g(s))|}{s} ds$$

for  $t \in [s_1, T)$ . It follows that there are constants  $D_0$  and  $D_1$  such that

$$\frac{|y(g(t))|}{t} \leq D_0 + D_1 \int_{s_1}^t s |p(s)| \frac{|y(g(s))|}{s} ds$$

for  $t \in [s_1, T)$ . By Gronwall's inequality we get

$$\frac{|y(g(t))|}{t} \leq D_0 \exp \left[ D_1 \int_{s_1}^t s |p(s)| ds \right] < \infty$$

for  $t \in [s_1, T)$ . This contradiction to (3) proves the theorem.

REMARK (1). This theorem extends Theorem (2.1) of Graef and Spikes [3] to retarded equations. Their method does not apply to equation (2).

REMARK (2). From now on the term "solution" will apply to continuously extendable solutions of equations under consideration.

#### 4. On nonoscillation of solutions

In this section we first find conditions for boundedness of positive part of oscillatory solutions of equation (2). We, then, use this information to find criteria under which all solutions of equation (2) become nonoscillatory.

LEMMA (4.1). *In addition to conditions outlined in Section (2) suppose*

- (i)  $q(t) \geq 0$ ;
- (ii)  $p(t) \geq 0, \int_0^\infty p(t)dt < \infty$ ;
- (iii)  $\int_0^\infty 1/r(t)dt < \infty$ .

*Then all oscillatory solutions of equation (2) are bounded above.*

PROOF. Let  $y(t)$  be an oscillatory solution of equation (2) defined on  $[t_0, \infty)$ . Suppose to the contrary that

$$(7) \quad \limsup_{t \rightarrow \infty} y(t) = \infty.$$

We choose  $s > t_0$  so large that

$$(8) \quad \int_s^\infty 1/r(t)dt < m,$$

$$(9) \quad m \int_s^\infty p(t)dt < 1,$$

and take  $t_1 \geq s$  for which  $g(t_1) \geq s$  and  $y(t_1) = 0$ . In view of (7) there is  $T > t_1$ , such that

$$\max \{y(t) : s \leq t \leq T\} = y(T) > 0.$$

Let  $[x_1, x_2]$  designate the smallest closed interval containing  $T$  such that  $y(x_1)$

$=y(x_2)=0$  and  $y(t)>0$  for  $t \in (x_1, x_2)$ . Put

$$(10) \quad M = \max \{y(t) : x_1 \leq t \leq x_2\}.$$

It is clear that

$$(11) \quad y(t) \leq M$$

and

$$(12) \quad |y(g(t))| \leq M$$

for  $t \in [x_1, x_2]$ . Let  $s_0 \in [x_1, x_2]$  be such that  $y(s_0)=M$ . Then

$$M = \int_{x_1}^{s_0} y'(t)dt = - \int_{s_0}^{x_2} y'(t)dt.$$

Thus

$$\begin{aligned} 2M &\leq \int_{x_1}^{x_2} |y'(t)|dt \\ &= \int_{x_1}^{x_2} (r(t))^{1/2}(r(t))^{-1/2}|y'(t)|^{1/2}|y'(t)|^{1/2}dt. \end{aligned}$$

On squaring and applying Schwarz's inequality we get

$$4M^2 \leq \left( \int_{x_1}^{x_2} \frac{dt}{r(t)} \right) \left( \int_{x_1}^{x_2} r(t)y'(t) \cdot y'(t) dt \right).$$

Integrating the second integral by parts we get

$$(13) \quad 4M^2 \leq \left[ \int_{x_1}^{x_2} \frac{dt}{r(t)} \right] \left[ - \int_{x_1}^{x_2} (r(t)y'(t))' y(t) dt \right].$$

Making use of equation (2) in (13) we get

$$(14) \quad 4M^2 \leq \left[ \int_{x_1}^{x_2} \frac{dt}{r(t)} \right] \left[ \int_{x_1}^{x_2} p(t)f(y(g(t)))y(t) dt - \int_{x_1}^{x_2} q(t)y(t) dt \right].$$

It follows that

$$4M^2 \leq \left[ \int_{x_1}^{x_2} \frac{dt}{r(t)} \right] \left[ \int_{x_1}^{x_2} p(t)y(t)y(g(t)) \frac{f(y(g(t)))}{y(g(t))} dt \right],$$

which in view of (8), (9), (11) and (12) gives

$$4M^2 \leq \left[ \int_{x_1}^{x_2} \frac{dt}{r(t)} \right] \left[ \int_{x_1}^{x_2} p(t)y(t)y^+(g(t)) \frac{f(y(g(t)))}{y(g(t))} dt \right],$$

$$4 \leq \int_{x_1}^{x_2} \frac{dt}{r(t)} \cdot m \int_{x_1}^{x_2} p(t) dt < 1.$$

This contradiction shows that  $y^+(t) = \max \{y(t), 0\}$  is bounded for  $t \in (s, \infty)$ . This completes the proof of the lemma.

**THEOREM (4.1).** *Suppose conditions of Lemma (4.1) hold. Further suppose  $\int_{a_0}^{\infty} q(t)dt = \infty$ . Then all solutions of equation (2) are nonoscillatory.*

**PROOF.** Suppose to the contrary that  $y(t)$  is an oscillatory solution of equation (2). By Lemma (4.1)  $y^+(t)$  is bounded. Now  $y'(t)$  must be oscillatory. Let  $a_0 > s$  be a zero of  $y'(t)$ . From equation (2)

$$\begin{aligned} r(t)y'(t) &= -\int_{a_0}^t p(x)f(y(g(x)))dx + \int_{a_0}^t q(x)dx \\ &= -\int_{a_0}^t p(x)y(g(x))\frac{f(y(g(x)))}{y(g(x))}dx + \int_{a_0}^t q(x)dx \\ &\geq -m\int_{a_0}^t p(x)y^+(g(x)) + \int_{a_0}^t q(x)dx \longrightarrow \infty \\ &\text{as } t \longrightarrow \infty \end{aligned}$$

since the first integral is bounded. Thus  $y'(t) > 0$  eventually and  $y(t)$  is nonoscillatory. The proof is now complete by contradiction.

**COROLLARY (4.1).** *If in Theorem (4.1) we have  $q \leq 0$  on  $(t_0, \infty)$ , all other conditions being the same, then all solutions of equation (2) are nonoscillatory.*

**PROOF.** If  $y(t)$  is a solution of equation (2), we set  $Z(t) = -y(t)$ . The same proof holds with  $\int_{a_0}^{\infty} -q(t)dt = \infty$ .

**THEOREM (4.2).** *Suppose conditions of Theorem (4.1) hold. Further suppose that  $r(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and*

$$(iv) \quad \liminf_{t \rightarrow \infty} \frac{\int_{t_0}^t q(s)ds}{r(t)} > 0.$$

*Then all solutions of equation (2) are eventually positive.*

**PROOF.** Let  $y(t)$  be a solution of equation (2). Then  $y(t)$  is nonoscillatory. Suppose to the contrary that  $y(g(t)) < 0$  for  $t > P > t_0$ . From equation (2)

$$(15) \quad (r(t)y'(t))' \geq q(t)$$

for  $t \geq P$ . Integrating (15) and dividing by  $r(t)$  we have

$$(16) \quad y'(t) \geq r(P)y'(P)/r(t) + \frac{\int_P^t q(t)dt}{r(t)}.$$

Condition (iv) and (16) imply that  $y'(t)$  is bounded away from zero, which means that  $y(t)$  is eventually positive. This contradiction proves the theorem.

REMARK (3). The following example shows that condition (iv) of Theorem (4.2) cannot be weakened if all other conditions hold.

EXAMPLE (1). Consider the equation

$$(17) \quad (t^3 y'(t))' + \frac{1}{(t-1)^2} y(t-1) = 1 - \frac{1}{(t-1)^3}, \quad t \geq 2.$$

All conditions except (iv) of Theorem (4.2) hold. It is easily verified that

$$\liminf_{t \rightarrow \infty} \frac{\int_2^t q(s) ds}{r(t)} = 0.$$

In fact equation (17) has  $y(t) = -1/t$  as a negative solution.

COROLLARY (4.2). *If  $q(t) \leq 0$  and condition (iv) is modified accordingly, then all solutions of equation (2) are eventually negative.*

REMARK (4). The solution of equation (17) approaches zero. It may seem that violation of condition (iv) of Theorem (4.2) might lead to negative solutions of equation (2) approaching zero. The following example shows that this is not the case.

EXAMPLE (2). Consider the equation

$$(18) \quad (t \exp(t) y'(t))' + \exp(-t - \pi) y(t - \pi) = 1 - \exp(-t - \pi) \exp(-2t).$$

Again condition (iv) is not satisfied but all others are. This equation has  $y(t) = -1 - \exp(-t)$  as a nonvanishing negative solution.

THEOREM (4.3). *Suppose conditions of Theorem (4.2) hold. Let  $y(t)$  be a solution of equation (2). Then  $y(t)$  is unbounded as  $t \rightarrow \infty$ .*

PROOF. By (iv),  $y(t)$  is eventually positive. Let  $T$  be large enough to allow  $y(t) > 0$  and  $y(g(t)) > 0$  for  $t \geq T$ . Suppose to the contrary that

$$(19) \quad y(t) \leq C, \quad t \geq T.$$

Integrating equation (2) we have

$$r(t) y'(t) - r(t) y'(T) + \int_T^t p(s) y(g(s)) \frac{f(y(g(s)))}{y(g(s))} ds = \int_T^t q(s) ds.$$

The above gives

$$(20) \quad r(t)y'(t) \geq r(T)y'(T) - mC \int_T^t p(s)ds + \int_T^t q(s)ds.$$

Dividing (20) by  $r(t)$  we get

$$(21) \quad y'(t) \geq r(T)y'(T)/r(t) - \left(mC \int_T^t p(s)ds\right)/r(t) + \left(\int_T^t q(s)ds\right)/r(t).$$

As  $t \rightarrow \infty$  we find from (21) that

$$y'(t) \geq \beta > 0$$

for some  $\beta$ , implying that  $y(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . This is a contradiction to (19) and the proof is complete.

### 5. Case of bounded $r(t)$

In this section, it is shown that when  $r(t)$  is bounded, then oscillatory solutions of equation (2) are precisely those that are “slowly oscillating.” More precisely, let  $y(t)$  be an oscillatory solution of equation (2). Let  $Z_y$  be the set:

$$Z_y = \{\alpha - \delta : \delta > \alpha, y(\alpha) = y(\delta) = 0, \alpha \text{ and } \delta \text{ being consecutive zeros of } y(t)\}.$$

The following theorem is an adaptation of Theorem 4 of this author in [10].

**THEOREM (5.1).** *Suppose  $\int_0^\infty 1/r(t)dt = \infty$ ,  $p(t) \geq 0$ ,  $\int_0^\infty p(t)dt < \infty$ ,  $q(t) \geq 0$ ,  $\int_0^\infty q(t)dt = \infty$ . Let  $y(t)$  be an oscillatory solution of equation (2). Then the associated set  $Z_y$  is unbounded.*

**PROOF.** Suppose to the contrary that  $Z_y$  is bounded, where we assume that the solution  $y(t)$  is defined on  $[t_0, \infty)$ . It means that there exists a constant  $N > 0$  such that for any pair  $x_1 < x_2$  of consecutive zeros of  $y(t)$  we have

$$(22) \quad x_2 - x_1 \leq N.$$

In a manner of the proof of Lemma (4.1) we will show that  $y^+(t)$  is bounded. In fact due to (22), there exists a constant  $B > 0$  such that

$$(23) \quad \int_\alpha^\delta 1/r(t)dt \leq B$$

for all  $\alpha, \delta$  such that  $y(\alpha) = y(\delta) = 0$ ,  $y(t) \neq 0$ ,  $t \in (\alpha, \delta)$ . Let now  $s > t_0$  be so large that for  $\alpha > s$

$$(24) \quad \int_\alpha^\delta \frac{dt}{r(t)} \cdot m \int_\alpha^\delta p(t)dt < 1.$$

(24) replaces conclusions (8) and (9) in the proof of Lemma (4.1). The whole proof now stays the same verbatim. The last inequality in the proof of Lemma (4.1) is essentially (24). Thus  $y(t)$  is bounded above. The rest of the proof follows from Theorem (4.1). The proof of Theorem (5.1) is now complete.

The following example clarifies the situation by satisfying this theorem.

EXAMPLE (3). Consider the equation

$$(25) \quad y''(t) + \frac{3}{t^2}y(t) = 5 + 4 \sin(\ln t) + 3 \cos(\ln t).$$

It has  $y(t) = t^2(1 + \sin(\ln t))$  as the oscillatory solution. Since the relative extremes of the function  $q(t) = 5 + 4 \sin(\ln t) + 3 \cos(\ln t)$  occur for  $\tan(\ln t) = 4/3$ ,  $q(t) \geq 0$  and  $\int_{\infty}^{\infty} q(t)dt = \infty$ , the zeros of  $y(t)$  occur as  $t_n = e^{n\pi + 3\pi/2}$ ,  $n = 1, 2, \dots$ . It is easily seen that  $(t_{n+1} - t_n) \rightarrow \infty$  as  $n \rightarrow \infty$  so that  $Z_y$  is unbounded.

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### References

- [1] T. Burton and R. Grimmer, On the Asymptotic Behavior of Solutions of  $x''(t) + a(t)f(x(t)) = e(t)$ , *Pacific J. Math.* **41** (1972), 77-88.
- [2] ———, Oscillation, Continuation and Uniqueness of Solutions of Retarded Differential Equations, *Trans. Amer. Math. Soc.*, **179** (1973), 193-209.
- [3] Joh R. Graef and Paul W. Spikes, Continuity, Boundedness and Asymptotic Behavior of Solutions of  $x'' + q(t)f(x) = r(t)$ , *Ann. Mat. Pura Appl.* **101** (1974), 307-320.
- [4] ———, Asymptotic Behavior of Solutions of a Second Order Nonlinear Differential Equation, *J. Differential Equations*, **17** (1975), 471-476.
- [5] Michael E. Hammett, Nonoscillation Properties of a Nonlinear Differential Equation, *Proc. Amer. Math. Soc.* **30** (1971), 92-96.
- [6] T. Kusano and H. Onose, Oscillations of Functional Differential Equations with Retarded Arguments, *J. Differential Equations*, **15** (1974), 269-277.
- [7] S. Londen, Some Nonoscillation Theorems for a Second Order Nonlinear Differential Equation, *SIAM J. Math. Anal.* **4** (1973), 460-465.
- [8] S. Rankin, Oscillation of a Forced Second Order Nonlinear Differential Equation, *Proc. Amer. Math. Soc.* (to appear).
- [9] Bhagat Singh, Asymptotic Nature of Nonoscillatory Solutions of  $n$ th Order Retarded Differential Equations, *SIAM J. Math. Anal.* **6** (1975), 784-795.
- [10] ———, Asymptotically Vanishing Oscillatory Trajectories in Second Order Retarded Equations, *SIAM J. Math. Anal.* **7** (1976), 37-44.



- [11] ———, Impact of Delays on Oscillation in General Functional Equations, Hiroshima J. Math. **5** (1975), 351–361.
- [12] ———, Nonoscillation of Forced Fourth Order Retarded Equations, SIAM J. Appl. Math. **28** (1975), 265–269.
- [13] A. Skidmore and John J. Bowers, Oscillatory Behavior of Solutions of  $y''+p(x)y=f(x)$ , J. Math. Anal. Appl., **49** (1975), 317–323.
- [14] A. Skidmore and W. Leighton, On the Differential Equation  $y''+p(x)y=f(x)$ , J. Math. Anal. Appl., **43** (1973), 46–55.
- [15] H. Teufel, Forced Second Order Nonlinear Oscillations, J. Math. Anal. Appl., **40** (1972), 148–152.
- [16] Curtis C. Travis, Oscillation Theorems for Second Order Differential Equations with Functional Arguments, Proc. Amer. Math. Soc. **30** (1972), 199–201.

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