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On Claw-Decomposition of a Complete Multi-Partite Graph[*)]

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1. Introduction

A multi-partite graph, denoted by $G_m(n_1, n_2,..., n_m)$, is a graph whose point set can be partitioned into m subsets $V_1, V_2,..., V_m$ with $n_1, n_2,..., n_m$ points each, such that every line of $G_m(n_1, n_2,..., n_m)$ joins different subsets. If $G_m(n_1, n_2,..., n_m)$ contains every line joining different subsets, then it is called a complete m-partite graph and is denoted by $K_m(n_1, n_2,..., n_m)$. A complete bipartite graph $K_2(1, c)$ is called a *claw* or a c-claw by specifically indicating its number of lines.

A graph G is said to be c-claw decomposable if it can be decomposed into pairwise line-disjoint claws with c lines each.

The problem of c-claw decomposability of the complete graph and that of the complete bigraph have been raised and cleared up by Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [2]. The purpose of this paper is to give an analogous claw-decomposition theorem for a complete m-partite graph $K_m(n, n,..., n)$ with m sets of n points each.

2. A claw-decomposition theorem

With respect to the claw-decomposition of a complete m-partite graph $K_m(n, n,..., n)$, we have the following theorem.

THEOREM. *$K_m(n, n,..., n)$ can be decomposed into pairwise line-disjoint c-claws if and only if* (i) $\binom{m}{2}n^2 \equiv 0 \pmod{c}$ *and* (ii) $mn \geq 2c$.

Since $K_m(n, n,..., n)$ turns out to be the complete graph K_m when $n=1$, our theorem is a generalization of the c-claw decomposition theorem for K_m which has been given in [2] and has been applied to the design of an efficient storage and retrieval system in [3]. This is also a partial generalization of the theorem for a complete bipartite graph $K_2(n_1, n_2)$ given in [2].

Suppose an arbitrary direction of adjacency is assigned on every line of

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$K_m(n, n,\dots, n)$. Since there are $\binom{m}{2}n^2$ lines in $K_m(n, n,\dots, n)$, the number of possible ways of the assignment of direction to $K_m(n, n,\dots, n)$ is $2^{\binom{m}{2}n^2}$. To each way of assignment there corresponds an $mn \times mn$ adjacency matrix

$$M = \|m_{ik,jl}\|, \ m_{ik,jl} = \begin{cases} 1 & \text{if } v_{ik} \text{ is adjacent to } v_{jl}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

which satisfies

$$\begin{cases} m_{ik,il} = 0 & \text{for all } \ i, k \ \text{ and } \ l, \\ m_{ik,jl} + m_{jl,ik} = 1 & \text{for all } \ i, j \ (\neq i), k \ \text{ and } \ l, \end{cases} \tag{2.2}$$

where v_{ik} is the label of the kth point in V_i and $ik=(i-1)n+k$.

Conversely, if a $0-1$ square matrix $M=\|m_{ik,jl}\|$ of size mn satisfies (2.2), it is an adjacency matrix of $K_m(n, n,\dots, n)$ for some way of the assignment of direction.

In addition to these considerations, we shall provide two lemmas for the proof of Theorem.

LEMMA 1. *$K_m(n, n,\dots, n)$ can be decomposed into pairwise line-disjoint c-claws if and only if there exists a way of the assignment of direction to each line in such a manner that every row sum of the corresponding adjacency matrix M is an integral multiple of c.*

PROOF. This Lemma can be easily verified since every set of c ones standing on the ikth row of M corresponds to a directed c-claw whose root, or point of outdegree c, is v_{ik}.

With respect to a necessary and sufficient condition for the existence of an adjacency matrix M having a given row sum vector, the following lemma for the m-partite tournament which has been given by J. W. Moon [1] is useful.

LEMMA 2. *There exists an adjacency matrix M of a directed $K_m(n, n,\dots, n)$ which has a given row sum vector $(a_{11},\dots, a_{1n}, a_{21},\dots, a_{m1},\dots, a_{mn})$ satisfying $\sum\sum a_{ik}=\binom{m}{2}n^2$ and $a_{i1}\geq a_{i2}\geq\cdots\geq a_{in}$ for all i if and only if the inequality*

$$\sum_{i=1}^{m}\sum_{k=1}^{p_i} a_{ik} \leq (m-1)np - \frac{1}{2}p^2 + \frac{1}{2}(p_1^2 + p_2^2 + \cdots + p_m^2) \tag{2.3}$$

holds for every set of m integers p_i satisfying $0\leq p_i\leq n$, where $p=p_1+p_2+\cdots+p_m$.

PROOF OF THEOREM. (*Necessity*) The condition (i) is obviously necessary. Suppose $mn<2c$ and assume that $K_m(n, n,\dots, n)$ is c-claw decomposable. The number of c-claws is $\binom{m}{2}n^2/c$ and is less than $(m-1)n$. Thus there exists at least one point which is not the root-point (or a point of degree c) of any claw and, consequently, its degree must be less than $(m-1)n$. This contradicts the fact that $K_m(n, n,\dots, n)$ is regular of degree $(m-1)n$. The condition (ii) is, therefore, necessary.

(*Sufficiency*) Suppose the conditions (i) and (ii) are satisfied and let $\binom{m}{2}n^2/c = mna+nb+q$, where $a=\left[\frac{(m-1)n}{2c}\right]$, $0\le b<m$ and $0\le q<n$. Put

$$a_{ik} = \begin{cases} (a+1)c & \text{for} \quad 1 \le (i-1)n+k \le nb+q\,, \\ ac & \text{for} \quad nb+q+1 \le (i-1)n+k \le mn\,, \end{cases} \tag{2.4}$$

then these mn integers are integral multiples of c and satisfy the condition of row sum vector in Lemma 2. The proof of sufficiency will, therefore, be completed from Lemmas 1 and 2 by showing that a_{ik}'s satisfy (2.3).

Case 1. $2c\le mn<2c+n$. In this case, since $a=0$ and $b=m-1$, the left side of (2.3) can be expressed by

$$\sum_{i=1}^{m}\sum_{k=1}^{p_i} a_{ik} = (p_0+q_m)c \tag{2.5}$$

for every set of m integers p_i satisfying $0\le p_i\le n$ and $p=p_1+p_2+\cdots+p_m$, where $p_0=p_1+p_2+\cdots+p_{m-1}$ and $q_m=\min(p_m, q)$. For the right side of (2.3), we have

$$\begin{aligned}(m-1)np &- \frac{1}{2}p^2 + \frac{1}{2}(p_1^2+p_2^2+\cdots+p_m^2) \\ &\ge (m-1)n(p_0+p_m) - \frac{1}{2}p_0^2 - p_0p_m + \frac{p_0^2}{2(m-1)}\,,\end{aligned} \tag{2.6}$$

since $p_1^2+p_2^2+\cdots+p_{m-1}^2\ge p_0^2/(m-1)$. Thus we have

$$\begin{aligned}(m-1)np &- \frac{1}{2}p^2 + \frac{1}{2}(p_1^2+p_2^2+\cdots+p_m^2) - \sum_{i=1}^{m}\sum_{k=1}^{p_i} a_{ik} \\ &\ge -\frac{m-2}{2(m-1)}p_0^2 + \{(m-1)n-p_m-c\}\,p_0 + (m-1)np_m - q_mc\,.\end{aligned} \tag{2.7}$$

The right side of (2.7) is a linear $(m=2)$ or a quadratic $(m>2)$ function $f(p_0)$ of p_0 which is concave and, since $f(0)=(m-1)np_m-q_mc\ge \frac{mn}{2}p_m-q_mc\ge 0$ and $f((m-1)n)=(q-q_m)c\ge 0$, it is nonnegative for every set of m integers p_i satisfying $0\le p_i\le n$. Thus a_{ik}'s satisfy (2.3).

Case 2. $2c+n\le mn$. In this case, $a\ge 1$ and $ac=\frac{(m-1)n}{2}-\frac{rc}{mn}$, where

$r = nb + q$. For the left side of (2.3), we have

$$\sum_{i=1}^{m}\sum_{k=1}^{p_i} a_{ik} \leq p(a+1)c = p\left\{\frac{(m-1)n}{2} + \frac{c}{mn}(mn-r)\right\}$$
$$\leq p\left\{\frac{(m-1)n}{2} + \frac{c}{mn}(mn-p)\right\} \tag{2.8}$$

for $p \leq r$ and

$$\sum_{i=1}^{m}\sum_{k=1}^{p_i} a_{ik} \leq pac + rc = \frac{(m-1)n}{2}p + \frac{rc}{mn}(mn-p)$$
$$\leq p\left\{\frac{(m-1)n}{2} + \frac{c}{mn}(mn-p)\right\} \tag{2.9}$$

for $p > r$. For the right side of (2.3), we have

$$(m-1)np - \frac{p^2}{2} + \frac{1}{2}(p_1^2 + p_2^2 + \cdots + p_m^2) \geq (m-1)np - \frac{1}{2}p^2 + \frac{p^2}{2m},$$

since $p_1^2 + p_2^2 + \cdots + p_m^2 \geq p^2/m$. Thus we have

$$(m-1)np - \frac{p^2}{2} + \frac{1}{2}(p_1^2 + p_2^2 + \cdots + p_m^2) - \sum_{i=1}^{m}\sum_{k=1}^{p_i} a_{ik}$$
$$\geq \frac{(mn-p)p}{2mn}\{(m-1)n - 2c\} \geq 0. \tag{2.10}$$

Thus a_{ik}'s satisfy (2.3).

This completes the proof.

References

[1] J. W. Moon, On the score sequence of an n-partite tournament, *Canad. Math. Bull.* **5** (1962), 51–58.

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[3] S. Yamamoto, H. Ikeda, S. Shige-eda, K. Ushio and N. Hamada, Design of a new balanced file organization scheme with the least redundancy, *Information and Control* **28** (1975), 156–175.

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